Wave Propagation in a Medium With Position-Dependent Permittivity and Permeability

Ashkan Roozbeh

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Prof. Dr. Elvan Yilmaz
Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Physics.

Prof. Dr. Mustafa Halilsoy
Chair, Department of Physics

We certify that we have read this thesis and that in our opinion, it is fully adequate, in scope and quality, as a thesis of the degree of Master of Science in Physics.

Asst. Prof. Dr. S. Habib Mazharimousavi
Supervisor

Examining Committee

1. Prof. Dr. Omar Mustafa
2. Asst. Prof. Dr. S. Habib Mazharimousavi
3. Asst. Prof. Dr. Mustafa Riza

ABSTRACT

The electromagnetic wave propagation in inhomogeneous media is studied. The wave equation in such a medium is obtained. By considering the permittivity as a position-dependent function (i.e, z-dependent) and the permeability as a constant (i.e, $\mu = k_m \mu_0$) the wave equation is determined. The wave function is reported exactly in a medium with smooth step dielectric constant, in terms of hypergeometric function. The asymptotic behaviors of the wave function are examined and the reflection and transmission coefficients are found in perfect agreement with the previous results. Moreover, a smooth double layer case is studied and the corresponding wave function is presented and plotted exactly in terms of Heun function. Besides, the asymptotic behavior of the wave function is appointed. It is noticeable that the result of this thesis published in an international journal.

ÖZ

Ayni cinsten (homojen) olmayan bir ortamda elektromagnetik dalga yayılması incelenmi ştir. Böyle bir ortamda dalga denklemi inşa edilmiştir. Magnetik geçirgenlik katsayısı (/mu) sabit, elektrik geçirgenlik katsayısı (/epsilon) ise z- yönüne bağlı olduğu durumda denklem çözülmüştür. Sürekli tek adım geçişli elektrik parameter durumunda çözüm hipergeometrik fonksiyonlar cinsinden ifade edilmiştir. Kırılma ve geçirme katsayıları önceki homojen durumlardakine indirgenecek şekilde tanımlanmıştır. Çift adım geçişli ortam probleminde ise elektromagnetik dalga Heun fonksiyonları cinsinden kesin olarak bulunmuştur. Asimtotik değerler sayısal incelenip dalga davranışı grafiksel çizilmiştir.

To My Family

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

Faraday's experiments are assumed as the initial point of the new age of electromagnetism. In 19th century, he proved that not only electricity and magnetism are not two distinct segregated phenomena but also they are closely related when they are time-varying quantities [3] .

Electromagnetic theory is a triumph of classical physics that was completed in a set of differential equations by Maxwell between 1855 and 1865. Maxwell's equations for electric field $\vec{E}$ and magnetic field $\vec{B}$ at any frequency are [10]

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \qquad & \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 \qquad & \vec{\nabla} \times \vec{B} &= \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \end{aligned} \tag{1.1}$$

Note that Maxwell's equations refer to a classical point that is conceived as an infinitesimal volume of a macroscopic field, but containing a large number of atoms. In equations (1.1), ρ is the total electric charge density, $\vec{J}$ is the total electric current density and ε_0 and μ_0 are permittivity and permeability of vacuum.

Permittivity, expresses dielectric properties which effect reflection of electromagnetic waves at interfaces and its attenuation within materials [22] . In other words, it gives the response of the medium to the application of an electric field. Similarly the permeability can be defined as the amount of magnetization when we apply a magnetic field to a medium. In the recent years, the problem of electromagnetic waves propagation through dispersive and nondispersive media has drawn a special attention motivated by numerous experiments taking place inside materials [8,9,12,15,16,18,23] .

As an example, we can point out the time-dependent linear medium where the electric permittivity and permeability vary with time. Here, it is important to note that a time-dependent dielectric permittivity system can lead to produce quanta of the electromagnetic field (photons) even from vacuum states [1] . This phenomenon is similar to pure quantum effects such as dynamical Casimir effect (attractive interaction between two perfectly conducting plates separated by a short distance in vacuum). Similarly the classical effects of time-dependent permittivity have been investigated recently, and lead to addition of some extra term in Ampere-Maxwell equation. Literally in other studies, the effect of temperature and frequency on the dielectric permittivity has been explored [22] . Moreover some methods have been developed for accurate implementation of frequency-dependent materials. Recently much attention has been devoted to the development of FDTD (Finite-Difference Time-Domain) methods for solving Maxwell's equations in dispersive media [24] .

The main concern of this study is to deal with the concepts of permittivity and permeability as some functions of position from a classical point of view. Having the electric permitivitty and permeability to be a continuous (isotropic) spatial function has many applications in physics, biology, electronics, meteorology and chemistry [2] .

As an example in chemistry, a novel method has been developed based on the dielectric-continuum solvation model with a position-dependent permittivity that leads to a new algorithm showing the exact solution for the Poisson electrostatic equation [17] . The inhomogeneous medium consists of a solute immersed in a non-uniform continuum medium. This technique frequently has been used to calculate the total electrostatic and the solvation free energy.

Although, in biology the continuum electrostatic model can describe successfuly electrostatic mediated phenomena on atomic scale, there is explicit disagreement about how to determine the permittivity in inhomogeneous media. It is common that in these systems we sharply divide the medium into solvent and solute region and choose two different permittivities for each one. The region between these two parts strongly affects the results of continuum calculations. An example of such a system is a lipid bilayer surrounded by water that the dielectric constant varies continuously from a large value in water to a lower value in the bilayer [19] .

Also, in heavy doped regions the dielectric constant changes with the density of impurity and so with the position. Such a region can be found in bipolar transistors, p-n junctions and solar cells [2] .

In the previous examples the behavior of an inhomogeneous medium in the presence of an external static electric field has been investigated by using Poisson-Boltzmann equation that can be written as

$$\vec{\nabla} \cdot \left[\varepsilon(r) \vec{\nabla} \Psi(r) \right] = -\rho_f(r) - \sum_r c_i^\infty z_i q \lambda(r) exp[\frac{-z_i q \Psi(r)}{\kappa_B T}] \tag{1.2}$$

Here $\varepsilon(r)$ is the permittivity as a function of position, $\Psi(r)$ is the electrostatic potential and $\rho_f(r)$shows the position-dependent charge density of the medium. Also, z_i and c_i show the charge and the concentration of ions, respectively T is the temperature, κ_B is the Boltzmann constant and $\lambda(r)$ is a factor that depends on the accessibility of a position to ions in the medium.

In the present work we shall study the wave behavior in an inhomogeneous medium with electric permittivity $\varepsilon(r)$ and magnetic permeability $\mu(r)$ which are isotropic functions of position. In the absence of external sources $(J_{free} = 0, \rho_{free} = 0)$ using Maxwell's equations which lead us to

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = -\left(\vec{E} \cdot \vec{\nabla}\right) \vec{\nabla}\bar{\varepsilon} - \left(\vec{\nabla}\bar{\varepsilon} \cdot \vec{\nabla}\right) \vec{E} - \vec{\nabla}\left(\bar{\varepsilon} + \bar{\mu}\right) \times \left(\vec{\nabla} \times \vec{E}\right) \tag{1.3}$$

Where $\bar{\varepsilon} = ln\varepsilon$ and $\bar{\mu} = ln\mu$. This equation is discussed in chapter 2.

Next, we shall take Ermakov equation into account (which is also derived from the Maxwell's equations for inhomogeneous transparent media in one dimension) [11] .

Different cases have been considered due to the variation of the permittivity (if it takes place on larger scale than the wavelength we shall use Eikonal equation, and if it happens on the smaller scale standard wave equation can be used), but in the recent article the intermediate case [11] where the permittivity variation takes place on the wavelength scale has been considered. The wave propagation has been considered only in one dimension with a permittivity gradient orthogonal to polarization. As a result, a nonlinear equation is derived from the field amplitude (recognized as the Ermakov-Pinney's equation) in the following form

$$\frac{\partial^2 A}{\partial z^2} + \frac{Q^2}{A^3} = -\varepsilon \kappa_0^2 A \tag{1.4}$$

Where $\kappa_0 = \frac{\omega}{c}$ and $Q = A^2 \frac{\partial q}{\partial z}$ is an exact invariant even for an arbitrary permittivity space dependence. The amplitude A and phase q are real quantities appear from the proposed solution for E as

$$E = Ae^{iq} \tag{1.5}$$

Several solutions have been proposed to satisfy equation (1.4) and clarify the wave behavior in inhomogeneous media. We report that the Ermakov equation's outcomes are in good agreement with ours in the current study.

Chapter 2

THE WAVE EQUATION

In this chapter, we recollect some basic concepts and definitions that are irrevocable to discern the motif. Then, we proceed to derive the wave equation in a medium with both constant and position-dependent permittivity and permeability.

2.1 Basic Concepts And Principles

The first well-known experiment in the history of electromagnetism was done by Petruus Peregrinus (Pierre der Maricourt) in the thirteenth century (which was an attempt to calculate the force, that was generated by a spherical magnet) . But the concept of energy transport was uncovered until 1887 (by Heinrich Hertz) and the discovery led to the unification of electrodynamics and optics [3] . Hertz defined the wave as a disturbance of a continuous, non-dispersive and non-absorptive medium that propagates with a fixed shape at constant velocity. Chronologically Maxwell's equations had been introduced formerly by James Clerk Maxwell in 1862.

In the most general form Maxwell's equations can be written as [10]

$$\begin{aligned} \vec{\nabla}\cdot\vec{B} &= 0 & \vec{\nabla}\cdot\vec{D} &= \rho_f \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{H} &= \frac{\partial\vec{D}}{\partial t}+\vec{J}_f \end{aligned} \tag{2.1}$$

This set of equations in a sourceless, infinite medium ($\rho_f = 0, \vec{J}_f = 0$), reads

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \cdot \vec{D} = 0 \tag{2.2}$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

Herein $\vec{D} = \varepsilon \vec{E}$ is the displacement vector, where $\vec{E}$ is the electric field. $\vec{H}$ is the auxiliary magnetic field which can be represented as $\vec{H} = \frac{\vec{B}}{\mu}$, where B is the magnetic field. Two crucial parameters ε and μ play the main role in this study.

Using the third equation in (2.1) and applying curl operator on the both sides we derive

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{B} \tag{2.3}$$

Which leads to

$$\vec{\nabla}\left[\vec{\nabla}\frac{1}{\varepsilon} \cdot \vec{D} + \frac{1}{\varepsilon}\vec{\nabla} \cdot \vec{D}\right] - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left[\vec{\nabla}\mu \times \vec{H} + \mu \vec{\nabla} \times \vec{H}\right] \tag{2.4}$$

Considering ε and μ to be constants and taking the time-dependent part of electric field to be $e^{i\omega t}$ one gets Helmholtz wave equation

$$(\nabla^2 + \mu\varepsilon\omega^2)\vec{E} = 0 \qquad (\nabla^2 + \mu\varepsilon\omega^2)\vec{B} = 0 \tag{2.5}$$

Now if we assume that the wave is travelling in z direction, by applying separation of variables in a nondispersive medium, one easily gets the plane wave equation as

$$E(z,t) = E_0 e^{i\kappa z - i\omega t} \tag{2.6}$$

$$B(z,t) = B_0 e^{i\kappa z - i\omega t}$$

Here κ is the wave number and the magnitude is $\sqrt{\mu\varepsilon}\omega$, consequently ν (the phase velocity) can be described as

$$\nu = \frac{\omega}{\kappa} = \frac{c}{n} \tag{2.7}$$

Note that c is the speed of light in vacuum and n is the index of refraction which equals $\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$ that is almost everywhere a position-dependent or a frequency-dependent function.

2.2 Wave Equation In Non-Homogeneous Media

We have reviewed the Maxwell's equations and the plane wave equation in a medium with constant permittivity and permeability. Now we treat the permittivity ε and permeability μ as position-dependent functions (i.e. z-dependent $\mu(z)$ and $\varepsilon(z)$) . The Maxwell's equations in such a medium obey the same form as before. Applying the same method, the third equation of (2.2) becomes

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \vec{\nabla} \times (\mu(z)\vec{H}) \tag{2.8}$$

Since

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon(z)}\vec{\nabla}\cdot\vec{D} + \vec{\nabla}\frac{1}{\varepsilon(z)}\cdot D = -\vec{\nabla}\bar{\varepsilon}\cdot\vec{E} \tag{2.9}$$

here $(\vec{\nabla}\cdot\vec{E})$ can be substituted with $(-\vec{\nabla}\bar{\varepsilon}\cdot\vec{E})$ if we assume $\bar{\varepsilon} = \ln\varepsilon(z)$, so that we express (2.8) as

$$\vec{\nabla}\left[\vec{\nabla}\bar{\varepsilon}\cdot\vec{E}\right] + \nabla^2\vec{E} = \frac{\partial}{\partial t}\vec{\nabla}\times\left[\mu(z)\vec{H}\right] \tag{2.10}$$

With some manipulations, we get

$$\vec{\nabla}\left[\vec{\nabla}\bar{\varepsilon}\cdot\vec{E}\right] + \nabla^2\vec{E} = \vec{\nabla}\mu(z)\times\frac{\partial\vec{H}}{\partial t} + \mu(z)\varepsilon(z)\frac{\partial^2\vec{E}}{\partial t^2} \tag{2.11}$$

Taking into account $\bar{\mu} = \ln\mu(z)$, we have

$$\nabla^2\vec{E} - \mu(z)\varepsilon(z)\frac{\partial^2\vec{E}}{\partial t^2} = -\vec{\nabla}\left[\vec{\nabla}\bar{\varepsilon}\cdot\vec{E}\right] + \vec{\nabla}\bar{\mu}\times\frac{\partial\vec{B}}{\partial t} \tag{2.12}$$

using the following identity [4]

$$\vec{\nabla}\left[\vec{A}\cdot\vec{B}\right] = \left(\vec{B}\cdot\vec{\nabla}\right)\vec{A} + \left(\vec{A}\cdot\vec{\nabla}\right)\vec{B} + \vec{B}\times\left(\vec{\nabla}\times\vec{A}\right) + \vec{A}\times\left(\vec{\nabla}\times\vec{B}\right) \tag{2.13}$$

(2.11) can be expressed as

$$\nabla^2\vec{E} - \mu(z)\varepsilon(z)\frac{\partial^2\vec{E}}{\partial t^2} = -\left(\vec{E}\cdot\vec{\nabla}\right)\vec{\nabla}\bar{\varepsilon} - \left(\vec{\nabla}\bar{\varepsilon}\cdot\vec{\nabla}\right)\vec{E} - \vec{\nabla}\left(\bar{\varepsilon}+\bar{\mu}\right)\times\left(\vec{\nabla}\times\vec{E}\right) \tag{2.14}$$

Knowing that the permittivity and permeability are only z-dependent the gradient operator acts like a simple derivative in $\hat{k}$ direction so that the latter equation reduces to

$$\frac{\partial^2 \vec{E}}{\partial z^2} - \mu(z)\varepsilon(z)\frac{\partial^2 \vec{E}}{\partial t^2} = -\left(\bar{\varepsilon}'\frac{\partial}{\partial z}\right)\vec{E} - \left(\vec{E}\cdot\vec{\nabla}\right)\bar{\varepsilon}'\hat{k} - \left(\bar{\varepsilon}' + \bar{\mu}'\right)\hat{k}\times\left(\vec{\nabla}\times\vec{E}\right) \quad (2.15)$$

Where ε' and μ' are derivatives of ε and μ with respect to z (from now on we write ε and μ instead of $\varepsilon(z)$ and $\mu(z)$) . It's obvious that our electromagnetic wave (no matter what direction it goes to) , is a function of (x, y, z, t). For further predigestion we presume it to be a z-dependent variable. Although this assumption seems to obscure at the first glance, due to the symmetry of the medium it's plausible.Therefore (2.14) can be written as

$$\left(\frac{\partial^2}{\partial z^2} - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)E_x = \bar{\mu}'\frac{\partial E_x}{\partial z} \quad (2.16)$$

$$\left(\frac{\partial^2}{\partial z^2} - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)E_y = \bar{\mu}'\frac{\partial E_y}{\partial z} \quad (2.17)$$

$$\left(\frac{\partial^2}{\partial z^2} - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)E_z = -\bar{\varepsilon}'E_z' - \bar{\varepsilon}''E_z \quad (2.18)$$

We can separate time-dependent and position-dependent parts of the wave function and write it as

$$\vec{E}(z,t) = \vec{\bar{E}}(z)e^{i\omega t} \quad (2.19)$$

Where ω is the frequency of the wave. As we know, electromagnetic waves are transverse waves which means that if we consider the propagation in z-direction the electric and magnetic components will be in x and y directions. Knowing about the symmetry of the medium we can rotate the system of coordinates and change the direction of electric and the magnetic components of the wave. Considering the propagation in z-direction, automatically (2.17) will be satisfied, while (2.15) and (2.16) yield

$$\left(\frac{d^2}{dz^2} + \mu\varepsilon\omega^2\right)\bar{E}_i(z) = \frac{\mu'}{\mu}\frac{d\bar{E}_i(z)}{dz} \tag{2.20}$$

Having $i = x, y$ the latter equation can be represented in one direction as

$$\left(\frac{d^2}{dz^2} + \mu\varepsilon\omega^2\right)\bar{E}_x(z) = \frac{\mu'}{\mu}\frac{d\bar{E}_x(z)}{dz} \tag{2.21}$$

We assumed the electric component of the electromagnetic wave to be in x direction and the other two components considered to be zero. It's crystal clear that if we take the permittivity $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ the latter equation can be simplified as

$$\left(\frac{d^2}{dz^2} + \frac{\omega^2}{c^2}\right)\bar{E}_x(z) = 0 \tag{2.22}$$

In the above equation ε_0 and μ_0 are vacuum permittivity and permeability, and the solution of the equation admits a plane wave propagating in z direction

$$\bar{E}_x = E_{x0}\, e^{\mp i\kappa z} \tag{2.23}$$

Where κ is the wave number ($\kappa = \frac{\omega}{c}$).

To recall the subject at hand, our main concern now is to solve equation (2.20) . For any further scrutiny we need to know the exact form of ε and μ. In the following two chapters we examine two different forms of ε and μ and try to solve equation (2.20) according to our hypothesis.

Chapter 3

SMOOTH STEP DIELECTRIC CONSTANT

In the previous chapter we found the wave equation in a medium with position-dependent properties and also explained that we need to determine the form of permittivity and permeability for further simplifications and manipulations. In this chapter we suppose to examine the smooth step dielectric constant, While we consider a permittivity function that is changing moderately through a medium.

3.1 The Wave Equation

In the previous chapter we got the following wave equation in the most general form for a medium with position-dependent properties

$$\left(\frac{d^2}{dz^2}+\mu\varepsilon\omega^2\right)\bar{E}_x(z)=\frac{\mu'}{\mu}\frac{d\bar{E}_x(z)}{dz} \tag{3.1}$$

where ω is the wave frequency, ε and μ are permittivity and permeability and μ' is the permittivity rate of change, as we the wave propagates in z direction. Firstly, we define the permeability and permittivity to be

$$\mu = k_m \mu_0 \tag{3.2}$$

$$\varepsilon = k_e(z)\varepsilon_0 \tag{3.3}$$

Herein $k_e(z)$ can be defined in the following form

$$k_e(z) = k_2 - \frac{\Delta K}{4}\left(1 - \tanh(az)\right)^2 \tag{3.4}$$

Here dimension of a is the inverse unit length $(1\frac{}{m),and}\Delta k = k_2 - k_1$, where k_1 and k_2 are

$$\begin{aligned} k_1 &= \lim_{z \to -\infty} k_e(z) \\ k_2 &= \lim_{z \to +\infty} k_e(z) \end{aligned} \tag{3.5}$$

If we consider $a = 0.6$ $(\frac{1}{m})$ the behaviour of $k_e(z)$ can be seen in figure 1, which is changing smoothly as we move through the medium

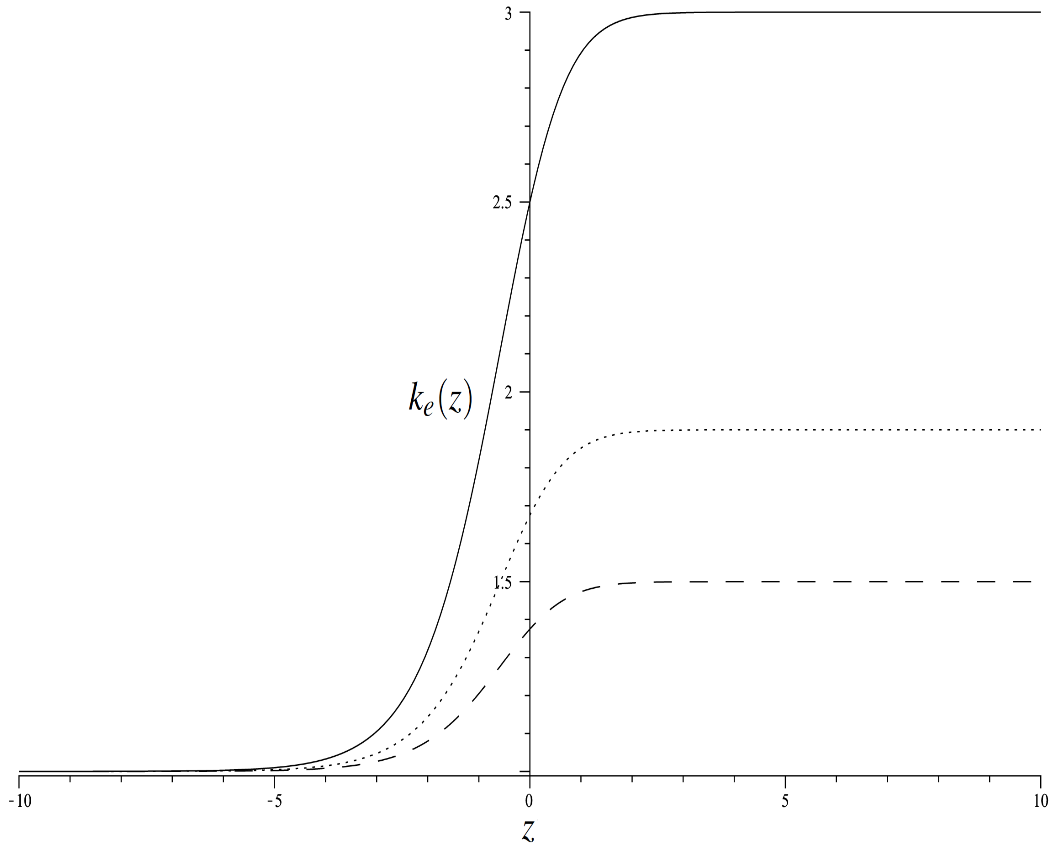

Figure 1. Shows the behavior of the permittivity function when z changes from -10 to $+10$, The plots are sketched when $k_1 = 1$ and $k_2 = 3$ with line, when $k_1 = 1$ and $k_2 = 1.9$ with dotted line and when $k_1 = 1$ and $k_2 = 1.5$ with dashed line.

Therefore, equation (3.1) can be expressed as

$$\left(\frac{d^2}{dz^2}+\frac{\omega^2}{c^2}k_m k_e(z)\right)\bar{E}_x(z)=0 \tag{3.6}$$

For further convenience, we make the following change of variables

$$\kappa^2=\frac{\omega^2}{c^2}k_m k_1 \tag{3.7}$$

$$\nu^2=\frac{\omega^2}{c^2}k_m k_2 \tag{3.8}$$

and define the dimensionless parameter

$$\xi=-e^{-2az} \tag{3.9}$$

Due to our new variables we redefine the wave function in the form

$$\bar{E}_x(z)=\left(-\xi\right)^{-i\nu}F(\xi) \tag{3.10}$$

And upon substitution into (3.6) it turns to (derivatives are with respect to z)

$$\xi F''+(1-2i\nu)F'+\frac{1}{4a^2}\left[\frac{(1-4a^2)\nu^2}{\xi}+\frac{\nu^2-\kappa^2}{1-\xi}-\frac{\nu^2-\kappa^2}{(1-\xi)^2}\right]F=0 \tag{3.11}$$

It's clear that we have two singularities at $\xi = 0$ and $\xi = 1$. To solve this problem we substitute

$$F(\xi) = \xi^{\sigma}(\xi - 1)^{\rho} G(\xi) \tag{3.12}$$

where

$$\sigma = \frac{i\nu(2a-1)}{2a} \tag{3.13}$$

and

$$\rho = \frac{1}{2}(1 - \frac{1}{a}\sqrt{a^2 + \nu^2 - \kappa^2}) \tag{3.14}$$

After some manipulations (3.11) can be simplified in the form of hypergeometric differential equation

$$\xi(\xi-1)G'' + \left[\frac{i\nu - a}{a} - (\frac{i\nu - 2a}{a} + \frac{1}{a}\sqrt{a^2+\nu^2-\kappa^2})\xi\right]G' \\ - \frac{i\nu - a}{2a^2}\left[a - \sqrt{a^2+\nu^2-\kappa^2}\right]G = 0 \tag{3.15}$$

Therefore equation (3.15) is the wave equation in a medium with a smooth step dielectric constant, and the wave function can explain wave behaviour in such a medium.

3.2 Solution of the Wave Equation

In the previous section we found that the wave equation (3.15) is a hypergeometric differential equation. In this section we solve this equation and explain the wave behaviour.

A priori, we recollect that an ordinary second-order linear differential equation in the form of hypergeometric differential equation can be written as [5,7]

$$\xi(\xi-1)G'' + \Big[(\alpha+\beta+1)\xi-\gamma\Big]G' + \alpha\beta G = 0 \tag{3.16}$$

Or in the self-adjoint form

$$\left(e^{-\xi}\xi^{\gamma}G'\right)' - \left(\alpha e^{-\xi}\xi^{\gamma-1}\right)G = 0 \tag{3.17}$$

With solutions, in the most general form, as

$$G = C_1F(\alpha,\beta;\gamma;\xi) + C_2\xi^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;\xi) \tag{3.18}$$

or

$$\begin{aligned} G = C_1(-\xi)^{-\alpha}F(\alpha,\alpha-\gamma+1;\alpha-\beta+1;\xi^{-1}) \\ + C_2(-\xi)^{-\beta}F(\beta-\gamma+1,\beta;\beta-\gamma;\xi^{-1}) \end{aligned} \tag{3.19}$$

Equation (3.18) can be used when γ is not an integer and (3.19) can be used when γ is an integer (here we use (3.18)). Comparing equation (3.16) with (3.15) one easily gets:

$$\alpha = \frac{1}{2a}\left[a - \sqrt{a^2 + \nu^2 - \kappa^2} - i(\nu + \kappa)\right] \tag{3.20}$$

$$\beta = \frac{1}{2a}\left[a - \sqrt{a^2 + \nu^2 - \kappa^2} - i(\nu - \kappa)\right] \tag{3.21}$$

and

$$\gamma = \frac{a - i\nu}{a} \tag{3.22}$$

Using equation (3.16), $G(\xi)$ admits

$$\begin{aligned} G(\xi) = C_1 F\Bigg(& \frac{1}{2a}\left[a - \sqrt{a^2 + \nu^2 - \kappa^2} - i(\nu + \kappa)\right], \\ & \frac{1}{2a}\left[a - \sqrt{a^2 + \nu^2 - \kappa^2} - i(\nu - \kappa)\right]; \frac{a - i\nu}{a}; \xi\Bigg) + C_2 \\ & \xi^{1-\gamma} F\Bigg(\frac{1}{2a}\left[a - \sqrt{a^2 + \nu^2 - \kappa^2} - i(\nu + \kappa)\right] - \frac{a - i\nu}{a} + 1, \\ & \frac{1}{2a}\left[a - \sqrt{a^2 + \nu^2 - \kappa^2} - i(\nu - \kappa)\right] - \frac{a - i\nu}{a} + 1; 2 - \frac{a - i\nu}{a}; \xi\Bigg) \end{aligned} \tag{3.23}$$

To find $\bar{E}_x(z)$ we should go backward, so the general solution for electric field in such a medium is

$$\bar{E}_x(z) = C_1(-1)^{\sigma+\rho}(-\xi)^{-i\nu\frac{1}{2a}}(1-\xi)^{\rho}F(\alpha,\beta;\gamma;\xi)+$$
$$C_2(-1)^{\sigma+\rho+1-\gamma}(-\xi)^{-i\nu\frac{1}{2a}}(1-\xi)^{\rho}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;\xi) \quad (3.24)$$

If we take $\bar{C_1} = C_1(-1)^{\sigma+\rho}$ and $\bar{C_2} = C_2(-1)^{\sigma+\rho+1-\gamma}$ our latter equation reduces to

$$\bar{E}_x(z) = \bar{C_1}(-\xi)^{-i\nu\frac{1}{2a}}(1-\xi)^{\rho}F(\alpha,\beta;\gamma;\xi)+$$
$$\bar{C_2}(-\xi)^{-i\nu\frac{1}{2a}}(1-\xi)^{\rho}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;\xi) \quad (3.25)$$

The next step is to consider the boundary conditions which are essential to determine two integration constants C_1 and C_2 . First, we assume that the electromagnetic wave is moving from $z = -\infty$ toward $z = +\infty$. It is evident that when $z \to +\infty$, $\xi = -e^{-2az} \to 0$ and

$$F(\alpha,\beta;\gamma,0) = 1 \quad (3.26)$$

Thus the previous equation reduces to

$$\lim_{z\to+\infty} \bar{E}_x(z) = \bar{C_1}e^{i\nu z} + \bar{C_2}e^{-i\nu z} \quad (3.27)$$

Since in this limit the electromagnetic wave propagates only in positive z direction, the second term vanishes. Therefore, we can set $\bar{C}_2 = 0$. In conclusion (3.25) becomes

$$\bar{E}_x(z) = \bar{C}_1(-\xi)^{-i\nu\frac{1}{2a}}(1-\xi)^{\rho}F(\alpha,\beta;\gamma;\xi) \tag{3.28}$$

and

$$\lim_{z\to+\infty}\bar{E}_x(z) = \bar{C}_1 e^{i\nu z} = E_{02}e^{i\nu z} \tag{3.29}$$

Where E_{02} is the amplitude of the transmitted wave. Obviously this result is completely in agreement with what we expected, due to our previous knowledge about wave transmission, from a medium. We have already checked the limit, when $z \to +\infty$. Now we continue the discussion in the other direction when $z \to -\infty$. Since $\xi = -e^{-2az}$ it admits that $\xi \to -\infty$ and we must find

$$\lim_{z\to-\infty}\bar{E}_x(z) \tag{3.30}$$

One can easily conclude that in this limit we should have two different terms in two different directions. The first is our transmitted wave in the possitive z direction and the second is the reflected one in the negative z direction. In this step, the first task that we should deal with is to find the limits of hypergeometric function when $z \to -\infty$. To examine this condition we should use the following property

$$F(\alpha,\beta;\gamma;\xi) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-1)^{\alpha}\xi^{-\alpha}F(\alpha,\alpha+1-\gamma;\alpha+1-\beta;\frac{1}{\xi})$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(-1)^{\beta}\xi^{-\beta}F(\beta,\beta+1-\gamma;\beta+1-\alpha;\frac{1}{\xi}) \quad (3.31)$$

This property helps us to eliminate hypergeometric functions since

$$\lim_{\xi\to-\infty} F(\alpha,\alpha+1-\gamma;\alpha+1-\beta;\frac{1}{\xi}) = 1 \quad (3.32)$$

and

$$\lim_{\xi\to-\infty} F(\beta,\beta+1-\gamma;\beta+1-\alpha;\frac{1}{\xi}) = 1 \quad (3.33)$$

Consequently (3.31) yields

$$\lim_{z,\xi\to-\infty} F(\alpha,\beta;\gamma;\xi) = \lim_{z,\xi\to-\infty} \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-1)^{\alpha}\xi^{-\alpha}$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(-1)^{\beta}\xi^{-\beta} \quad (3.34)$$

Using the following formula

$$\bar{E}_x(z) = E_{02}(-\xi)^{\frac{-i\nu}{2a}}(1-\xi)^{\rho}F(\alpha,\beta;\gamma;\xi) \quad (3.35)$$

With substitution one gets

$$\lim_{z,\xi\to-\infty}\bar{E}_x(z)=E_{02}(\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-\xi)^{-\alpha-\frac{i\nu}{2a}+\rho}+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(-\xi)^{-\beta-\frac{i\nu}{2a}+\rho}) \quad (3.36)$$

In the next step according to the equations (3.20), (3.21), (3.14), (3.8) we have α, β , ρ and ν therefore

$$-\alpha-\frac{i\nu}{2a}+\rho=+i\frac{\kappa}{2a} \quad (3.37)$$

and

$$-\beta-\frac{i\nu}{2a}+\rho=-i\frac{\kappa}{2a} \quad (3.38)$$

As a result, (3.36) can be written as below

$$\lim_{z,\xi\to-\infty}\bar{E}_x(z)=E_{02}\left\{\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}e^{-i\kappa z}+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}e^{i\kappa z}\right\} \quad (3.39)$$

As we illustrated before we expect to have two different waves, transmitted wave and reflected wave in positive and negative direction. Respectively (3.39) must be equal to

$$\lim_{z,\xi\to-\infty} \bar{E}_x(z) = E'_{01}e^{-i\kappa z} + E_{01}e^{i\kappa z} \tag{3.40}$$

By analogy one concludes that the amplitude of transmitted wave is

$$E_{01} = \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}E_{02} \tag{3.41}$$

and the amplitude of reflected wave is

$$E'_{01} = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}E_{02} \tag{3.42}$$

Now we find two new quantities, reflection and transmission coefficients. The reflection coefficient R can be described as the fraction of the reflected wave to the incident wave, and the transmission coefficient T is the fraction of the transmitted wave to the incident wave.Therefore

$$R = \frac{E'_{01}}{E_{01}} = \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} \tag{3.43}$$

and

$$T = \frac{E_{02}}{E_{01}} = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)\Gamma(\alpha-\beta)} \tag{3.44}$$

3.3 An Investigation of a Sharp Step Dielectric

In the previous section we defined and derived the reflection and the transmission coefficients in a medium that the permittivity is position-dependent conforming the function that was described in (3.4) . In the following section we examine the correctness of our results at the interface of two dielectrics with different permittivities. As remembered, in this condition we describe the reflection and transmission coefficient [6]

$$R = \frac{E_r}{E_i} \tag{3.45}$$

and

$$T = \frac{E_t}{E_i} \tag{3.46}$$

The boundary condition for the normal magnetic field yields

$$\frac{n_1}{C} sin(\theta_i)(E_i + E_r) = \frac{n_2}{C} sin(\theta_t) E_t \tag{3.47}$$

Herein, n is the optical index of the space. Similarly for the tangential magnetic field

$$\frac{n_1}{\mu_1 C} cos(\theta_i)(E_0 - E_r) = \frac{n_2}{\mu_2 C} cos(\theta_t) E_t \tag{3.48}$$

From these two equations we can define

$$R = \frac{\frac{n_1}{\mu_1} cos(\theta_i) - \frac{n_2}{\mu_2} cos(\theta_t)}{\frac{n_1}{\mu_1} cos(\theta_i) + \frac{n_2}{\mu_2} cos(\theta_t)} \tag{3.49}$$

Now if we consider a condition where $\mu_1 = \mu_2$ (our case) , the latter equation yields

$$R = \frac{n_1 cos(\theta_i) - n_2 cos(\theta_t)}{n_1 cos(\theta_i) + n_2 cos(\theta_t)} \tag{3.50}$$

In the case of normal incidence, transmission and reflection, where $\theta_t = \theta_i = \theta_r = 0$ it can be simplified as

$$R = \frac{n_1 - n_2}{n_1 + n_2} \tag{3.51}$$

Also we may calculate the transmission coefficient by the same steps

$$T = \frac{2\frac{n_1}{\mu_1} cos(\theta_i)}{\frac{n_1}{\mu_1} cos(\theta_i) + \frac{n_2}{\mu_2} cos(\theta_t)} \tag{3.52}$$

Then we assume $\mu_1 = \mu_2$ we have

$$T = \frac{2n_1 cos(\theta_i)}{n_1 cos(\theta_i) + n_2 cos(\theta_t)} \tag{3.53}$$

Finally for the normal incidence, transmission and reflection, one gets

$$T = \frac{2n_1}{n_1 + n_2} \tag{3.54}$$

Now we go back to our main argument. We should show that if we are at the interface of two dielectric, equations (3.43) and (3.44) can be written exactly in the form of (3.51) and (3.54). To do so we need to take the limits of equations (3.43) and (3.44) when $a \to +\infty$ (as $a \to +\infty$ in equation (3.4) $\tanh(az) \to 1$). Therefore, $k_e(z) = k_2 = const$ and it means that we have two media (say two dielectrics) with different permittivities, thus

$$\lim_{a\to+\infty} R = \lim_{a\to+\infty} \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} = \frac{\kappa-\nu}{\kappa+\nu} \tag{3.55}$$

Which after substitution of (3.7) and (3.8) it reads

$$\lim_{a\to+\infty} R = \frac{\sqrt{k_m k_1} - \sqrt{k_m k_2}}{\sqrt{k_m k_1} + \sqrt{k_m k_2}} = \frac{n_1 - n_2}{n_1 + n_2} \tag{3.56}$$

The expression we obtained is completely in agreement with (3.51) for the reflection coefficient.

To get the transmission coefficient we should follow the same way. Using (3.41) and taking the limit when $a \to +\infty$ we get

$$\frac{2\kappa}{\kappa+\nu} = \frac{2\sqrt{k_m k_1}}{\sqrt{k_m k_1}+\sqrt{k_m k_2}} \tag{3.57}$$

That leads to

$$T = \frac{2n_1}{n_1+n_2} \tag{3.58}$$

Where n is called the optical index or index of refraction and it is a dimensionless quantity. The latter equation is exactly what we had in standard electromagnetics or optics as the transmission coefficient and the reflection coefficient (equation (3.54)). If we take $z = 0$ to be fixed at the boundary of the two surfaces for $z > 0$, optical index is $\sqrt{k_2 k_m}$ and for $z < 0$ is $\sqrt{k_1 k_m}$. To see how the amplitude of the electric field is changing we plot it with respect to z.

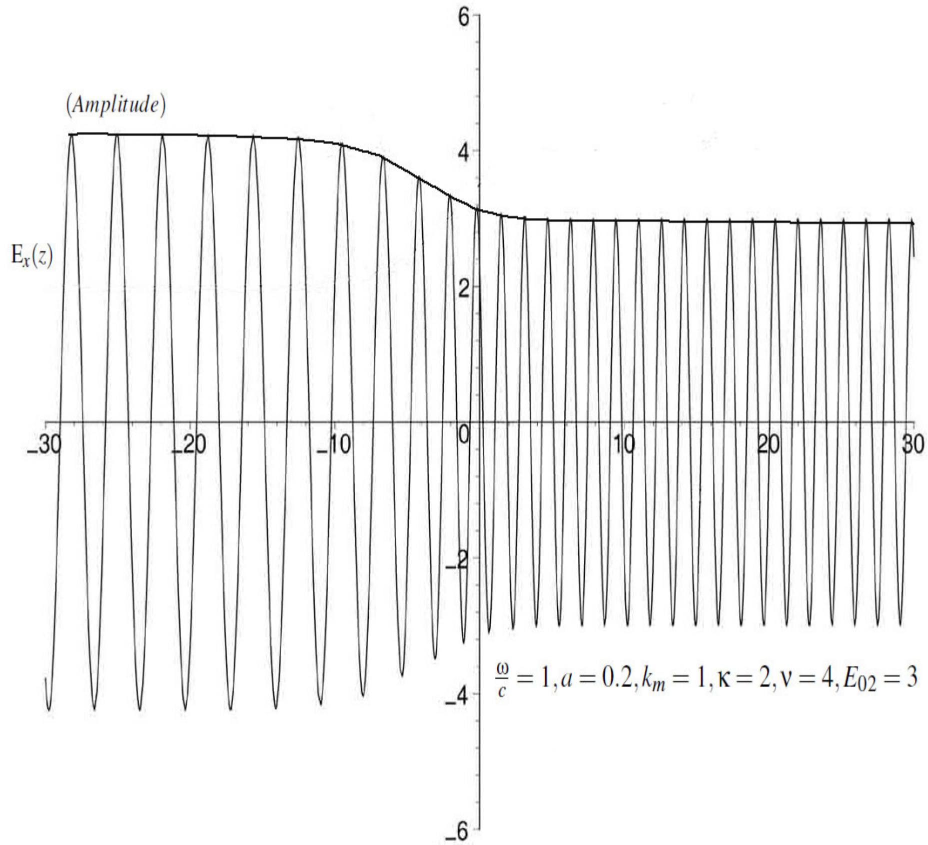

Figure 2. Shows the smooth change in dielectric constant and amplitude of the wave

3.4 Behavior of the Magnetic Field

To complete this chapter we write the complete form of electric field as

$$\vec{E}(z,t) = \hat{x}E_{02}(-\xi)^{-\frac{i\nu}{2a}}(1-\xi)^{\rho}F(\alpha,\beta;\gamma;\xi)e^{i\omega t} \tag{3.59}$$

In this equation we just added the time-dependent part of the wave function to equation (3.28).

Using Maxwell's equations ($\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$) one can easily get the magnetic field of a plane wave moving in $\hat{z}$ direction. As we expected the magnetic field is in $\hat{y}$ direction.

$$\begin{aligned}\vec{B}(z,t) = \hat{y}\frac{iE_{02}}{2\omega a}(-\xi)^{\frac{-i\nu}{2a}}(1-\xi)^{\rho} \\ \left\{\frac{2a\alpha\beta}{\gamma}F(\beta+1,\alpha+1;\gamma+1;\xi)+(\frac{i\nu}{\xi}-\frac{2a\rho}{1-\xi})F(\alpha,\beta;\gamma;\xi)\right\}e^{i\omega t}\end{aligned} \tag{3.60}$$

Chapter 4

SMOOTH DOUBLE LAYER

In the previous chapter we considered the smooth step dielectric constant problem and we found the solutions for the general equation that we had obtained in chapter 2, namely (2.20). After that we calculated the solutions asymptotic behavior to examine their validity.

In this chapter we'll debate another permittivity that is also position-dependent and discuss a smooth double-layer problem.

4.1 The Wave Equation

We start with the equation that we have obtained in the second chapter in the most general form for a medium with position-dependent permittivity and permeability (both are z dependent)

$$\left(\frac{d^2}{dz^2}+\mu\varepsilon\omega^2\right)\bar{E}_x(z)=\frac{\mu'}{\mu}\frac{d\bar{E}_x(z)}{dz} \tag{4.1}$$

Now we assume that the electromagnetic wave is passing through a double-layer thick shell. To satisfy the condition we choose the permeability μ of the medium to be constant ($k_m = const$) and the permittivity of the medium $\varepsilon(z) = \varepsilon_0 k_e(z)$ with

$$k_e(z) = k_1 + \frac{\Delta k}{2} \left\{ \tanh(az) - \tanh\Big(a(z-L)\Big) \right\} \tag{4.2}$$

Where, a is a positive constant. Δk is defined as $\Delta k = k_2 - k_1$ and L is the thickness of a flat double layer dielectric of dielectric constant k_2 located inside another medium of dielectric constant k_1.

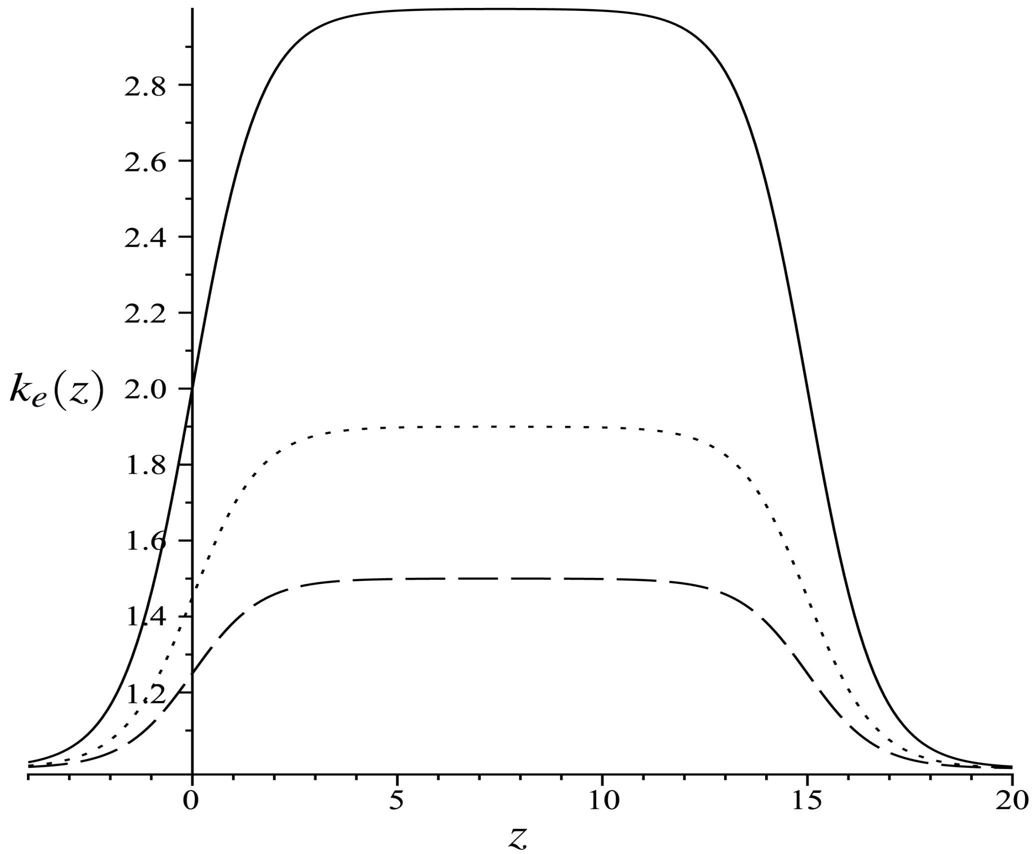

Figure 3. Shows the behavior of the permittivity function when z changes from -4 to $+20$, The plots are sketched when $k_1 = 1$ and $k_2 = 3$ with line, when $k_1 = 1$ and $k_2 = 1.9$ with dotted line and when $k_1 = 1$ and $k_2 = 1.5$ with dashed line ($a = 0.6, L = 15$).

Thus after substitution of $k_e(z)$, equation (4.1) leads to

$$\left(\frac{\partial^2}{\partial z^2}+\left\{\kappa^2+\frac{\nu^2-\kappa^2}{2}\left[\tanh(az)-\tanh\Big(a(z-L)\Big)\right]\right\}\right)\bar{E}_x(z)=0 \tag{4.3}$$

where

$$\kappa^2=\frac{\omega^2}{c^2}k_m k_1 \tag{4.4}$$

and

$$\nu^2=\frac{\omega^2}{c^2}k_m k_2 \tag{4.5}$$

Also we define

$$\lambda=e^{2aL} \tag{4.6}$$

So that in (4.3) after some manipulations we get

$$\tanh(az)-\tanh(az-aL)=\frac{2\lambda-2}{(1-\xi)(1-\lambda\xi)} \tag{4.7}$$

with

$$\xi = -e^{-2az} \tag{4.8}$$

Finally (4.3) yields

$$\frac{d^2\bar{E}_x(z)}{dz^2} + \left[\kappa^2 + \frac{(\nu^2-\kappa^2)(\lambda-1)}{\lambda(\xi-1)(\xi-\frac{1}{\lambda})}\right]\bar{E}_x(z) = 0 \tag{4.9}$$

Next we substitute

$$\bar{E}_x(z) = (-\xi)^{-i\nu}F(\xi) \tag{4.10}$$

in equation (4.10) to get

$$\begin{aligned} &-4a^2\xi^2F''(\xi) + (8i\nu a^2\xi - 4a^2\xi)F'(\xi) + \\ &\qquad \left[4a^2\nu^2 - \kappa^2 - \frac{(\nu^2-\kappa^2)(\lambda-1)}{\lambda(\xi-1)(\xi-\frac{1}{\lambda})}\right]F(\xi) = 0 \end{aligned} \tag{4.11}$$

Which can be simplified to read

$$F''(\xi)+\frac{1-2i\nu}{\xi}F'(\xi)+ \\ \frac{1}{4a^2\lambda\xi}\left[\frac{\lambda(\xi-1)(\xi-\frac{1}{\lambda})(\kappa^2-4a^2\nu^2)+(\nu^2-\kappa^2)(\lambda-1)}{\xi(\xi-1)(\xi-\frac{1}{\lambda})}\right]F(\xi)=0 \quad (4.12)$$

For further calculations, we may use the well-behaved function

$$F(\xi)=\xi^{\sigma}H(\xi) \quad (4.13)$$

in (4.12) to imply

$$\xi^{\sigma}H''+2\sigma\xi^{\sigma-1}H'+\sigma(\sigma-1)\xi^{\sigma-2}H+\frac{1-2i\nu}{\xi}(\sigma\xi^{\sigma-1}H+\xi^{\sigma}H')+ \\ \frac{1}{4a^2\lambda\xi}\left[\frac{\lambda(\xi-1)(\xi-\frac{1}{\lambda})(\kappa^2-4a^2\nu^2)+(\nu^2-\kappa^2)(\lambda-1)}{\xi(\xi-1)(\xi-\frac{1}{\lambda})}\right]\xi^{\sigma}H=0 \quad (4.14)$$

Note that σ is given by

$$\sigma=-\frac{i\kappa}{2a}+i\nu \quad (4.15)$$

So that (4.14) can be written in the form

$$\xi^{\sigma}H'' + (1 - \frac{i\kappa}{a})\xi^{\sigma-1}H' + \frac{1}{4a^2}\left[\frac{(\nu^2-\kappa^2)(\lambda-1)}{\lambda(\xi-1)(\xi-\frac{1}{\lambda})}\right]\xi^{\sigma-2}H = 0 \tag{4.16}$$

Finally the wave equation for the smooth double-layer problem becomes

$$H'' + \frac{1-\frac{i\kappa}{a}}{\xi}H' + \left[\frac{(\nu^2-\kappa^2)(\lambda-1)}{4a^2\lambda\xi^2(\xi-1)(\xi-\frac{1}{\lambda})}\right]H = 0 \tag{4.17}$$

It's obvious that we have singularities at $\xi = \frac{1}{\lambda}$, $\xi = 0$ and $\xi = 1$. Solving this homogeneous, second order differential equation needs further discussions and manipulations. We do it in the following section.

4.2 Solution of the Wave Equation

In the previous section we derived the wave equation (4.17). This equation satisfies the general condition of Heun function. The general form of the Heun function can be written as [13,20,21]

$$W''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-p}\right)W'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-p)}W(z) = 0 \tag{4.18}$$

With the condition

$$\gamma + \delta + \varepsilon = \beta + \alpha + 1 \tag{4.19}$$

We now define the solution of equation (4.17) in terms of Heun function

$$W(z) = C_1 HeunG(p, q, \alpha, \beta, \gamma, \delta, z) +$$
$$C_2 z^{1-\gamma} HeunG(p, q-(p\delta+\varepsilon)(\gamma-1), \beta-\gamma+1, \alpha-\gamma+1, 2-\gamma, \delta, z) \quad (4.20)$$

Where C_1 and C_2 are integration constants.If we compare (4.17) with (4.18) we get

$$p = \frac{1}{\lambda} \quad (4.21)$$

$$q = \frac{(\kappa^2 - \nu^2)(\lambda - 1)}{4a^2\lambda} \quad (4.22)$$

$$\alpha = \delta = 0 \quad (4.23)$$

$$\beta = \frac{-i\kappa}{a} \quad (4.24)$$

$$\gamma = 1 - \frac{i\kappa}{a} \quad (4.25)$$

and

$$z = \xi \quad (4.26)$$

Also from (4.19) it's obvious that

$$\varepsilon = 0 \tag{4.27}$$

Therefore, pursuant to (4.20), the electric field

$$\begin{aligned} E(z) = C_1(-\xi)^{\frac{-i\kappa}{2a}} HeunG(\frac{1}{\lambda}, \frac{(\kappa^2-\nu^2)(\lambda-1)}{4a^2\lambda}, 0, \frac{-i\kappa}{a}, \frac{a-i\kappa}{a}, 0, \xi) + \\ C_2(-\xi)^{\frac{i\kappa}{2a}} HeunG(\frac{1}{\lambda}, \frac{(\kappa^2-\nu^2)(\lambda-1)}{4a^2\lambda}, 0, \frac{i\kappa}{a}, \frac{a+i\kappa}{a}, 0, \xi) \end{aligned} \tag{4.28}$$

This equation is the most general solution that can be written as the wave function in this medium.

We now determine C_1 and C_2. We use the asymptotic behavior of the Heun function to assign the integration constants. We know that $HeunG(p,q,\alpha,\beta,\gamma,\delta,0) = 1$. Now if we assume that the wave is moving from $-\infty$ to $+\infty$ when $z \to \infty$, $\xi \to 0$ and $\kappa \to \nu$ thus (4.28) can be simplified as

$$\lim_{z\to+\infty} E(z) = C_1 e^{i\nu z} + C_2 e^{-i\nu z} \tag{4.29}$$

One can easily presume that C_2 must be 0 and $C_1 = E_{03}$. As a result we have

$$E(z) = E_{03}(-\xi)^{\frac{-i\kappa}{2a}} HeunG(\frac{1}{\lambda}, \frac{(\kappa^2-\nu^2)(\lambda-1)}{4a^2\lambda}, 0, \frac{-i\kappa}{a}, \frac{a-i\kappa}{a}, 0, \xi) \tag{4.30}$$

Herein E_{03} in the amplitude of the wave when it propagates in $\hat{z}$ ($z \rightarrow +\infty$) direction. To specify the magnitude of E_{03} we should find behavior of the wave function when the electromagnetic wave propagates in $z \rightarrow -\infty$ direction. From now on we want to find a way to describe

$$\lim_{z\to-\infty} Heun\left(\frac{1}{\lambda}, \frac{(\kappa^2-\nu^2)(\lambda-1)}{4a^2\lambda}, 0, \frac{-i\kappa}{a}, 1-\frac{i\kappa}{a}, 0, \xi\right) \tag{4.31}$$

The behaviour of this function is illustrated in figure 3. The incoming electromagnetic wave from z smaller than 0 encounters with the first layer at z = 0. $Re(E_x(z))$ has similar structure after crossing the second layer at $z = 4$. It is clear that the oscillation is changing between these two layers.

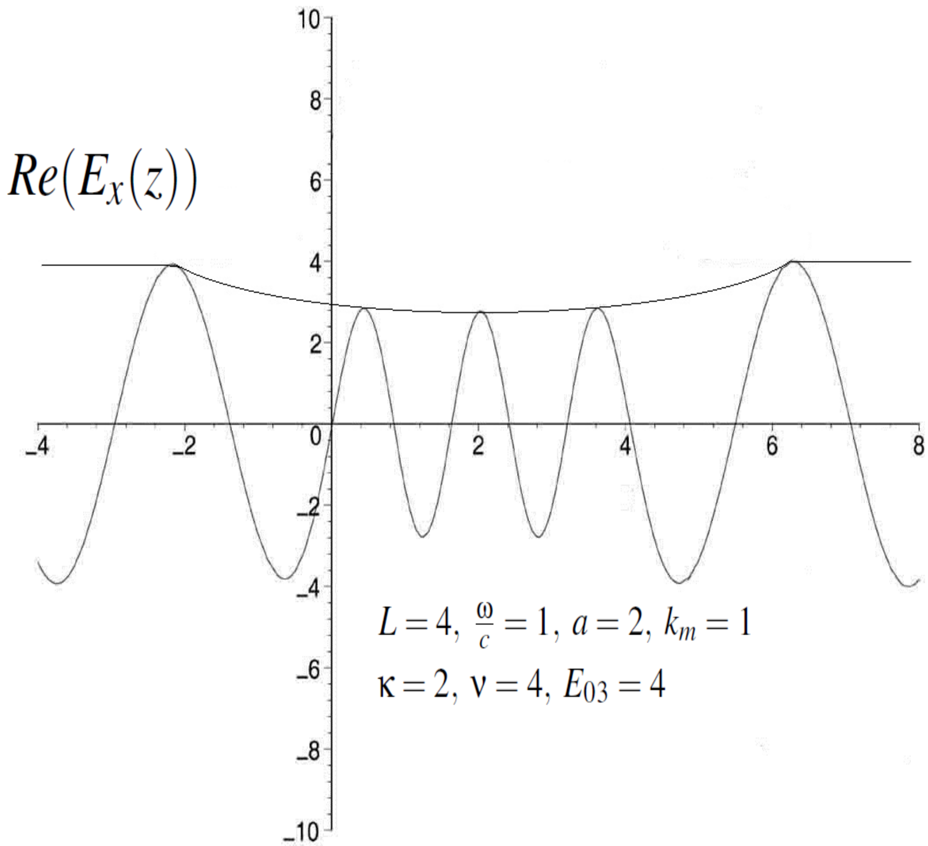

Figure 4. The incoming electromagnetic wave from z smaller than 0 encounters with the first layer at $z = 0$. $Re(E_x(z))$ has similar structure after crossing the second layer at $z = 4$. When z is between 0 and 4, the oscillatory behaviour evidently changes.

4.3 Reflection and Transmission from a Double Layer

In the previous section we derived the wave equation in the form (4.30). Now we shall inquire its asymptotic behavior where $\xi \to -\infty$.

Since the Heun function can be expressed in terms of any arbitrary function according to the relations between parameters, it would be adequate if we simply leave $E(\xi)$ in the form (4.30). But for more investigation on the reflection and transmission coefficient it will be fruitful to take a quick survey over this problem as a classical optics issue.

Therefore, we will examine the reflection and transmission features of a time harmonic electromagnetic wave normally incident at a layered dielectric and find the total reflection and transmission coefficients. Figure 4 clarifies the phenomenon, schematically.

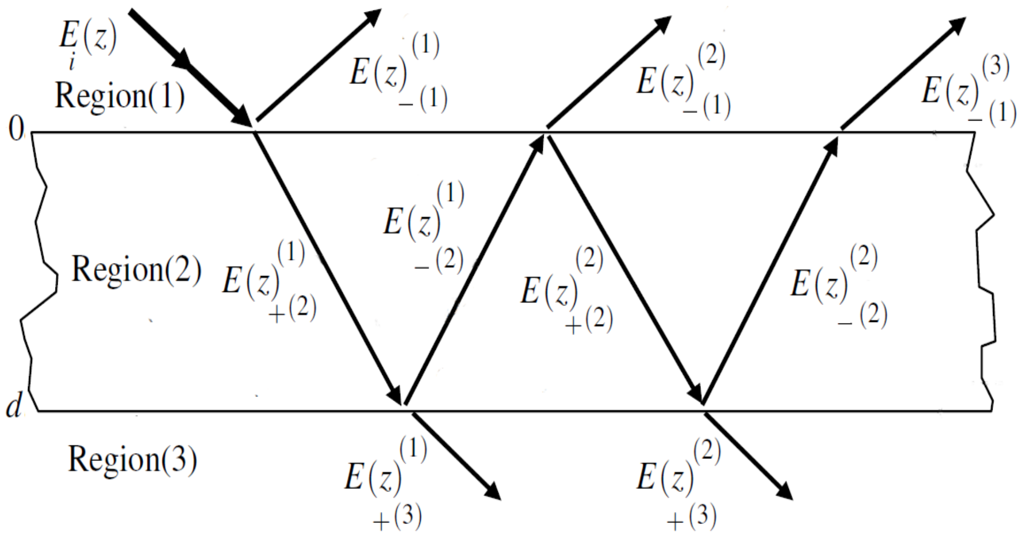

Figure 5.Smooth double layer(reflection and transmission waves)

We assume the thickness of the layer to be d and permittivities to be

$$\varepsilon_2 = \varepsilon_r \varepsilon_0$$
$$\varepsilon_1 = \varepsilon_3 = \varepsilon_0 \tag{4.32}$$

Where ε_r is the relative permittivity. The amplitudes of the total transmitted and reflected fields can be written as

$$E_t(z) = \sum_n E^n(z)_{+3}$$
$$E_r(z) = \sum_n E^n(z)_{-1} \tag{4.33}$$

Here, n (number of region) varies from 1 to ∞ which is a wave index and the signs of indices show the direction of the wave propagation.

When the wave incident to the boundary between the first and the second medium some parts can be reflected and some can be transmitted, the same phenomenon takes place at the interface of medium 2 and 3. Note that the phase for each individual term differs from the others by a factor Kd for each crossing slab. Next we consider the reflection coefficient at the first joint to be R_1 in $+z$ direction and since we know medium 1 is similar to medium 3 we have

$$R_2 = -R_1 \qquad\qquad R_1 = -R_2 \tag{4.34}$$

Similarly it is clear that the transmission in $+z$ direction coefficient would be

$$T_1^+ = 1 + R_1 \tag{4.35}$$

And in $-z$ direction is

$$T_1^- = 1 - R_1 \tag{4.36}$$

In this form we can write the total reflected electric field as (here $j = \sqrt{-1}$)

$$\begin{aligned} E_r(z) = E_i[R_1 + T_1^+ R_2 T_1^- e^{dj2k} + T_1^+ R_2 T_1^- e^{dj4k}(-R_2 R_1) + \\ T_1^+ R_2 T_1^- e^{dj6k}(-R_2 R_1)^2 + ...] \end{aligned} \tag{4.37}$$

After some manipulations we can write it as

$$E_r(z) = E_i \left\{ R_1 - (1 - R_1^2) R_1 e^{dj2k} \left\{ 1 + (R_1^2 e^{dj2k}) + (R_1^2 e^{dj2k})^2 + ... \right\} \right\} \tag{4.38}$$

Therefore, the total reflection coefficient is

$$R = \frac{E_r(z)}{E_i} = R_1 \left\{ 1 - \frac{(1 - R_1^2) e^{dj2k})}{1 - R_1^2 e^{dj2k}} \right\} \tag{4.39}$$

Which yields

$$R = \frac{R_1(1 - e^{dj2k})}{1 - R_1^2 e^{dj2k}} \tag{4.40}$$

By using the same method the total transmitted electric field can be derived

$$E_t(z) = E_i[T_1^+ T_1^- e^{jkd} + T_1^+ T_1^- e^{j3kd}(-R_1 R_2) + T_1^+ T_1^- e^{j5kd}(-R_1 R_2)^2 + ...] \tag{4.41}$$

That is equal to

$$E_t(z) = E_i \left\{ (1 - R_1^2) e^{jkd} \left\{ 1 + (R_1^2 e^{2jkd}) + (R_1^2 e^{2jkd})^2 + ... \right\} \right\} \tag{4.42}$$

Which is

$$T = \frac{E_t(z)}{E_i} = \frac{(1 - R_1^2) e^{jkd}}{1 - R_1^2 e^{jkd}} \tag{4.43}$$

The reflection and transmission phenomenon in a double layer have been a charming and practical issue specially in classical optics for many years.

Further investigations on the Huen function in future, can help us to expand equation (4.30) in a way that leads us to derive equations (4.40) and (4.43).

Chapter 5

CONCLUSION

In this thesis the electromagnetic wave propagation in non homogeneous media has been studied. We derived the most general form of the wave equation in chapter 2 with constant permeability and position-dependent permittivity (it can be a function of x, y or z). It is remarkable that this wave equation can be simplified in the form of plane wave equation for a constant ε. We wrote the wave function in terms of the hypergeometric functions and derived T and R (the transmission and the reflection coefficients). They are in good agreement with known transmission and the reflection coefficients of a plane wave that enters a new medium (dielectric) with a different permittivity constant. The results are analytically exact and schematically presented. Moreover, a smooth step dielectric constant was examined and the solution was presented in terms of the Heun functions.

This work can be extended to cover a wide rang of problems ranging from biology and chemistry to meteorology. For example, it would be valuable to use this method in spherical and cylindrical coordinates to obtain the wave function in non homogenous media.

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