# WEAKLY COFINITELY SUPPLEMENTED MODULES

A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of
Dokuz Eylűl University
In Partial Fullfilment of the Requirements for
the Degree of Master of Sciences in Mathematics



by
Engin BŰYŰKAŞIK



June, 2001
İZMİR

## M. Sc. THESIS EXAMINATION RESULT FORM

We certify that we have read the thesis, entitled "WEAKLY COFINITELY SUPPLEMENTED MODULES" completed by ENGİN BÜYÜKAŞIK under supervision of PROF. DR. REFAİL ALİZADE and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Refail Alizade

Supervisor

Assoc. Prof. Dr. A. Nihat Baden

Committee Member

Assist. Prof. Dr. Ünal Ufuktepe

Committee Member

Approved by the

Graduate School of Natural and Applied Sciences

Prof. Dr. Cahit Helvacı

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my supervisor, Professor R. Alizade for suggesting the topic of this dissertation and all his guidance, help and advice.

Engin BÜYÜKAŞIK

# ABSTRACT

In this thesis we introduce and study Weakly Cofinitely Supplemented modules. We prove that every $M-$generated module is weakly cofinitely supplemented if $M$ is weakly cofinitely supplemented. We give a condition under which weakly cofinitely supplemented modules are cofinitely supplemented. We also investigate modules whose finitely generated submodules have weak supplements. In particular, we prove that any small cover of a weakly finitely supplemented module is weakly finitely supplemented.

# ÖZET

Bu tezde zayıf dual-sonlu tümlenmiş modülleri ele alıyoruz. $M$ modülü zayıf dual-sonlu tümlenmiş ise, her $M$-üretilmiş modülün de zayıf dual-sonlu tümlenmiş olduğunu kanıtlıyoruz. Zayıf dual-sonlu tümlenmiş modüllerin dual sonlu üretilmiş modüllerle çakışması için yeter bir koşul veriyoruz. Ayrıca, tüm sonlu üretilmiş alt modüllerinin zayıf tümleyenlerinin bulunduğu modülleri inceliyoruz. Zayıf sonlu tümlenen modüllerin küçük örtüsünün de zayıf sonlu tümlenen olduğunu kanıtlıyoruz.

# CONTENTS

# CHAPTER ONE
# INTRODUCTION

The problem of decomposition of a module into a direct sum of its submodules is a fundamental one in the module theory. It is well known that not every submodule of a module is a direct summand. Moreover, we can not state that for every submodule $N$ of a module $M$ there is a minimal submodule $K$ of $M$ satisfying $M = N + K$. If this is the case, we say that $K$ is a *supplement* of $N$. Minimality of $K$ is equivalent to the condition $N \cap K \ll K$. Reducing the last condition to $N \cap K \ll M$, we get the definition of a weak supplement. If every submodule of $M$ has a supplement (weak supplement), we say that $M$ is *supplemented* (respectively, *weakly supplemented*). There is a great number of papers dealing with supplements and weak supplements. We mention the monographs [Wisbauer], [Müller, Mohammed] containing many results on supplements. Weak supplements were studied in the papers Zöschinger (1978) and Lomp (1999). If every submodule $N$ of $M$ with finitely generated $M/N$ has a supplement (weak supplement) we say that $M$ is *cofinitely supplemented* (*weakly cofinitely supplemented*, respectively). Cofinitely supplemented modules were investigated in Alizade, Bilhan & Smith (to appear).

The main goal of this thesis is to study weakly cofinitely supplemented modules. In Chapter 2 we prove that factor modules and sums of weakly cofinitely supplemented modules are weakly cofinitely supplemented. In particular, if $M$ is weakly cofinitely supplemented, then any $M-$generated module is weakly cofinitely supplemented. We give a condition under which weakly cofinitely supplemented modules are cofinitely supplemented. On the other hand we prove that if $RadM \ll M$, then $M$ is weakly cofinitely supplemented if and only if every maximal submodule of $M$ has a weak supplement.

If every finitely generated submodule of $M$ has a weak supplement, we say that $M$ is *weakly finitely supplemented.* In Chapter 3 we prove that under some conditions the sum of two weakly finitely supplemented modules is weakly finitely supplemented. Furthermore we show that any small cover of weakly finitely supplemented modules is weakly finitely supplemented. We also give an example of weakly finitely supplemented modules which are not weakly supplemented.

## 1.1 Preliminaries

A relation $\leq$ on a set $P$ is a *partial order* on $P$ in case it is reflexive ($a \leq a$), transitive ($a \leq b$ and $b \leq c \Rightarrow a \leq c$), and antisymmetric ($a \leq b$ and $b \leq a \Rightarrow a = b$). A pair $(P, \leq)$ consisting of a set a partial order on the set is called a *a partially ordered set* or a *poset.* If the partial order is a total order ($a \leq b$ or $b \leq a$ for every pair $(a, b)$, then the poset is a chain.

Let $P$ be a poset and let $A \subseteq P$. An element $e \in A$ is a *greatest (least)* element of $A$ in case $a \leq e$ ($e \leq a$) for all $a \in A$. An element $b \in P$ is an *upper bound (lower bound)* for $A$ in case $a \leq b$ ($b \leq a$) for all $a \in A$. If the set of upper bounds of $A$ has a least element, it is called the *least upper bound* of $A$; if the set of lower bounds has a greatest element, it is called the *greatest lower bound* of $A$. A *lattice* is a poset $P$ in which every pair of $P$ has both a least upper bound and a greatest lower bound in $P$.

**Lemma 1.1.1.** *(Zorn's Lemma) Let $X$ be non-empty partially ordered set. If every chain has an upper bound, then $X$ has a maximal element.*

Troughout this thesis will consider unitary left $R-$modules (i.e $1.m = m$ for every element $m$ of the module and the identity 1 of $R$), where $R$ is an associative ring with identity.

All definitions used, but not given in this thesis and results given in Chapter 1 can be found in Wisbauer (1991), Anderson & Fuller (1992), , Ribenboim(1967).

We will often use the following theorems, describing the properties of homomorphisms without any mentioning.

**Theorem 1.1.2.** *(Homomorphism Theorem) If $f : M \to N$ is a homomorphism and $S$ is a submodule of an $R-$module $M$ with $S \leq \operatorname{Ker} f$, then there is exactly one homomorphism $g : M/S \to N$ with $f = g \circ \pi$, where $\pi : M \to M/S$ is the canonical epimorphism. Furthermore, if $S = Kerf$, then $g$ is monomorphism. If $f$ is epimorphism then $g$ is also epimorphism.*

*Proof.* Define $g : M/S \to N$ as, $g(m+S) = f(m)$ then $m_1 + S = m_2 + S$ implies that $m_1 - m_2 \in S \leq Kerf$ so $0 = f(m_1 - m_2)$, then $f(m_1) = f(m_2)$ and thus $g$ is well defined.
In addition to that $g \circ \pi(m) = g(\pi(m)) = g(m + S) = f(m)$.

Suppose there exists another map $h : M/S \to N$ such that $f = h \circ \pi$ which implies $h \circ \pi = g \circ \pi$ and since $\pi$ is an epimorphism,
$h(m + S) = h(\pi(m)) = g(\pi(m)) = g(m + S)$. Therefore $h = g$ □

**Theorem 1.1.3.** *Let $N$ be a submodule of $K$, then there exists an epimorphism $f : M/N \to M/K$.*

*Proof.* Define the map $f : M/N \to M/K$ as $f(m + N) = m + K$. Then suppose that $m_1 + N$ and $m_2 + N$, so $m_1 - m_2 \in N \leq K$ implies that $m_1 + K = m_2 + K$. Hence the map is Properly defined. Surjectivity is straight forward. □

**Theorem 1.1.4.** *(First Isomorphism Theorem) Let $M$ and $N$ be $R-$modules and $\phi : M \to N$ be a homomorphism, then $M/Kerf \cong Im\phi$. In particular if $\phi$ is an epimorphism then $M/Kerf \cong N$*

*Proof.* By 1.1.2, $g : M/Kerf \to N$ is monomorphism and so $g :\to Im\phi$ is an isomorphism with the map $\phi : M \to N$. □

**Theorem 1.1.5.** *(Second Isomorphism Theorem) Let $M$ and $N$ be $R-$modules then $(M + N)/N \cong M/(M \cap N)$.*

*Proof.* Let's define$f : M \to (M+N)/N$ as $f(m) = m + N$. Then clearly $f$ is an epimorphism. $Kerf = m \in M \mid m \in N = M \cap N$. Therefore by 1.1.4, $(M+N)/N \cong M/(M \cap N)$. □

**Theorem 1.1.6.** *(Third Isomorphism Theorem) Let $M$ and $N$ be $R-$modules and $V$ be a submodule of both $M$ and $N$, then $(M/V)/(N/V) \cong M/N$.*

*Proof.* Define the epimorphism as in 1.1.3 $f : M/V \to M/N$ Then $Kerf = m + V \mid m \in N = N/V$. Therefore by 1.1.4 $(M/V)/(N/V) \cong M/N$. □

Modular Law will also be used in this thesis.

**Lemma 1.1.7.** *(Modular Law) Let $A$, $B$ and $C$ be submodules of an $R-$module $M$ with $B \leq C$, then $(A+B) \cap C = (A \cap C) + B$.*

*Proof.* For any $x$ from $(A+B) \cap C$, it can be represented as $x = a + b = c$ for some $a, b, c$ from $A, B, C$ respectively. This implies $a = b - c$. Since $B \leq C$, $a \in C$. Thus $x = a + b \in (A \cap C) + B$. Converse is straight forward. □

**Definition 1.1.8.** We say that a module $M$ is *finitely generated*, if there is a finite set $F = \{x_1, ..., x_n\}$ where each $\{x_i\}(1 \leq i \leq n)$ is from $M$ such that any element $x$ from $M$ can be written as $x = r_1x_1 + ... + r_nx_n$ where each $r_i(1 \leq i \leq n)$ is from $R$.

**Definition 1.1.9.** An $R-$ module $M$ is called *free*, if there is a basis $\{x_i\}_{i \in I}$. That is every element $m \in M$ can be written uniquely as $m = \sum_{i \in I} r_i x_i$ where $r_i \in R$ $(i \in I)$.

**Proposition 1.1.10.** *Let $f : M \to N$ be a homomorphism and $U$ be finitely generated submodule of $M$. Then $f(U)$ is a finitely generated submodule of $N$.*

*Proof.* Let $\{u_1, ...., u_n\}$ be a spanning set for $K$ and $u' \in f(U)$. Then there exists $u - k \in M$, where $u \in U$ and $k \in Kerf$ such that $f(u-k) = f(u) = u'$. Since $u \in U$, we have $u = r_1u_1 + ... + r_nu_n$, for $u_i \in U$ and $r_i \in R$ $1 \leq i \leq n$.

Therefore $u' = f(u) = f(r_1u_1 + ... + r_nu_n) = f(r_1)f(u_1) + ... + f(r_n)f(u_n)$. This shows that $u'$ is a linear combination of $\{f(u_1), ...., f(u_n)\}$. Hence $f(U)$ is finitely generated. □

**Corollary 1.1.11.** *If a module $M$ is finitely generated then any factor module of $M$ is also finitely generated.*

*Proof.* Let $M/N$ be a factor module of $M$. Since $M$ is finitely generated any element of $M$ can be written as $m = r_1x_1 + ... + r_nx_n$, where $\{x_1, ..., x_n\}$ is a spanning set for $M$. If we choose an element $m + N$ from $M/N$ then
$m + N = (r_1x_1) + ... + (r_nx_n + N)$. Therefore $\{x_1 + N, ..., x_n + N\}$ becomes a spanning set for $M/N$. □

**Definition 1.1.12.** A submodule $U$ of a module $M$ is called *maximal* if whenever $U \leq K \leq M$ for a submodule $K$ then $K = U$ or $K = M$.

**Note:**Maximal submodules are maximal elements in the set of all proper submodules.

**Theorem 1.1.13.** *Let $M$ be an $R-$module and $N$ be its submodule. For the natural epimorphism $\phi : M \to M/N$, if $B$ is a maximal submodule in $M/N$ then $\phi^{-1}(B)$ is also maximal submodule in $M$.*

*Proof.* Suppose that there exists a proper submodule $K$ of $M$ such that $\phi^{-1}(B) \leq K$. Then $\phi(\phi^{-1}(B)) \leq \phi(K)$, since $\phi$ is an epimorphism $B \leq \phi(K)$. By the maximality of $B$, $\phi(K) = M/N$. This implies $\phi^{-1}(\phi(K)) = \phi^{-1}(M/N)$ and $K + Ker\phi = M$. That is $M = K$. □

An arbitrary module need not have a maximal submodule, but for finitely generated modules we have the following important result.

**Theorem 1.1.14.** *If $M$ is a finitely generated module, then for any submodule $N$ of $M$, there exists a maximal submodule $P$ of $M$ such that $N \leq P$.*

*Proof.* Denote the collection of all proper submodules of $M$ containing $N$ by $\Gamma$. Then $\Gamma$ is not empty because $N \in \Gamma$. We may partially order $\Gamma$ by inclusion. Let $\Delta$ be a non-empty chain of $\Gamma$. Denote by $N_0$ the union of all elements of $\Delta$. Then $N \leq N_0$, and it may be verified that $N_0$ is a submodule of $M$. Note this verification uses the fact that $\Delta$ is totally ordered. $N_0$ is a proper submodule of $M$. Assume the contrary, and let $\{m_1, ..., m_n\}$ be a set of generators of $M$. Then each $m_i$ belongs to $N_0$, therefore is contained in some member of $\Delta$. So by total ordering of $\Delta$, there must be a member of $\Delta$ which contains every $m_i$, i.e. which is the whole of $M$, contradiction. Thus $N_0 \in \Delta$. It follows by Zorn's Lemma that $\Gamma$ contains a maximal member $P$. □

**Corollary 1.1.15.** *For an $R-$module $M$, if $M/N$ is finitely generated then $N$ is contained in a maximal submodule of $M$.*

*Proof.* By 1.1.12 $M/N$ has a maximal submodule $P/N$. By 1.1.11, $\phi^{-1}(P/N)$ is a maximal submodule containing $N$. □

## 1.2 Small Submodules

A submodule $K$ of an $R-$module $M$ is called *superflous* or *small* in $M$, written $K \ll M$, if, for every submodule $L \leq M$, the equality $K + L = M$ implies $L = M$.

The main properties of small submodules are given in the following Proposition.

**Proposition 1.2.1.** *Let $M$ be an $R-$module and $N \leq M$.*

*(1) $N$ is small if and only if $N + K \neq M$ for any proper submobule $K$ of $M$.*

*(2)Let $N$ be a small submodule of a module $M$, then any submodule of $N$ is also small in $M$.*

*(3)Let $N$ be a module and $A$ be a submodule of $N$. If $A$ is small in $N$, then it is small in any module containing $N$.*

*(4) Let $X$ be small in $A$ and $Y$ small in $B$, then $X+Y$ is small in $A+B$.*

*(5) Let $f: M \to N$ be a homomorphism and let $A$ be a submodule of $M$. If $A$ is small in $M$, then $f(A)$ is small in $f(M)$.*

*(6) Let $K \leq L \leq M$. Then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*

*(7) $L_1 + L_2 + ... + L_n \ll M$ if and only if $L_i \ll M(1 \leq i \leq n)$*

*(8) If $L$ is a direct summand of $M$, then $K \leq L$ if and only if $K \ll M$.*

*Proof.* (1) $\Rightarrow$ Suppose that $N+K=M$,then by definition of small submodule $K=M$.

$\Leftarrow$ Assume that $N+K=M$ but by the assumption this is true for just the case $K=M$.

(2) Let $K \leq N$ and $K+X=M$.Then we get $N+X=M$. Since $N$ is a small submodule of $M$, it follows that $X=M$ and this implies $K \ll M$.

(3) Let $M$ be a module containing $N$ and suppose that $A+X=M$ for some submodule $X$ of $M$. Then by the modular law, $A+(N \cap X)=N$ And since $A$ is small in $N$, $N \cap X = N$. So $N \leq X$. Hence $A \leq X$. Therefore we get $X = A + X = M$.

(4) Suppose that $(X+Y)+Z=A+B$ where $Z$ is a subgroup of $A+B$. By (3) $X$ is small in $A+B$ and so we get $Y+Z=A+B$. By (3) $Y$ is also small in $A+B$ so $Z=A+B$. Therefore $X+Y$ is small in $A+B$.

(5)Suppose that $f(A)+X=M$ with $X \leq f(M)$. Then $f^{-1}(f(A))+f^{-1}(X) =$

$f^{-1}(f(M)) = M$ and then $A + \operatorname{Ker} f + f^{-1}(X) = M$ implies $A + f^{-1}(X) = M$. Since $A \ll M$, $f^{-1}(X) = M$ and hence $f(f^{-1}(X)) = f(M)$ implying, $X \cap \operatorname{Im} f = f(M)$. Therefore $X = f(M)$.

(6) $\Rightarrow$ By (2) $K \ll M$. Suppose that $L/K + X/K = M/K$ where $X/K$ is a submodule of $M/K$, then $L + X = M$ and by the assumption $X = M$. That is $X/K = M/K$.

$\Leftarrow$ Let $L+X = M$ then $(L+X)/K = M/K$ that is $L/K + (X+K)/K = M/K$. So $X + K = M$ and again by the assumption $X = M$.

(7) $\Rightarrow$ By (2).

$\Leftarrow$ Let $L_1 + L_2 + ... + L_n + X = M$. Since $L_1 \ll M$, $L_2 + ... + L_n + X = M$,so in this way after $n-1$ steps we will get $L_n + X = M$, finally since $L_n \ll M$, we have $X = M$.

(8) $\Rightarrow$ By (3)

$\Leftarrow$ Let $K + X = L$. Since $L$ is a direct summand of $M$, there exists $L_0 \leq M$ such that $L \oplus L_0 = M$. So $K + X + L_0 = M$ then by the assumption $X + L_0 = M$ and $X \oplus L_0 = M$. Therefore by modular law $L = L \cap (X + L_0) = X + (L \cap L_0) = X$. □

## 1.3 The Radical

In the study of rings and modules, (Jacobson) radical plays an important role. We present two equivalent definitions of the radical.

**Theorem 1.3.1.** *Let $M$ be an $R-$module. Then*

$$RadM = \bigcap\{K \leq M | K \text{ is maximal in } M\} = \sum\{L \leq M | L \text{ is small in } M\}.$$

*Proof.* See Anderson & Fuller (1992) 9.13 □

The *radical* of a module $M$ is defined as a sum (intersection) of all small (maximal) submodules of $M$ and denoted by $RadM$.

**Proposition 1.3.2.** *Let $\{N_i\}_{i\in I}$ be a family of submodules of $M$ containing submodule $K$ of $M$. Then we have*

$$\bigcap_{i\in I}(N_i/K) = (\bigcap_{i\in I} N_i)/K$$

*Proof.* ($\subset$) Let $a+K \in \cap_{i\in I}(N_i/K)$ then $a+K \in N_i/K$ and $a \in N_i$ for all $i \in I$. Therefore $a+K \in (\bigcap_{i\in I} N_i)/K$

($\supset$) Straight forward. □

**Proposition 1.3.3.** *The radical of the factor module $M/RadM$ is equal to zero, that is $O_{M/RadM}$ is the only small submodule of $M/RadM$.*

*Proof.* $Rad(M/RadM)$ is the intersection of all maximal submodules of $M/RadM$. Let $\{M_i/RadM\}_{i\in I}$ be the collection of all maximal submodules of $M/RadM$. Then by the proposition above

$$\bigcap_{i\in I}(M_i/RadM) = (\bigcap_{i\in I} M_i)/RadM = RadM/RadM = 0.$$

□

Many statements for a module $M$ can be provided $RadM \ll M$, so we describe a situation when it is true.

**Proposition 1.3.4.** *If every proper submodule $U$ of $M$ is contained in a maximal submodule then $RadM \ll M$.*

*Proof.* Suppose that $RadM + X = M$ for some proper submodule $X$ of $M$. Then $X$ is contained in some maximal submodule $P$ of $M$, but by the definition of

$RadM$; $RadM \leq P$, so $RadM + X = M \leq P$. This is a contradiction. Therefore $X = M$. □

This proposition together with 1.1.14 gives the following corollary.

**Corollary 1.3.5.** *If $M$ is finitely generated then $RadM \ll M$*

We are going to prove that the condition of 1.3.5 is satisfied for all semisimple modules.

A submodule $S$ of $M$ is said to be *simple*, if it has no nontrivial proper submodule.

**Theorem 1.3.6.** *For an $R$ module $M$ the following statements are equivalent:*

*(1) $M$ is simple,*
*(2) $M$ is generated by one of its nonzero elements,*
*(3) $M \cong R/I$, where $I$ is a maximal ideal of $R$.*

*Proof.* See , Ribenboim(1967) pg:47 □

**Theorem 1.3.7.** *For an $R-$module $M$, the followings are equivalent:*
(1) *$M$ is the direct sum of a family of simple modules;*
(2) *$M$ is generated by a family of simple submodules;*
(3) *Every submodule of $M$ is a direct summand;*
(4) *Every factor module of $M$ is a direct factor.*

*Proof.* See , Ribenboim(1967) 4.8 □

**Definition 1.3.8.** A module $M$ is *semisimple* if it satisfies the equivalent conditions of the theorem above.

**Proposition 1.3.9.** *If $M$ is semisimple then every proper submodule of $M$ is contained in a maximal submodule.*

*Proof.* Let $U$ be a submodule of $M$ then by 1.3.7(3) $U$ is a direct summand of $M$. And $M = U \oplus (\oplus_{i\in I} U_i)$ for submodules $U_i$ of $M$ with $i \in I$. Then the factor module $M/(U \oplus (\oplus_{i\in I-\{j\}} U_i))$ is isomorphic to $U_j$. So, $U \oplus (\oplus_{i\in I} U_i)$ is maximal in $M$ by 1.3.6 □

**Corollary 1.3.10.** *For any semisimple module $M$, $RadM = 0$.*

**Proposition 1.3.11.** *For any $f \in Hom(M, N)$ $f(RadM) \leq RadN$*

*Proof.* Let $K$ be a small submodule of $M$. Then By 1.2.1(5) $f(K)$ is a small submodule of $N$. □

## 1.4 Coherent Modules

**Definition 1.4.1.** A sequence $M$ of modules is a collection $\{M_n, f_n\}_{n\in Z}$ where for each integer $n$, $M_n$ is a module and $f_n : M_n \to M_{n-1}$ is a homomorphism:

$$M : .... \longrightarrow M_{n+1} \longrightarrow M_n \xrightarrow{f_n} M_{n-1} \longrightarrow ....$$

It is said to be *exact* if $Kerf_n = Imf_{n+1}$ for all $n \in Z$.

**Definition 1.4.2.** An exact sequence of the form:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called *short exact sequence.*

Sometimes we need the condition on a module which is stronger than being finitely generated.

**Definition 1.4.3.** A module $M$ is called *finitely presented* if

$(i)$ $M$ is finitely generated and

$(ii)$ in every exact sequence $0 \to K \to L \to M \to 0$, with $L$ is finitely generated, $K$ is also finitely generated.

**Definition 1.4.4.** A module $M$ is called *coherent* if

$(i)$ $M$ is finitely generated and

$(ii)$ any finitely generated submodule of $M$ is finitely presented.

**Proposition 1.4.5.** *If $M$ is coherent and and $K$, $L$ are finitely generated submodules of $N$, then $K \cap L$ is finitely generated.*

*Proof.* Let consider the external direct sum of submodules $K$ and $N$

$$K \oplus N = \{(k,n) | k \in K, \ n \in N\}$$

and the homomorphisms $f : K \oplus N \to K + N$, $g : K \cap N \to K \oplus N$ which are defined as $f(k,n) = k + n$, $g(x) = (x, -x)$. Then the following sequence is exact:

$$0 \longrightarrow K \cap N \xrightarrow{f} K \oplus N \xrightarrow{g} K + N \longrightarrow 0$$

Really, if $(x, -x) \in Img$ then $f(x,-x) = x + (-x) = 0$ and if $(k,n) \in Kerf$ then $f(k,n) = k + n = 0$ so $n = -k$ i.e $(k,n) = (k,-k)$ and since $n = -k \in K$, $k \in K \cap N$. So $g(k) = (k,-k)$. Since $M$ is coherent and $K \oplus N$ and $K + N$ are finitely generated, $K \cap N$ is finitely generated. $\square$

## 1.5 Self-Projective Modules

**Definition 1.5.1.** $P$ is called a *projective* $R$-module if for any epimorphism $f : A \to B$ and homomorphism $g : P \to B$, there exists a homomorphism $h : P \to A$ such that $g = f \circ h$.

**Definition 1.5.2.** Let $M$ and $P$ be $R-$modules. $P$ is called *M-projective* if every diagram with exact row

$$\begin{tikzcd} & P \arrow[d] & \\ M \arrow[r, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

can be extended commutatively by a morphism $h : P \to M$.

If $P$ is $P-$projective, then $P$ is called *self-projective.*

**Proposition 1.5.3.** *Let $M$ be an $R-$module and let $(P_k)_{k\in K}$ be an indexed set of $R-$modules. Then $\oplus_K P_k$ is $M$-projective if and only if each $P_k$ is $M-$projective.*

*Proof.* $\Rightarrow$: Let $P = \oplus_K P_k$ be $M-$projective and $k \in K$ and let $f : M \to B$ be any epimorphism, $g : P_k \to B$ a homomorphism and $p_k : P \to P_k$, $i_k : P_k \to P$. Since $P$ is $M-$projective, for the homomorphism $g \circ p_k : P \to B$, there exists a homomorphism $h : P \to M$ such that $g \circ p_k = f \circ h$. Now let us define a homomorphism $e : P_k \to M$, $e = h \circ i_k$. In this case we get,
$f \circ e = f \circ h \circ i_k = g \circ p_k \circ i_k = g \circ 1_{p_k} = g$. Hence $P_k$ is $M-$projective.

$\leftarrow$: Let us consider the epimorphism $f : M \to B$ and the homomorphism $g : P \to B$. Since $P_k$ is $M-$projective for all $k \in K$, then for the homomorphism $g \circ i_k : P_k \to B$ there exists a homomorphism $h_k : P_k \to M$ such that $g \circ i_k = f \circ h_k$. Now for any $k \in K$ there exists a homomorphism $h : P \to M$ such that $h_k = h \circ i_k$. So we get $g \circ i_k = f \circ h_k = f \circ h \circ i_k$. Therefore $g = f \circ h$. Hence $P = \oplus P_k$ is $M-$projective. □

**Proposition 1.5.4.** *Let $U$ be an $R$ module. If*

$$0 \to M^1 \to M \to M^2 \to 0$$

*is an exact sequence and $U$ is $M-$projective, then $U$ is projective relative to both $M^1$ and $M^2$*

*Proof.* See [3] 16.12 □

*Note:* If $M$ is an $R-$module and $U$ is a submodule of $M$, then the sequence $0 \to U \to M \to M/U \to 0$ is exact.
So from Proposition 1.5.4 we conclude that; if $P$ is $M$-projective then $P$ is projective reletive to all submodules of $M$.

CHAPTER TWO

# WEAKLY COFINITELY SUPPLEMENTED MODULES

## 2.1 Weakly Supplemented Modules

Let $M$ be an $R$-module. The *supplement* of a submodule $N$ of $M$ is defined as a submodule $L$ of $M$ minimal with respect to $N + L = M$. This is equivalent to $N + L = M$ and $N \cap L \ll L$. And we say that $M$ is *supplemented* if every submodule has a supplement in $M$. Clearly semisimple modules are supplemented. A module $M$ is called *artinian* if every strictly decreasing sequence of submodules is finite. Every artinian module is supplemented. Really, if $U$ is a submodule of $M$ then there is a submodule $V$ of $M$ such that $U + V = M$. Suppose there is a submodule $V_0$ of $V$ with $U + V_0 = M$, $U + V_1 = M$ and $V_1 \subset V_0$...continuing in this way we have a descending chain of submodules $V \supset V_0 \supset V_1 \supset .... \supset V_n$... But since $M$ is artinian, this sequence is finite. Denote the module at the end by $V_n$. Therefore $V_n$ is a supplement of $U$ in $M$.

More generally, a submodule $N$ of $M$ has a *weak supplement* $L$ in $M$ if $N+L = M$ and $N \cap L \ll M$, and $M$ is called *weakly supplemented* if every submodule $N$ of $M$ has a weak supplement.

**Definition 2.1.1.** A submodule $K$ of $M$ is called *essential (or large)* ,abbreviated $K \trianglelefteq M$ in case for every submodule $K \leq M$, $K \cap L = 0$ implies $L = 0$. A monomorphism $f : K \to M$ is said to be *essential* in case $Imf \trianglelefteq M$.

**Definition 2.1.2.** Let $N$ be a submodule of $M$. If $N' \leq M$ is maximal with respect to $N \cap N' = 0$, then we say that $N'$ is an $M-$*complement* of $N$.

**Proposition 2.1.3.** *For a proper submodule $N \leq M$, the following statements are equivalent:*

(1) $M/N$ is semisimple;
(2) For every $L \leq M$ there exist a submodule $K \leq M$ such that $L + K = M$ and $L \cap K \leq N$;
(3) There exist a decomposition $M = M_1 \oplus M_2$ such that $M_1$ is semisimple, $N$ is an essential submodule of $M_2$ and $M_2/N$ is semisimple.

*Proof.* (1) $\Rightarrow$ (3) Let $M_1$ be a complement of $N$. $M_1 \cong (M_1 \oplus N)/N$ is a direct summand in $M/N$, hence semisimple and there is a semisimple submodule $M_2/N$ such that $(M_1 \oplus N)/N \oplus M_2/N$. Thus $M = M_1 + M_2$ and $M_1 \cap M_2 \leq N \cap M_1 = 0$ implies $M = M_1 \oplus M_2$ Since $M_1$ is a complement of $N$ we have the natural isomorphism $N \cong (M_1 \oplus N)/M_1 \trianglelefteq M/M_1 \cong M_2$ that $N \trianglelefteq M2$.

(3) $\Rightarrow$ (1) Clear, since $M/N \cong (M_1 \oplus M_2/N)$.

(1) $\Rightarrow$ (2) Clear, since $(L + N)/N$ is a direct summand in $M/N$

(2) $\Rightarrow$ (1) Let $L/N \leq M/N$; then there exists a submodule $K \leq M$ such that $L + K = M$ and $L \cap K \leq N$. Thus $L/N \oplus (K + N)/N = M/N$. Hence every submodule of $M/N$ is a direct summand. $\square$

**Definition 2.1.4.** A module $M$ is called a *semilocal module* if $M/RadM$ is semisimple.

**Definition 2.1.5.** We call $N$ a *small cover* of a module $M$ if there exist an epimorphism $f : N \to M$ such that $Kerf \ll N$. Then $f$ is called a *small epimorphism.*

**Proposition 2.1.6.** *If $f : M \to N$ is a small epimorphism and $K$ is a small submodule of $N$, then $f^{-1}(K)$ is a small submodule of $M$.*

*Proof.* Since $0 \in K$, $f^{-1}(K) = f^{-1}(K) + Kerf$. Let $f^{-1}(K) + Kerf + X = M$. Since $Kerf \ll M$ we get $f^{-1}(K)+X = M$ and $f(f^{-1}(K))+f(X) = K+f(X) = f(M) = N$ and since $K \ll N$ we get $f(X) = N$ and so $f^{-1}(f(X)) = f^{-1}(N)$. Thus, $X = M$. $\square$

**Proposition 2.1.7.** *Assume $M$ to be weakly supplemented. Then:*

(1) *$M$ is semilocal;*

(2) *$M = M_1 \oplus M_2$ with $M_1$ is semisimple, $M_2$ semilocal and $RadM$ is esential in $M$;*

(3) *Every factor module of $M$ is weakly supplemented;*

(4) *Any small cover of $M$ is weakly supplemented;*

(5) *Every supplement in $M$ and every direct summand of $M$ is weakly supplemented.*

*Proof.* (1) Since $M$ is weakly supplemented, for every $L \leq M$ there exists a weak supplement $K \leq M$ such that $L + K = M$ and $L \cap K \leq RadM$. Therefore by 2.1.3 $M/RadM$ is semisimple i.e. $M$ is semilocal.

(2) clear by 2.1.3

(3) Let $f : M \to N$ be an epimorphism and $K \leq N$, then $f^{-1}(K)$ has a weak supplement $L$ in $M$ i.e. $f^{-1}(K) + L = M$ and $f^{-1}(K) \cap L$
$M$. Thus $f(f^{-1}(K) + f(L) = K + f(L) = f(M) = N$ and $f(f^{-1}(K) \cap f(L) \leq K \cap f(L) \ll N$. This shows that $f(L)$ is a weak supplement of $K$ in $N$.

(4) Let $N$ be a small cover of $M$ and $f : N \to M$ be a small epimorphism. Since $Kerf \ll N$, $f^{-1}(K) \ll N$ for every $K \ll M$. Let $L \leq N$. Then $f(L)$ has a weak supplement $X$ in $M$. Thus $f^{-1}(f(L)) + f^{-1}(X) = L + f^{-1}(X) = N$ and $L \cap f^{-1}(X) \ll N$. This implies $f^{-1}(X)$ is a weak supplement of $L$ in $N$.

(5) If $N \leq M$ is a supplement in $M$, then $N + K = M$ for some $K \leq M$ and $K \cap N \ll N$. By (3), $M/K \cong N/(N \cap K)$ is weakly supplemented and by (4),

$N$ is weakly supplemented. Direct summands are supplements and hence weakly supplemented. □

**Corollary 2.1.8.** *An $R-$ module $M$ with $RadM = 0$ is weakly supplemented if and only if $M$ is semisimple.*

*Proof.* This follows by 2.1.7 (1). □

The following technical lemma is needed to show that every finite sum of weakly supplemented modules is weakly supplemented.

**Lemma 2.1.9.** *Let $K$ and $M_1$ be submodules of an $R-$module $M$. Assume $M_1$ is weakly supplemented and $M_1 + K$ has a weak supplement in $M$. Then $K$ has a weak supplement in $M$*

*Proof.* By assumption $M_1 + K$ has a weak supplement $N \leq M$, i.e $M_1 + K + N = M$ and $(M_1 + K) \cap N \ll M$. Since $M_1$ is weakly supplemented, $(K + N) \cap M_1$ has a weak supplement $L \leq M_1$. So
$M = M_1 + K + N = L + (K + N) \cap M_1 + K + N = K + L + N$
and $(K + N) \cap L = [(K + N) \cap M_1] \cap L \ll M_1$. And
$K \cap (L + N) \leq (K + L) \cap N + (K + N) \cap L \leq (K + M_1) \cap N + (K + N) \cap L \ll M$.
Hence $N + L$ is a weak supplement of $K$ in $M$ □

**Proposition 2.1.10.** *Let $M = M_1 + M_2$, where $M_1$ and $M_2$ are weakly supplemented, then $M$ is weakly supplemented.*

*Proof.* For every submodule $N \leq M$, $M_1 + (M_2 + N)$ has the trivial weak supplement 0 and by 2.1.9 $M_2 + N$ has a weak supplement in $M$. Applying the Lemma 2.1.9 again we get a weak supplement for $N$. □

**Corollary 2.1.11.** *Every finite sum of weakly supplemented modules is weakly supplemented.*

## 2.2 Hollow Modules

**Definition 2.2.1.** A module $M$ is said to be *hollow* if $M \neq 0$ and every proper submodule is small in $M$. $M$ is called *local* if it has exactly one maximal submodule that contains all proper submodules.

Clearly hollow modules are (weakly) supplemented. Really for any submodule $K$ of $M$, $K + M = M$ and $K \cap M = N \ll M$.

Every simple module is hollow.
Every local module is hollow and so (weakly) supplemented.

**Proposition 2.2.2.** *The following statements are equivalent:*

*(1) $M$ is local;*
*(2) $M$ is hollow and cyclic (or finitely generated);*
*(3) $M$ is hollow and $RadM \neq M$.*

*Proof.* See Wisbauer (1991) 41.4 □

**Definition 2.2.3.** $M$ is said to have *finite hollow dimension* if there exists an epimorphism $g : M \to \oplus_{i=1}^{n} H_i$ where all the $H_i$ $i(i = 1, ..., n)$ are hollow and the kernel of $g$ is small in $M$. Then $n$ is called the hollow dimension of $M$ and we write $hdim(M) = n$.

The relationship between the concepts 'hollow dimension' and 'weakly supplemented' is expressed in the following theorem.

**Theorem 2.2.4.** *Consider the following properties:*

*$(i)$ $M$ has finite hollow dimension;*
*$(ii)$ $M$ is weakly supplemented;*
*$(iii)$ $M$ is semilocal.*

*Then* $(i) \Rightarrow (ii) \Rightarrow (iii)$ *holds. If* $RadM \ll M$ *then* $(iii) \Rightarrow (ii)$ *holds. If* $M$ *is finitely generated, then* $(iii) \Rightarrow (i)$ *holds.*

*Proof.* $(i) \Rightarrow (ii)$ There is a small epimorphism $f : M \to \oplus_{i=1}^{n} H_i$ with hollow modules $H_i$. Since hollow modules are weakly supplemented we get by 2.1.11 $\oplus_{i=1}^{n} H_i$ is weakly supplemented. Since $f$ is a small epimorphism we get by 2.1.7(4) that $M$ is weakly supplemented.

$(ii) \Rightarrow (iii)$ By 2.1.7(1)

$(iii) \Rightarrow (ii)$ If $RadM \ll M$, then this follows by 2.1.7(4)

$(iii) \Rightarrow (i)$ If $M$ is finitely generated then $RadM \ll M$ and so, $M$ is a small cover of $M/RadM$ which is a finite sum of simple (and therefore, hollow) modules. Therefore $M$ has a finite hollow dimension. □

## 2.3 Weakly Cofinitely Supplemented Modules

**Definition 2.3.1.** Let $U \leq M$. If $M/U$ is finitely generated, then $U$ is called *cofinite.*

**Definition 2.3.2.** A module $M$ is called *weakly cofinitely supplemented* if every cofinite submodule of M has a weak supplement in $M$.

The following Lemma shows that without loss of generality weak supplements of cofinite submodules can be regarded as finitely generated.

**Lemma 2.3.3.** *Let* $M$ *be a module and* $U$ *a cofinite submodule of* $M$*. If* $V$ *is a weak supplement of* $U$ *in* $M$*, then* $U$ *has a finitely generated weak supplement in* $M$ *contained in* $V$*.*

*Proof.* $M/U \cong V/V \cap U$ so $V/V \cap U$ is finitely generated. Let
$V/V \cap U = \{x_1 + V \cap U, ...., x_n + V \cap U\}$. We get
$< \{x_1, ..., x_n\} > +U =< \{x_1, ..., x_n\} > +V \cap U + U = V + U$ and
$< \{x_1, ..., x_n\} > \cap U \leq V \cap U \ll M$. Therefore $< \{x_1, ..., x_n\} >$ is a finitely generated weak supplement of $U$. □

**Proposition 2.3.4.** *A homomorphic image of a weakly cofinitely supplemented module is weakly cofinitely supplemented .*

*Proof.* Let $f : M \to N$ be a homomorphism and $M$ be a weakly cofinitely supplemented module. Suppose that $X$ is a cofinite submodule of $f(M)$, then
$M/f^{-1}(X) \cong (M/Kerf)/(f^{-1}(X)/Kerf)$ with $M/Kerf \cong f(M)$ and
$f^{-1}(X)/Kerf \cong X$. Thus $M/f^{-1}(X)$ is finitely generated. Since $M$ is weakly cofinitely supplemented $f^{-1}(X)$ has a weak supplement $U$ in $M$,
i.e. $f^{-1}(X) + U = M$ and $f^{-1}(X) \cap U \ll M$. So $f(f^{-1}(X)) + f(U) = f(M)$ and since $X$ is a submodule of $f(M)$, $f(f^{-1}(X)) = X$ and so $X + f(U) = M$. In addition to that $f(f^{-1}(X)) \bigcap f(U) \ll f(M)$ by 1.2.1
Therefore $X \cap f(U) \ll f(M)$. □

**Corollary 2.3.5.** *Any factor module of a weakly cofinitely supplemented module is weakly cofinitely supplemented .*

*Proof.* Let $M$ be a weakly cofinitely supplemented module. Consider the factor module $M/N$ of $M$ where $N$ is a submodule of $M$. Consider the canonical epimorphism $\pi : M \to M/N$, then by Lemma 2.3.4 $\pi(M) = M/N$ is weakly cofinitely supplemented . □

To prove that sums of weakly cofinitely supplemented module are weakly cofinitely supplemented we use the following standard lemma (Wisbauer (1991) 41.2).

**Lemma 2.3.6.** *Let $N$ and $U$ be submodules of $M$ with weakly cofinitely supplemented $N$ and cofinite $U$. If $N + U$ has a weak supplement in $M$, then $U$ also has a weak supplement in $M$.*

*Proof.* Let $X$ be a weak supplement of $N+U$ in $M$. Then we have

$$N/N\cap(X+U)\cong(N+X+U)/[N\cap(X+U)]\cong M/(X+U)\cong M/U/(X+U)/U.$$

The last term is finitely generated module, hence $N\cap(X+U)$ has a weak supplement $Y$ in $N$ i.e.

$$Y+N\cap(X+U)=N,\quad Y\cap N\cap(X+U)=Y\cap(X+U)\ll M.$$

Since

$$M=U+X+N=U+X+Y+N\cap(X+U)=U+X+Y,$$

$Y$ is a weak supplement of $U+X$ in $M$. Since $Y+U\leq N+U$ and $Y+U+X=M$, $X$ is a weak supplement of $Y+U$. Therefore $U\cap(X+Y)\leq X\cap(Y+U)+Y\cap(X+U)\ll M$. So $X+Y$ is a weak supplement of $U$ in $M$. □

**Proposition 2.3.7.** *Arbitrary sum of weakly cofinitely supplemented modules is a weakly cofinitely supplemented module.*

*Proof.* Let $M=\sum_{i\in I}M_i$ and let $N$ be cofinite submodule of $M$. Then $M/N$ has a generating set $\{x_1+N,....,x_r+N\}$ so any $x_i$ with $i\in I$ can be expressed as $x_i=m_{i1}+m_{i2}+....+m_{is(i)}$ with each $m_{ik}$ is from some $M_{ik}$ where $i_k\in I$. Any element $m+N$ from $M/N$ can be expressed as: $m+N=r_1x_1+....+r_nx_n+N$. So $m=r_1(m_{11}+...+m_{1s(1)})+...+r_n(m_{n1}+...+m_{ns(n)})+N$, where $n\in N$. Thus $M=\sum_{j\in J}M_j+N$ with finite $J=\{1_1,....,1_{s(1)},2_1,...,n_{s(n)}\}$ Let us write

$$M=\sum_{j\in J}M_j+N=M_{11}+\sum_{j\in J-\{11\}}M_j+N.$$

We know that $M_{11}$ is weakly cofinitely supplemented, and $M_{11}+\sum_{j\in J-\{11\}}M_j+N$ has trivially weak supplement 0. Therefore by the Lemma 2.3.6 $\sum_{j\in J-\{11\}}M_j+N$ has a weak supplement. Since $J$ is finite by repeated use of the Lemma 2.3.6, we deduce that $N$ has a weak supplement. It follows that $M$ is weakly cofinitely supplemented. □

**Definition 2.3.8.** Let $M$ be an $R-$module if there is an epimorphism $f:\oplus M\to N$, then we say that $N$ is *$M-$generated.*

**Corollary 2.3.9.** *If $M$ is weakly cofinitely supplemented, then any $M-$generated module is weakly cofinitely supplemented.*

*Proof.* By 2.3.7 $\oplus M$ is weakly cofinitely supplemented and by 2.3.4 the image of the epimorphism $f : \oplus M \to N$ is weakly cofinitely supplemented □

**Definition 2.3.10.** A module $M$ is called *cofinitely supplemented* if every cofinite submodule of $M$ has a supplement in $M$.

Clearly any cofinitely supplemented module is weakly cofinitely supplemented. Now we study a situation when the converse is true.

**Lemma 2.3.11.** *Let $M$ be an $R-$module and $U$ be a submodule of $M$ such that $M/U$ is finitely generated. If $U$ has a weak supplement $V$ in $M$ and for every finitely generated submodule $K$ of $V$, $RadK = K \cap RadM$, then $U$ has a finitely generated supplement in $M$.*

*Proof.* $V$ is a weak supplement of $U$ in $M$, i.e $U + V = M$ and $U \cap V \ll M$. Since $M/U$ is finitely generated, by Lemma 2.3.3 $U$ has a finitely generated weak supplement $K \leq V$ in $M$, i.e. $M = U + K$ and $U \cap K \ll M$. Then $U \cap K \leq RadM$. Therefore $U \cap K \leq K \cap RadM = RadK$. But $RadK \ll K$ by 1.3.5, so $U \cap K \ll K$, i.e. $K$ is a supplement of $U$ in $M$. □

**Theorem 2.3.12.** *Let $M$ be an $R-$module such that for every finitely generated submodule $K$ of $M$, $RadN = N \cap RadM$. Then $M$ is weakly cofinitely supplemented if and only if $M$ is cofinitely supplemented.*

*Proof.* $\Rightarrow$ Let $K$ be a cofinite submodule of $M$. Since $M$ is weakly cofinitely supplemented, $K$ has a weak supplement $N$ in $M$ and by 2.3.3 we can assume $N$ is finitely generated submodule. Now we can write $K+N = M$ and $K \cap N \ll M$. Since $N$ is finitely generated by 1.3.5 $RadN \ll N$ and by hypothesis $RadN = N \cap RadM$. Obviously $K \cap N \leq N \cap RadM$ and so $K \cap N \leq RadN \ll N$. Therefore $K \cap N \ll N$ and this implies that $N$ is a supplement for $K$. Hence $M$ is cofinitely supplemented.

$\Leftarrow$ Clear, since every supplement is a weak supplement. $\square$

**Corollary 2.3.13.** *Let $M$ be a finitely generated module such that for every submodule $N$ of $M$, $RadN = N \cap RadM$. Then $M$ is weakly supplemented if and only if $M$ is supplemented.*

*Proof.* $\Rightarrow$: Let $U$ be a submodule of $M$. Since $M$ is finitely generated $M/U$ is also finitely generated and since $RadN = N \cap RadM$ for every submodule $N$ of $M$, $U$ has a supplement in $M$ by 2.3.11.

$\Leftarrow$: obvious. $\square$

**Proposition 2.3.14.** *Let $M$ be an $R-$module and and $M/RadM$ be weakly cofinitely supplemented . Then every cofinite submodule of $M/RadM$ is a direct summand.*

*Proof.* Let $K/RadM$ be a cofinite submodule $M/RadM$. Since $M/RadM$ is weakly cofinitely supplemented , $K/RadM$ has a weak supplement$L/RadM$ in $M/RadM$ i.e. $K/RadM + L/RadM = M/RadM$ and $K/RadM \cap L/RadM \ll M/RadM$. By 1.3.5 $M/RadM$ doesn't have nontrivial small submodule. Therefore $K/RadM \cap L/RadM = 0_{M/RadM}$. Hence $K/RadM$ is a direct summand. $\square$

**Theorem 2.3.15.** *Suppose $M$ is an $R-$module with $RadM \ll M$ and $M/RadM$ is weakly cofinitely supplemented. Then $M$ is weakly cofinitely supplemented .*

*Proof.* Let $U$ be a cofinite submodule of $M$. Then

$$M/(U + RadM) \cong (M/U)/((U + RadM)/U)$$

is finitely generated i.e. $U + RadM$ is cofinite.

On the other hand

$$(M/RadM)/(U + RadM)/RadM \cong M/(U + RadM)$$

is finitely generated and so $(U + RadM)/RadM$ is a cofinite submodule of $M/RadM$.

Since $M/RadM$ is weakly cofinitely supplemented, $(U + RadM)/RadM$ has a weak supplement $K/RadM$ in $M/RadM$ i.e.

$(U + RadM)/RadM \;+\; K/RadM \;=\; M/RadM$ and

$(U + RadM)/RadM \;\cap\; K/RadM \;=\; (U \cap K + RadM)/RadM \ll M/RadM$.

Now we get $M = U + RadM + K = U + K$. And since $M/RadM$ has only the trivial small submodule we get $U \cap K + RadM = RadM$, that is $U \cap K \leq RadM$ and since $RadM \ll M$; $U \cap K$ is also small in $M$. This shows that $K$ is a weak supplement of $U$ in $M$. Hence $M$ is weakly cofinitely supplemented. □

**Theorem 2.3.16.** *Let $M$ be an $R-$module with $RadM \ll M$. Then the following statements are equivalent.*

(1) *$M$ is weakly cofinitely supplemented,*

(2) *$M/RadM$ is weakly cofinitely supplemented,*

(3) *Every cofinite submodule of $M/RadM$ is a direct summand,*

(4) *Every maximal submodule of $M/RadM$ is a direct summand,*

(5) *Every maximal submodule of $M$ has a weak supplement.*

*Proof.* (1) $\Rightarrow$ (2) By 2.3.5

(2) $\Rightarrow$ (3) By 2.3.14

(3) $\Rightarrow$ (4) Maximal submodules are cofinite so by the assumption they are direct summands.

(4) $\Rightarrow$ (5) If $K$ is a maximal submodule $M$ then, $K/RadM$ is maximal submodule of $M/RadM$. So for a submodule $L/RadM$ we get $K/RadM \oplus L/RadM$ and $K/RadM \cap L/RadM \;=\; (K \cap L)/RadM \;=\; 0_{M/RadM}$. Therefore $K \cap L \leq RadM \ll M$. It implies that $L$ is a weak supplement of $K$.

(5) $\Rightarrow$ (4) Let $N/RadM$ be a maximal submodule of $M/RadM$. Then $N$ is maximal in $M$ and so $N + K = M$ and $N \cap K \ll M$ for a submodule $K$ of $M$. So $N/RadM \oplus (K + RadM)/RadM = M/RadM$ and $N/RadM \cap (K + RadM)/RadM = [N \cap (K + RadM)]/RadM = 0_{M/RadM}$. Hence $N/RadM$ is a direct summand of $M/RadM$.

(4) $\Rightarrow$ (3) Let $N/RadM$ be a cofinite submodule of $M/RadM$ and $K/RadM$ be a complement of $N/RadM$ in $M/RadM$. Then $N/RadM \oplus K/RadM$ is essential in $M$.
Suppose that $N/RadM \oplus K/RadM \neq M/RadM$, since there is an epimorphism $f : (M/RadM)/(N/RadM) \to (M/RadM)/(N/RadM \oplus K/RadM)$, $(M/RadM)/(N/RadM \oplus K/RadM)$ is also finitely generated. Thus $N/RadM \oplus K/RadM$ is contained in a maximal submodüle $P/RadM$ of $M/RadM$. By the assumption $P/RadM$ is a direct summand of $M/RadM$. So there exists a submodule $L/RadM$ such that $M/RadM = P/RadM \oplus L/RadM$. Then $(N/RadM \oplus K/RadM) \cap L/RadM = 0$, and since $N/RadM \oplus K/RadM$ is essential in $M/RadM$, $L/RadM = 0$ contradiction.

(3) $\Rightarrow$ (2) Let $K/RadM$ be a cofinite submodule of $M/RadM$. By hypothesis $K/RadM$ is a direct summand i.e $K/RadM \oplus L/RadM$ for a submodule $L/RadM$ and so $L/RadM$ is a weak supplement for $K/RadM$.

(2) $\Rightarrow$ (1) By Theorem 2.3.12 $\square$

## 2.4 Weakly Finitely Supplemented Modules

**Definition 2.4.1.** Let $M$ be a module. If every finitely generated submodule of $M$ has a weak supplement in $M$ then $M$ is called *weakly finitely supplemented.*

**Lemma 2.4.2.** *Let $M$ be a finitely generated module, $M_1$ and $U$ be finitely generated submodules and $M_1$ be weakly finitely supplemented. If $M_1 + U$ has a weak supplement $X$ such that $M_1 \cap (X + U)$ is finitely generated, then $U$ has a weak supplement in $M$.*

*Proof.* Let $X$ be a weak supplement of $M_1 + U$ in $M$ i.e. $M_1 + U + X = M$ and $(M_1+U)\cap X \ll M$. $M_1\cap(U+X) \leq M_1$ and by the assumption $M_1\cap(U+X)$ is finitely generated and $M_1$ is weakly finitely supplemented. So $M_1\cap(X+U)$ has a weak supplement $Y$ in $M_1$, that is, $M_1\cap(X+U)+Y = M_1$ and $M_1\cap(X+U)\cap Y = (X + U) \cap Y \ll M_1$.
We get $M_1 + U + X = M_1 \cap (X + U) + Y + U + X = U + X + Y$. And $U\cap(X+Y) \leq (U+Y)\cap X+(U+X)\cap Y \leq (U+M_1)\cap X+(U+X)\cap Y \ll M$. This implies that $X + Y$ is a weak supplement of $U$ in $M$. □

Unfortunately a sum of weakly finitely supplemented modules need not be weakly finitely supplemented, but we can prove the following proposition.

**Proposition 2.4.3.** *Let $M$ be an $R-$module and $M = M_1 + M_2$, with $M_1$, $M_2$ finitely generated and weakly finitely supplemented. If either*

*(i) $M$ is coherent, or*

*(ii) $M$ is self-projective and $M_1 \cap M_2 = 0$.*
*Then $M$ is weakly finitely supplemented.*

*Proof.* (i) Let $U$ be any finitely generated submodule of $M$. Clearly $M_1 + M_2 + U$ has the trivial weak supplement. Let consider the submodule $M_2 \cap (M_1 + U + 0)$.

If $M$ is coherent, then by 1.4.5 $M_2 \cap (M_1 + U + 0)$, as an intersection of finitely generated submodules is finitely generated and by the lemma 2.4.2 $M_1 + U$ has a weak supplement $X$ in $M$. By 2.3.3 we may consider $X$ as a finitely generated submodule of $M$.

Now $M_1 \cap (X + U)$ is also finitely generated by 1.4.5 and again applying the lemma 2.4.2 we can say $U$ has a weak supplement in $M$.

$(ii)$ If $M$ is self-projective and $M_1 \cap M_2 = 0$, then

$$(M_1 + U)/M_2 \cap (M_1 + U) \cong M/M_2 \cong M_1$$

is $M$-projective by 1.5.3 and by 1.5.4 $(M_1 + U)$-projective. Now we can say $M_2 \cap (M_1 + U)$ is a direct summand of $M_2$ and so finitely generated. Hence by the lemma 2.4.2 $M_1 + U$ has a weak supplement $X$ in $M$. By 2.3.3 we may consider $X$ as a finitely generated submodule of $M$.

Now

$$(X + U)/M_1 \cap (X + U) \cong (M_1 + X + U)/M_1 \cong M/M_1 \cong M_2$$

is $M$-projective by 1.5.3 and by 1.5.4 $(X + U)$-projective. So $M_1 \cap (X + U)$ is a direct summand of $M_1$ and so finitely generated. Therefore by the lemma 2.4.2 $U$ has a weak supplement. And this completes the proof. □

**Proposition 2.4.4.** *If $M$ is weakly finitely supplemented and $L$ is a finitely generated submodule of $M$, then $M/L$ is weakly finitely supplemented.*

*Proof.* Let $K/L$ be a finitely generated submodule of $M/L$ and $K/L = \{k_1 + L, ..., k_n + L\}$. By the assumption $L$ is also finitely generated, say $L = \{l_1, ..., l_m\}$. So we get $K =< \{k_1, ..., k_n\} > +L =< \{k_1, ..., k_n, l_1, ..., l_m\} >$ Since $M$ is weakly finitely supplemented, $K$ has a weak supplement $N$ in $M$. That is $K + N = M$ and $K \cap N \ll M$.

So we can write $K/L + (N + L)/L = (K + N)/L = M/L$ and $[K/L \cap (N+L)]/L = K \cap (N+L)/L = (K \cap N + L)/L$, this is the homomorphic image of $K \cap N$ under the canonical homomorphism $\pi : M \to M/L$. Since $K \cap N$ is small, by 1.2.1 it follows that $\pi(K \cap N) = (K \cap N + L)/L$ is small in $M/L$. Therefore $M/L$ is weakly finitely supplemented. □

**Corollary 2.4.5.** *Let $M$ be weakly finitely supplemented and finitely generated and $M-$projective or coherent. Then for any finitely generated submodule of $K \leq M^n$, with $n \in N$, the factor module $M^n/K$ is weakly finitely supplemented.*

*Proof.* By 2.4.3 $M^n$ is weakly finitely supplemented and by 2.4.4 $M^n/K$ is weakly finitely supplemented. □

**Proposition 2.4.6.** *Let $M$ be a weakly finitely supplemented module. Then any small cover of $M$ is weakly finitely supplemented.*

*Proof.* Let $N$ be a small cover of $M$ i.e.there is an epimorphism $f : N \to M$ with $Kerf \ll N$, and let $K$ be a finitely generated submodule of $N$. Then by 1.1.10 $f(K)$ is also finitely generated and so it has a weak supplement $L$ in $M$ i.e. $f(K) + L = M$ and $f(K) \cap L \ll M$.
Now, $K + f^{-1}(L) = f^{-1}(f(K)) + f^{-1}(L) = f^{-1}(M) = N$ and by 2.1.6 $K \cap f^{-1}(L) \ll N$. This implies that $f^{-1}(L)$ is a weak supplement of $K$ in $N$. Therefore $N$ is weakly finitely supplemented. □

**Corollary 2.4.7.** *If $L \ll M$ and $M/L$ is weakly finitely supplemented then $M$ is weakly finitely supplemented.*

*Proof.* Consider the epimorphism $f : M \to M/L$. Then $Kerf = L \ll M$ so $M$ is a small cover of $M/L$ and so by proposition above $M$ is weakly finitely supplemented. □

To give an example of a weakly finitely supplemented module which are not weakly supplemented we will consider Von Neumann regular ring.

**Definition 2.4.8.** $R$ is called *Von Neumann regular* ring if for all $a \in R$ there exists $x \in R$ such that $axa = a$.

For simplicity we shall refer to these rings as *regular* rings.

**Theorem 2.4.9.** *The following statements are equivalent:*

(1) *$R$ is a regular ring;*

(2) *Every principal ideal $Ra$ is generated by an idempotent;*

(3) *For every principal ideal $Ra$ of $R$, there exists $b \in R$ such that $R = Ra \oplus Rb$;*

(4) *Every principal ideal $Ra$ is a direct summand of $R$;*

(5) *Every finitely generated ideal is a direct summand.*

*Proof.* See , Ribenboim(1967) Chapter III 3.16. □

**Theorem 2.4.10.** *The Radical of a regular ring is equal to zero.*

*Proof.* See , Ribenboim(1967) Chapter III □

**Proposition 2.4.11.** *Regular ring which is not semisimple is weakly finitely supplemented but not weakly supplemented.*

*Proof.* Let $N$ be a finitely generated ideal of $R$. By 2.4.9(5) $N$ is a direct summand i.e. $R = N \oplus K$ and $K \cap N = 0 \ll R$ for an ideal $K$ of $R$. This shows that $K$ is a weak supplement of $N$. Therefore $R$ is weakly finitely supplemented. Now suppose that $R$ is weakly supplemented and let $L$ be any ideal of $R$. Then $L + K = M$ and $L \cap K \ll R$ for some $K \leq R$. By the theorem above $Rad_R R = 0$ and so $L \cap K = 0$. It follows that $L$ is a direct summand. And by 1.3.7 $R$ is semisimple, contradiction. □

# REFERENCES


Alizade, R., Bilhan, G. & Smith, P. F. (to appear). Modules whose maximal submodules have supplements. Communication in Algebra, to appear.

Alizade, R. & Pancar, A.(1999). Homoloji Cebire Giriş. Samsun: Ondokuz Mayıs Üniversitesi Yayınları.

Anderson, F. W. & Fuller, K. R.(1992). Rings and Categories of Modules. New York: Springer Verlag.

Bilhan, G. (1999). Cofinitely Supplemented Modules. Ph.D thesis: Dokuz Eylül University Natural Sciences Institute, İzmir, Turkey.

Lomp, C.(1999). On semilocal modules and rings, Communication in Algebra, 27, no. 4, 1921-1935.

Ribenboim, P.(1967). Rings and Modules. New York: John Wiley & Sons.

Wisbauer, R.(1991). Foundations of Modules and Rings . Amsterdam: Gordon and Breach.

Zöschinger, H.(1978). Invarianten wesentlicher Überdeckungen, Math. Annalen, 237, 193-202.