WIDOM FACTORS

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July, 2014

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In this thesis we recall classical results on Chebyshev polynomials and logarithmic capacity. Given a non-polar compact set K, we define the n-th Widom factor $W_n(K)$ as the ratio of the sup-norm of the n-th Chebyshev polynomial on K to the n-th degree of its logarithmic capacity. We consider results on estimations of Widom factors. By means of weakly equilibrium Cantor-type sets, $K(\gamma)$, we prove new results on behavior of the sequence $(W_n(K))_{n=1}^{\infty}$.

By K. Schiefermayr[1], $W_n(K) \geq 2$ for any non-polar compact $K \subset \mathbb{R}$. We prove that the theoretical lower bound 2 for compact sets on the real line can be achieved by $W_{2^s}(K(\gamma))$ as fast as we wish.

By G. Szegő[2], rate of the sequence $(W_n(K))_{n=1}^{\infty}$ is slower than exponential growth. We show that there are sets with unbounded $(W_n(K))_{n=1}^{\infty}$ and moreover for each sequence $(M_n)_{n=1}^{\infty}$ of subexponential growth there is a Cantor-type set which Widom factors exceed M_n for infinitely many n.

By N.I. Achieser[3][4], limit of the sequence $(W_n(K))_{n=1}^{\infty}$ does not exist in the case K consists of two disjoint intervals. In general the sequence $(W_n(K))_{n=1}^{\infty}$ may behave highly irregular. We illustrate this behavior by constructing a Cantor-type set K such that one subsequence of $(W_n(K))_{n=1}^{\infty}$ converges as fast as we wish to the theoretical lower bound 2, whereas another subsequence exceeds any sequence $(M_n)_{n=1}^{\infty}$ of subexponential growth given beforehand.

Keywords: Logarithmic capacity, Chebyshev numbers, Cantor sets.

ÖZET

WIDOM FAKTÖRLERİ

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Bu tezde Chebyshev polinomları ve logaritmik kapasite üzerine klasik sonuçları tekrar ettik. Polar olmayan tıkız bir küme için n-inci Widom faktörünü, $W_n(K)$, K üzerindeki n-inci Chebyshev polinomunun sup-normunun Knın logaritmik kapasitesinin n-inci kuvvetine oranı olarak tanımladık. Widom faktörlerinin kestirimleri üzerine sonuçları inceledik. Zayıf dengeli Cantor tipi kümeler, $K(\gamma)$, aracılığı ile $(W_n(K))_{n=1}^{\infty}$ dizisinin davranışı üzerine yeni sonuçlar kanıtladık.

K. Schiefermayr[1] tarafından her polar olmayan tıkız $K \subset \mathbb{R}$ için $W_n(K) \geq 2$ olduğu gösterildi. Reel doğru üzerindeki tıkız kümeler için teorik alt sınır olan 2'ye istediğimiz hızda $W_{2^s}(K(\gamma))$ ile ulaşılabileceğini kanıtladık.

G. Szegő[2] tarafından $(W_n(K))_{n=1}^{\infty}$ dizisinin büyüme hızının üstel büyümeden yavaş olduğu gösterildi. $(W_n(K))_{n=1}^{\infty}$ dizisi sınırlı olmayan kümeler olduğunu ve dahası alt-üstel büyüyen her $(M_n)_{n=1}^{\infty}$ dizisi için Widom faktörleri M_n'yi sonsuz sayıda n değerinde aşan bir Cantor tipi küme olduğunu gösterdik.

N.I. Achieser[3][4] tarafından K'nın iki ayrık aralıktan oluştuğu durumda $(W_n(K))_{n=1}^{\infty}$ dizisinin limitinin var olmadığı gösterildi. Genelde $(W_n(K))_{n=1}^{\infty}$ dizisi son derece düzensiz davranabilir. Biz bu durumu Cantor tipi bir küme kurarak resmettik, öyle ki $(W_n(K))_{n=1}^{\infty}$ dizisinin bir altdizisi teorik alt sınır olan 2'ye istediğimiz hızda yakınsarken başka bir altdizi önceden verilmiş alt-üstel herhangi bir $(M_n)_{n=1}^{\infty}$ dizisini aşıyor.

Anahtar sözcükler: Logaritmik kapasite, Chebyshev sayıları, Cantor kümeleri.

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Contents

Chapter 1

Introduction

Let K be a compact subset of the complex plane consisting of infinitely many points. Among all monic (i.e. its leading coefficient is 1) polynomials of degree n, there exists unique monic polynomial of degree n which deviates least from zero on K. This unique polynomial $T_{n,K}$ is called nth Chebyshev polynomial on K. Chebyshev polynomials are named after P.L. Chebyshev who first studied Chebyshev polynomials on $[-1,1]$. Supremum norm of $T_{n,K}$ on K, denoted by $t_n(K)$ is called nth Chebyshev number of K. Chebyshev polynomials have applications in different branches of mathematics such as numerical analysis, number theory, potential theory, approximation theory and orthogonal polynomials. They can be also seen as solution of best approximation problem of monomial z^n from set of algebraic polynomials of smaller degree in the space of continuous functions on K with supremum norm. Best approximation problem, defined as determination of a point of a given set with minimum distance to a given point not belonging this set in a metric space, is one of fundamental topics in approximation theory.

In Chapter 2 we consider some classical results on best approximation and Chebyshev polynomials. One of them, existence of $lim_{n\to\infty} t_n(K)^{\frac{1}{n}}$ was proved in 1923 by M. Fekete[5]. This limit is called Chebyshev constant of K. At the end of Chapter 2 we give a proof of this result.

Exact value of Chebyshev constant is equal to logarithmic capacity, which was first proved by G. Szegő[2] in 1924 . In Chapter 3 we focus on some fundamental concepts of logarithmic potential theory such as logarithmic potential, equilibrium measure, logarithmic capacity and transfinite diameter. Our aim is to present logarithmic capacity $Cap(.)$ in detail and prove Szegő's result.

Many mathematicians considered a relation between $t_n(K)$ and $Cap^n(K)$. Because of the fundamental paper by H. Widom[6] we suggest the name *Widom factor* of non-polar K, $W_n(K)$ as the ratio of the sup-norm of the nth Chebyshev polynomial on K to the n-th degree of its logarithmic capacity for all $n \in \mathbb{N}$. In [6] H. Widom studies orthogonal and Chebyshev polynomials on a system of Jordan curves. Two leading experts in the field V. Totik and P. Yuditskii say 'Widom's paper had a huge impact on the theory of extremal polynomials, in particularly on the theory of orthogonal polynomials.'[7].

Widom Factors are invariant under dilation and translation, but exact values of them are known only for a few cases; $W_n(\overline{\mathbb{D}}) = 1$, $W_n([-1, 1]) = 2$. In addition, even in simple cases $(W_n(K))_{n=1}^{\infty}$ may behave irregularly. N.I. Achieser[3] proved in 1932 nonexistence of $\lim_{n\to\infty}(W_n(K))_{n=1}^{\infty}$ in the case K consists of two disjoint closed intervals. Therefore people try to estimate $(W_n(K))_{n=1}^{\infty}$ for different sets. In Chapter 4 after showing some examples and properties of Widom Factors we consider known lower and upper estimates of $(W_n(K))_{n=1}^{\infty}$ for different cases.

In Chapter 5 we recall construction of *weakly equilibrium Cantor-type sets*, denoted by $K(\gamma)$, and calculate $W_{2^s}(K(\gamma))$. *Weakly equilibrium Cantor-type sets* recently introduced by A. Goncharov[8] are Cantor-type subsets of real line constructed as intersection of polynomial inverse images depending on a given beforehand sequence of positive real numbers less than 1/4. We prove that the value 2, which is the smallest accumulation point for sequences of Widom factors for compact sets on the real line, can be achieved by $W_{2^s}(K(\gamma))$ as fast as we wish.

In general $(W_n(K))_{n=1}^{\infty}$ is not bounded from above both for complex and real cases. However, by Szegő's result rate of $(W_n(K))_{n=1}^{\infty}$ is slower than exponential growth for any non-polar, compact $K \subset \mathbb{C}$. We proved this is the best possible case in general. In Chapter 6 we construct $K(\gamma)$ with preassigned growth of $W_{2^s}(K(\gamma))$. This will lead to the following result: it is not possible to find a sequence $(M_n)_{n=1}^{\infty}$ of subexponential growth and a constant $C > 0$ such that the inequality

$$W_n(K) \leq C \cdot M_n$$

is valid for all non-polar compact sets and for all $n \in \mathbb{N}$.

In the last chapter we construct a Cantor-type set K with highly irregular behavior of Widom factors. Namely, one subsequence of $(W_n(K))_{n=1}^{\infty}$ converges as fast as we wish to the theoretical lower bound 2, whereas another subsequence exceeds any sequence $(M_n)_{n=1}^{\infty}$ of subexponential growth given beforehand.

Chapter 2

Best Approximation and Chebyshev Polynomials

2.1 Best Approximation in Metric Spaces

Definition 2.1. Let (X, d) be a metric space, $Y \subset X$ and $f \notin Y$. Then $g_0 \in Y$ is a *best approximation* to f out of Y if for every $g \in Y$, $d(f, g_0) \leq d(f, g)$. The set of best approximations to f out of Y is denoted by $M_f(Y)$.

Theorem 2.2. *[9, Th.1.1] Let (X, d) be a metric space and $Y \subset X$ be compact. Then for every $f \notin Y$ there exists best approximation g_0 to f out of Y, so $M_f(Y)$ is not empty.*

Theorem 2.3. *[10, p.20] Let $(X, ||.||)$ be a normed linear space and Y be a finite dimensional linear subspace. Then for every $f \notin Y$ there exists best approximation g_0 to f out of Y.*

Definition 2.4. Let X be a linear space. A set $B \subset X$ is said to be *convex* if $\alpha b_1 + (1 - \alpha)b_2 \in B$ for every $b_1, b_2 \in B$, for every $\alpha \in [0, 1]$.

Remark 2.5. For every normed space $(X, ||.||)$ the unit ball

$$K(0, 1) := \{x : ||x|| \leq 1\}$$

is convex.

Definition 2.6. A normed space $(X, ||.||)$ is said to be *strictly convex* if $K(0, 1)$ is *strictly convex*, i.e. if $f \neq g$ and $||f|| = ||g|| = 1$, then $||f + g||<2$.

Remark 2.7. $C[a, b], c_0, l_\infty$ are not strictly convex.
$l_p, L_p((a, b))$ are strictly convex for $1<p<\infty$.

Proposition 2.8. *[9, Th.2.2] Let $(X, ||.||)$ be a normed space, $Y \subset X$ be convex and $f \in X$. Then $M_f(Y)$ is convex.*

Proposition 2.9. *[9, Th.2.4] Let $(X, ||.||)$ be strictly convex, normed linear space, $Y \subset X$ be convex and $f \in X$. Then either $M_f(Y) = \emptyset$ or $M_f(Y) = \{g_0\}$.*

Proposition 2.10. *[9, Th.2.3] Let $(X, ||.||)$ be strictly convex, normed linear space, $Y \subset X$ be convex, compact and $f \in X$. Then $M_f(Y) = \{g_0\}$.*

Proposition 2.11. *[10, p.23] Let $(X, ||.||)$ be strictly convex, normed linear space, $Y \subset X$ be convex, finite dimensional subspace and $f \in X$. Then $M_f(Y) = \{g_0\}$.*

2.2 Uniform Best Approximation in Space of Continuous Functions

By Theorem 2.3 for every $f \in C[a, b]$ there exists polynomial $p_n \in \mathcal{P}_n$ such that $inf_{p \in \mathcal{P}_n} ||f - p||_{[a,b]} = ||f - p_n||_{[a,b]}$. This polynomial is called *polynomial of best approximation* of degree n to f. The distance is denoted by $E_n(f, [a, b]) := ||f - p_n||_{[a,b]}$. Recall that

$$||f||_K := \sup_{x \in K} |f(x)|$$

where the function f is well-defined on the set K.

Definition 2.12. Let $f \in C[a, b]$. $(x_k)_{k=1}^N$ are called *alternation points* for f if $|f(x_k)| = ||f||_{[a,b]}$ for $k = 1, 2, \ldots, N$ and $f(x_{k+1}) = -f(x_k)$ for $k = 1, 2, \ldots, N-1$. The set of *alternation points* is called *alternation set*.

Remark 2.13. For $f \in C[a, b]$ alternation set is not unique, but number of elements of all alternation sets of f is the same.

Lemma 2.14. *Let $f \in C[a,b]$ and p_n be the polynomial of best approximation to f out of $\mathcal{P}_n$. Then there exists at least 2 alternation points for $f - p_n$.*

This evident lemma is the first step to the following fundamental *Chebyshev alternation theorem.* Note that even if it is called Chebyshev alternation theorem, it was proven independently by H.F. Blichfeldt[11] and P. Kirchberger[12].

Theorem 2.15. *[13, Th.3.A] Let $f \in C[a,b]$. Then $p_n \in \mathcal{P}_n$ is the polynomial of best approximation to f if and only if alternation set $(x_k)_1^N$ for $f - p_n$ contains at least $n+2$ points, i.e. $N \geq n+2$.*

In 1948 A.N. Kolmogorov gave characterization of best approximation in space of continuous functions on a compact Hausdorff topological space for both real and complex cases.

Theorem 2.16. *[13, Th.3.2.1] [14] Let K be a compact Hausdorff topological space and X_n be an n-dimensional subspace of $C(K)$, the set of complex valued continuous functions on K. A function $p \in X_n$ is a best approximation to $f \in C(K)$ if and only if for each $q \in X_n$*

$$\max_{x \in K_0} Re\{(f(x) - p(x))\overline{q(x)}\} \geq 0,$$

where $K_0 = \{x \in K : |f(x) - p(x)| = ||f - p||_K\}$

Theorem 2.17. *[13, Th.3.2.2] Let K be a compact Hausdorff topological space and X_n be an n-dimensional subspace of $C_{\mathbb{R}}(K)$, the set of real valued contiuous functions on K. A function $p \in X_n$ is a best approximation to $f \in C_{\mathbb{R}}(K)$ if and only if for each $q \in X_n$*

$$\max_{x \in K_0} \{(f(x) - p(x))q(x)\} \geq 0,$$

where $K_0 = \{x \in K : |f(x) - p(x)| = ||f - p||_K\}$

In order to deal with uniqueness of best approximation in $C(K)$ we should consider *Haar System.*

Definition 2.18. Let K be a compact Hausdorff topological space. A set $\Phi = (\varphi_k)_{k=1}^N$ in $C(K)$ is called a *Haar System* on K if it satisfies following conditions.

- K contains at least N points.
- Every nontrivial linear combination of elements of Φ, $P = \sum_{k=1}^{N} c_k \varphi_k$ has at most $N-1$ distinct zeros.

Theorem 2.19. *[13, Th.3.4.2] Let K be a compact Hausdorff topological space and $(\varphi_k)_{k=1}^N$ be a Haar system on K. Then for every $f \in C(K)$ there exists unique best approximation to f out of span$(\varphi_k)_{k=1}^N$.*

2.3 Chebyshev Polynomials

Definition 2.20. $T_n(x) = \cos n\theta$ is called *the classical Chebyshev polynomial of degree n*, where $x = \cos\theta, \theta \in [0, \pi]$ *and* $n \in \mathbb{N}_0 := \{0, 1, 2...\}$.

Some properties of the classical Chebyshev polynomials are as follows.

1) $T_n(x) \in \mathcal{P}_n$.
2) $T_{n+1} = 2xT_n - T_{n-1}$.
3) $T_n(x) = \frac{1}{2}[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n]$, $|x| \geq 1$.
4) $T_n(x)$ has $n+1$ extrema on [-1,1], namely $x_k = \cos(\frac{k\pi}{n})$, satisfying $T_n(x_k) = (-1)^k$, $k = 0, 1, .., n$. Therefore $|T_n| \leq 1$ on $[-1, 1]$.
5) $\widetilde{T}_n := 2^{-n+1}T_n \in \mathcal{M}_n$, where $\mathcal{M}_n$ is the set of monic algebraic polynomials of degree n.

Classical Chebyshev polynomials satisfy many other properties and have applications in different branches of mathematics. Above mentioned properties and more on classical Chebyshev polynomials can be found in [15]. We will focus on one of extremal properties of classical Chebyshev polynomials.

Chebyshev's Theorem . *[13, Th.3.6.1] The monic classical Chebyshev polynomial of degree n, $\widetilde{T}_n = \frac{T_n}{2^{n-1}}$ is the monic polynomial of least deviation on $[-1,1]$, i.e. $||\widetilde{T}_n||_{[-1,1]} \leq ||P||_{[-1,1]}$ for any monic polynomial P of degree n.*

Proof. Let us assume there exists $P \in \mathcal{M}_n$ such that $||P|||_{[-1,1]} < ||\widetilde{T}_n||_{[-1,1]} = 2^{-n+1}$. Define $Q := \widetilde{T}_n - P$. Q is a polynomial of degree at most $n-1$, since both $\widetilde{T}$ and P are monic. On $n+1$ extrema of $\widetilde{T}_n$, namely $x_k = \cos(\frac{k\pi}{n})$, we have $\widetilde{T}_n(x_k) = (-1)^k 2^{-n+1}$ and $|P(x_k)| < 2^{-n+1}$ for $k = 0, 1, .., n$. Therefore P alternates sign at least n times, i.e. P has at least n distinct zeros, which contradicts the fact Q is a polynomial of degree at most $n-1$. $\square$

It is natural to generalize Chebyshev polynomials to other compact sets in direction of Chebyshev's Theorem.

Definition 2.21. Let K be a compact subset of $\mathbb{C}$ and $T_{n,K}$ be the monic polynomial with degree n such that $||T_{n,K}||_K \leq ||P||_K$ for any monic polynomial P of degree n. Then $T_{n,K}$ is called *nth Chebyshev polynomial on* K and $||T_{n,K}||_K$ is called *nth Chebyshev number of* K, denoted by $t_n(K)$.

Remark 2.22. For any compact $K \subset \mathbb{C}$ containing at least $n+1$ points there exists unique $T_{n,K}$.

- (existence) By Theorem 2.3 $X = (\cup_{n=0}^{\infty} \mathcal{P}_n, ||.||_K)$, $Y = \mathcal{P}_{n-1}$, $f = z^n$, $g_0 = T_{n,K}$.
- (uniquness) By Theorem 2.19 $(\varphi)_1^N = (z^{k-1})_1^N$, $f = z^n$.

Example 2.23. By Chebyshev's Theorem, classical Chebyshev polynomials are Chebyshev polynomials on $[-1,1]$, i.e. $T_{n,[-1,1]} = \widetilde{T}_n$, $t_n([-1,1]) = 2^{-n+1}$.

Example 2.24. $T_{n,\overline{\mathbb{D}}} = z^n$, $t_n(\overline{\mathbb{D}}) = 1$.
Let us verify this by contradiction. Assume there exists $p_{n-1} \in \mathcal{P}_{n-1}$ such that $||z^n - p_{n-1}||_{\overline{\mathbb{D}}} < 1$. Then for every $z = e^{it} \in \partial\mathbb{D}$, $0 \leq t \leq 2\pi$, we have $|z^n - p_{n-1}(z)| < 1$ and hence $|Re[z^n - p_{n-1}(z)]| = |\cos(nt) - Re[p_{n-1}(z)]| < 1$. $\cos(nt)$ has $2n$ extrema on $[0, 2\pi)$, namely $t_k = \frac{k\pi}{n}$, $k = 0, 1, \ldots, 2n-1$, satisfying $\cos(nt_k) =$

$(-1)^k$. Therefore $Re[p_{n-1}]$ has at least $2n-1$ zeros on $[0, 2\pi)$, since $|\cos(nt) - Re[p_{n-1}(z)]|<1$. However this contradicts the fact $Re[p_{n-1}]$ is a trigonometric polynomial of degree at most $n-1$.

Proposition 2.25. *$T_{n,\lambda K+a}(z) = \lambda^n T_{n,K}(\frac{z-a}{\lambda})$ and $t_n(\lambda K + a) = \lambda^n t_n(K)$ for every $n \in \mathbb{N}$, where $\lambda>0$, $a \in \mathbb{C}$.*

Proof. Let $n \in \mathbb{N}$ be fixed. Define $P_n(z) := T_{n,K}(\frac{z-a}{\lambda})$. Observe that $\lambda^n P_n(z) \in \mathcal{M}_n$ and $||T_{n,K}||_K = ||P_n||_{\lambda K+a}$. Therefore

$$t_n(K) = ||T_{n,K}||_K = ||P_n||_{\lambda K+a} \geq \frac{1}{\lambda^n}||T_{n,\lambda K+a}||_{\lambda K+a} = \frac{1}{\lambda^n}t_n(\lambda K + a).$$

Similarly define $Q_n(z) := T_{n,\lambda K+a}(\lambda z + a)$. Observe that $\frac{Q_n(z)}{\lambda^n} \in \mathcal{M}_n$ and $||T_{n,\lambda K+a}||_{\lambda K+a} = ||Q_n||_K$. Therefore

$$t_n(\lambda K + a) = ||T_{n,\lambda K+a}||_{\lambda K+a} = ||Q_n||_K \geq \lambda^n ||T_{n,K}||_K = \lambda^n t_n(K).$$

Two inequalities imply $t_n(\lambda K + a) = \lambda^n t_n(K)$ and by uniqueness of Chebyshev polynomials we get the result. □

The problem of finding *nth degree Chebyshev polynomial on* $K \subset \mathbb{C}$ is equivalent to the problem of finding best approximation to z^n from the set of algebraic polynomials of degree at most $n-1$ in the normed space $(C(K); ||.||)$ with sup-norm. Therefore characterizations of Chebyshev polynomials can be stated as corollaries of Chebyshev Alternation (Th.2.15) and Kolmogorov (Th.2.16) Theorems.

Theorem 2.26. *Let $a, b \in \mathbb{R}$ and $a<b$. A polynomial $p_n \in \mathcal{M}_n$ is the Chebyshev polynomial of $[a, b]$ if and only if alternating set $(x_k)_1^N$ for p_n on $[a, b]$, i.e. the set $\{x_k \in [a, b] : |p_n(x_k)| = ||p||_{[a,b]},\ p_n(x_{k+1}) = -p_n(x_k)\}$ contains at least $n+1$ points.*

Theorem 2.27. *Let $K \subset \mathbb{C}$ be compact. A polynomial $p_n \in \mathcal{M}_n$ is the Chebyshev polynomial of K if and only if for each polynomial q of degree at most $n-1$,*

$$\max_{x \in K_0} Re\{p(x)\overline{q(x)}\} \geq 0,$$

where $K_0 = \{x \in K : |p(x)| = ||p||_K\}$.

By positivity of Chebyshev numbers one can show that logaritms of Chebyshev numbers are subadditive.

Proposition 2.28. *Let $K \subset \mathbb{C}$ be compact. Then*

$$\log t_{m+n}(K) \leq \log t_m(K) + \log t_n(K)$$

for every $m, n \in \mathbb{N}$.

Proof.

$$t_{m+n}(K) = ||T_{m+n}||_K \leq ||T_m T_n||_K \leq ||T_m||_K . ||T_n||_K = t_m(K) t_n(K)$$

Since $t_n(K)>0$ for every $n \in \mathbb{N}$, we take logarithm of both sides and get the result. $\square$

Logartihmic subadditivity and the following lemma imply existence of $\lim_{n\to\infty} t_n(K)^{\frac{1}{n}}$, which was first proved by M. Fekete[5] in 1923.

Lemma 2.29. *If $a_{m+n} \leq a_m + a_n$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{n}$ exists.*

Proof. Assume $\inf\limits_n \dfrac{a_n}{n} = -\infty$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{n} = -\infty$, since existence of a subsequence converging to a real number or diverging to $+\infty$ contradicts the inequality $a_{m+n} \leq a_m + a_n$. Therefore without loss of generality assume $\inf\limits_n \dfrac{a_n}{n} = \alpha$. Then $\forall$ $\epsilon>0$, $\exists$ m such that $\frac{a_m}{m}<\alpha+\epsilon$. Any $n \in \mathbb{N}$ can be written in the form $n = qm + r$ $(0 \leq r \leq m-1)$ and we have

$$a_n = a_{qm+r} \leq a_{qm} + a_r \leq q a_m + a_r.$$

Therefore

$$\alpha \leq \frac{a_n}{n} \leq \frac{q a_m + a_r}{qm + r} = \frac{a_m}{m}\frac{qm}{qm+r} + \frac{a_r}{n} < (\alpha+\epsilon)\frac{qm}{qm+r} + \frac{a_r}{n}$$

and we get $\limsup\limits_{n\to\infty} \dfrac{a_n}{n}<\alpha+\epsilon$. Since $\epsilon>0$ is arbitrary, $\limsup\limits_{n\to\infty} \dfrac{a_n}{n} \leq \alpha$ and $\lim\limits_{n\to\infty} \dfrac{a_n}{n} = \alpha$. $\square$

Fekete's Theorem. *[5] The limit of the sequence $(t_n(K)^{1/n})_{n=1}^{\infty}$ exists.*

Proof. By Proposition 2.28 we can apply Lemma 2.29 to the sequence $(\log t_n(K))_{n=1}^{\infty}$. Continuity of logarithm function on positive real axis and positivity of Chebyshev numbers imply existence of $\lim_{n\to\infty} t_n(K)^{\frac{1}{n}}$. $\square$

Definition 2.30. The expression $\lim_{n\to\infty} t_n(K)^{\frac{1}{n}}$ is called *Chebyshev constant* of K and is denoted by $cheb(K)$.

Chapter 3

Some Concepts From Logarithmic Potential Theory

In this chapter we recall some fundamental concepts of logarithmic potential theory such as potential, energy and equilibrium measure. We mainly focus on logarithmic capacity and finish with showing equivalance of Chebyshev constant with logarithmic capacity, which is due to Szegő. These concepts are classical results of logarithmic potential theory. In Sections 2 and 3 we closely follow one of the main sources of the field [16].

3.1 Potential, Energy and Equilibrium Measure

Definition 3.1. Let $M_F := \{\mu \textit{ Borel measure} |\ \mu\mathbb{C}<\infty,\ supp\mu \text{ is compact}\}$ and $\mu \in M_F$. Then *logarithmic potential of* μ is the function $U^\mu : \mathbb{C} \to (-\infty, \infty]$ defined by

$$U^\mu(z) := \int \log \frac{1}{|z-\omega|} d\mu(\omega) \tag{1.1}$$

Example 3.2. Let $\mu := \sum_{k=1}^n \delta_{a_k}$ where

$$\delta_{z_0}(z) := \begin{cases} 1, & \text{if } z = z_0, \\ 0, & \text{if } z \neq z_0. \end{cases}$$

The potential of μ is

$$U^{\mu}(z) = -\log|\prod_{k=1}^{n}(z-a_k)|.$$

Definition 3.3. Let $\mu \in M_F$. Its *logarithmic energy* $I(\mu) \in (-\infty, \infty]$ is defined by

$$I(\mu) := \int U^{\mu}(z)d\mu(z) = \int\int \log\frac{1}{|z-\omega|}d\mu(\omega)d\mu(z) \tag{1.2}$$

Example 3.4. Let $\mu \in M_F$ with $supp\mu = (z_k)_{k=1}^{N}$. Logarithmic energy of μ is $I(\mu) = +\infty$.

This example illustrates existence of compact sets such that logarithmic energy of every Borel measure supported on this set is $+\infty$. Such sets are called polar.

Definition 3.5. Let $M_F(K) := \{\mu \ \textit{Borel measure}|\ \mu\mathbb{C}<\infty,\ supp\mu \subseteq K\}$, where $K \subset \mathbb{C}$ is compact. A Borel set E is called *polar* if $I(\mu) = +\infty$ for every $\mu \in M_F(K)$ for every compact $K \subseteq E$.

Definition 3.6. A property is said to hold *nearly everywhere*$(n.e)$ on a subset S of $\mathbb{C}$ if it holds everywhere on $S\backslash E$ for some Borel polar set E.

Definition 3.7. Let K be a compact subset of $\mathbb{C}$ and $M(K) := \{\mu \ \textit{Borel measure}|\ \mu K = 1,\ supp\mu \ \textit{is compact},\ supp\mu \subseteq K\}$. The measure $\mu_K \in M(K)$ is called *equilibrium measure for* K if $I(\mu_K) = \inf_{\mu\in M(K)} I(\mu)$.

Equilibrium measure is well-defined by following theorem and by definition, it can be concluded that K is polar if and only if every $\mu \in M(K)$ is equilibrium measure for K.

Theorem 3.8. *[17, Th.I.1.3, Cor.I.4.5] For every compact $K \subset \mathbb{C}$ there exists μ_K. If K is not polar then its equilibrium measure μ_K is unique. In addition for non-polar K support of μ_K is a subset of exterior boundary of K.*

Frostman theorem, which is also called fundamental theorem of potential theory in some sources, gives the relation between logarithmic potential and logarithmic energy of an equilibrium measure.

Frostman Theorem. *[18, (A.8)] Let $K \subset \mathbb{C}$ be compact. Then*

- $U^{\mu_K}(z) \leq I(\mu_K)$ *on* $\mathbb{C}$ *and*
- $U^{\mu_K}(z) = I(\mu_K)$ *on* $K \backslash E$*, where E is a polar, F_σ subset of ∂K.*

Inverse of second part of Frostman Theorem is valid with fewer restriction.

Theorem 3.9. *[18, Th.A.1] Let $K \subset \mathbb{C}$ be compact and $\mu \in M(K)$ satisfy $I(\mu){<}\infty$, $U^\mu = c$ on $K \backslash E$ with polar E and constant c. Then $\mu = \mu_K$ and $I(\mu_K) = c$.*

By means of Theorem 3.9 we can find equilibrium measures of unit disc and $[-1, 1]$.

Example 3.10. If $K := [-1, 1]$, the arcsine measure $d\mu := \frac{dx}{\pi\sqrt{1-x^2}}$ is the equilibrium measure for K, since

$$\begin{aligned} U^\mu(z) &= \int \log \frac{1}{|z-\omega|} d\mu(\omega) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{1}{|z - \cos\theta|} d\theta \\ &= \begin{cases} \log 2, & \text{if } z \in [-1, 1], \\ \log 2 - \log|z + \sqrt{z^2 - 1}|, & \text{otherwise .} \end{cases} \end{aligned}$$

In addition we see that $I(\mu) = \log 2$.

Example 3.11. If $K := \overline{\mathbb{D}}$, the normalized arclength measure $d\mu := \frac{d\lambda_{arc}}{2\pi}$ is the equilibrium measure for K, since

$$\begin{aligned} U^\mu(z) &= \int \log \frac{1}{|z-\omega|} d\mu(\omega) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{1}{|z - re^{i\theta}|} d\theta \\ &= \begin{cases} \log 1, & \text{if } |z| \leq 1, \\ \log \frac{1}{|z|}, & \text{if } |z| > 1. \end{cases} \end{aligned}$$

Here, $I(\mu) = 0$.

3.2 Logarithmic Capacity

Definition 3.12. The *logarithmic capacity* of a subset E of $\mathbb{C}$ is given by

$$Cap(E) := \sup_{\mu \in M(E)} e^{-I(\mu)}. \tag{2.1}$$

In particular if E is compact with equilibrium measure μ_E, then $Cap(E) = e^{-I(\mu_E)}$.

Remark 3.13. K is *polar* if and only if $Cap(K) = 0$.

From its definition it can be shown that logarithmic capacity is a monotone set function multiplicative with respect to modulus and invariant under translation.

Theorem 3.14. *Let $E, F \subset \mathbb{C}$.*

1. *If $E \subseteq F$, then $Cap(E) \leq Cap(F)$.*
2. *$Cap(aE + b) = |a| Cap(E)$ for every $a, b \in \mathbb{C}$.*

In addition this monotone set function is continuous on nested sequences with the restriction of being compact on decreasing sequences.

Theorem 3.15.

1. *If $K_1 \supset K_2 \supset \dots$ are compact subsets of $\mathbb{C}$ and $K := \cap_{n=1}^{\infty} K_n$, then*

$$Cap(K) = \lim_{n \to \infty} Cap(K_n).$$

2. *If $B_1 \subset B_2 \subset \dots$ are Borel subsets of $\mathbb{C}$ and $B := \cup_{n=1}^{\infty} B_n$, then*

$$Cap(B) = \lim_{n \to \infty} Cap(B_n).$$

Even if it is multiplicative with respect to modulus, logarithmic capacity is not weakly subadditive.

Proposition 3.16. *There is no constant d satisfying*

$$Cap(E \cup F) \leq d(Cap(E) + Cap(F)),$$

where $E, F \subset \mathbb{C}$.

However, in terms of its logarithm some estimations can be made on logarithmic capacity of union.

Theorem 3.17. *Let $E := \cup_{n=1}^{m} E_n$ be the union of Borel subsets E_n of $\mathbb{C}$ and $d>0$, where $m \in \mathbb{N} \cup \{\infty\}$.*

1. *If $diam(E) \leq d$, then*

$$\frac{1}{\log(\frac{d}{Cap(E)})} \leq \sum_{n=1}^{m} \frac{1}{\log(\frac{d}{Cap(E_n)})}$$

2. *If $dist(E_j, E_k) \geq d$ whenever $j \neq k$, then*

$$\frac{1}{\log^{+}(\frac{d}{Cap(E)})} \geq \sum_{n=1}^{m} \frac{1}{\log^{+}(\frac{d}{Cap(E_n)})}$$

Note that in first part logarithm is always nonnegative since $Cap(E) \leq diam(E) \leq d$.

Remark 3.18. Theorem 3.17 is valid if we assume $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$. Therefore first part of it implies bounded countable union of polar sets is polar. In addition by using this fact and second part of Theorem 3.15 we can show countable union of polar sets is polar.

By means of their equilibrium measures we can compute logarithmic capacity of $[-1,1]$ and $\overline{\mathbb{D}}$ and by Theorem 3.14 any line segment and disc.

Example 3.19. Let E be a line segment with length l. Then

$$Cap(E) = Cap([-\frac{l}{2}, \frac{l}{2}]) = \frac{l}{2} Cap([-1,1]) = \frac{l}{2} e^{-\log 2} = \frac{l}{4}$$

Example 3.20.

$$Cap(\overline{B_r(z_0)}) = Cap(\overline{B_r(0)}) = rCap(\overline{\mathbb{D}}) = re^{-0} = r$$

Logarithmic capacity of different sets can be computed if it is possible to find conformal mappings satisfying necessary conditions.

Theorem 3.21. *Let K_1, K_2 be compact subsets of $\mathbb{C}$ and Ω_1, Ω_2 be defined as connected components of $\overline{\mathbb{C}} \backslash K_1$ and $\overline{\mathbb{C}} \backslash K_1$ including point $\{\infty\}$ respectively, where $\overline{\mathbb{C}}$ denotes the extended complex plane. If there exists a meromorphic function f mapping Ω_1 to Ω_2 and satisfying*

$$f(z) = z + O(1)$$

as z goes to ∞, then

$$Cap(K_2) \leq Cap(K_1).$$

In addition, if f is a conformal mapping of Ω_1 onto Ω_2, then

$$Cap(K_2) = Cap(K_1).$$

It is also easy to compute logarithmic capacity of polynomial inverse images.

Theorem 3.22. *Let $K \subset \mathbb{C}$ be compact and $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n. Then*

$$Cap(p^{-1}(K)) = (\frac{Cap(K)}{|a_n|})^{\frac{1}{n}}.$$

By this theorem we can easily compute logarithmic capacity of union of two intervals symmetric with respect to the origin.

Example 3.23. Let $K := [-b, -a] \cup [a, b]$, where $0<a<b$. Observe that inverse image of $p = z^2$ on $[a^2, b^2]$ is exactly K. Therefore

$$Cap(K) = Cap(p^{-1}([a^2, b^2])) = (Cap([a^2, b^2]))^{\frac{1}{2}} = \frac{\sqrt{b^2 - a^2}}{2}.$$

As we see computation of exact values of logarithmic capacity is possible only for a few cases. However we can estimate logarithmic capacity from both sides for more general sets.

Theorem 3.24. *Let $K \subset \mathbb{C}$ be compact.*

1. *If $f : K \to \mathbb{C}$ is a Lipschitz α map, i.e. it satisfies $|f(z) - f(w)| \leq M|z - w|^\alpha$, where $\alpha, M>0$, then*

$$Cap(f(K)) \leq MCap^\alpha(K).$$

2. If K is connected with $diam(K) = d$, then

$$Cap(K) \geq \frac{d}{4}.$$

3. If K is a rectifiable curve with length l, then

$$Cap(K) \leq \frac{l}{4}$$

4. If $K \subset \mathbb{R}$, then

$$Cap(K) \geq \frac{\lambda_1(K)}{4},$$

where λ_1 stands for linear Lebesgue measure.

5. If $K \subset \partial\overline{\mathbb{D}}$, then

$$Cap(K) \geq \sin(\frac{a}{4}),$$

where a is the arc-length of K.

6. If $diam(K) = d$, then

$$Cap(K) \leq \frac{d}{2}.$$

7. If $K \subset \mathbb{C}$, then

$$Cap(K) \geq \sqrt{\frac{\lambda_2(K)}{\pi}},$$

where λ_2 stands for area Lebesgue measure.

As we have seen so far logarithmic capacity of a countable set is zero. On the other hand, logarithmic capacity of an interval containing set is positive. Therefore it is natural to ask whether Cantor sets are of positive logarithmic capacity. There exist both types of Cantor sets, of zero and positive logarithmic capacity. In Chapter 4 we will construct *weakly equilibrium Cantor-type sets* for which it is possible to compute logarithmic capacity, but let us finish this section by considering estimation of logarithmic capacity of *generalized Cantor sets* from both sides.

Definition 3.25. Let $s := (s_n)_{n=1}^{\infty}$ be a sequence of real numbers satisfying $0<s_n<1$ for all $n \in \mathbb{N}$. Construct K_1 by deleting the open interval of length s_1 from the center of $[0, 1]$ and continue to construct K_n by deleting open intervals of length $s_n l_{n-1}$ from center of each $(n-1)th$ level intervals of length l_{n-1}. Finally take intersetion of $K_n s$ and define *generalized Cantor set* corresponding to the sequence $s = (s_n)_{n=1}^{\infty}$ as

$$K(s) := \cap_{n=1}^{\infty} K_n$$

It is possible to estimate logaritmic capacity of this set from below and above in terms of $(s_n)_{n=1}^{\infty}$.

Theorem 3.26. *Let $K(s)$ be a generalized Cantor set corresponding to the sequence $s = (s_n)_{n=1}^{\infty}$ such that $0<s_n<1$ for all $n \in \mathbb{N}$. Then*

$$\frac{1}{2}\prod_{n=1}^{\infty}(s_n(1-s_n))^{\frac{1}{2^n}} \leq Cap(K(s)) \leq \prod_{n=1}^{\infty}(1-s_n)^{\frac{1}{2^n}}.$$

Finally let us give examples of polar and non-polar Cantor sets.

Example 3.27. Take $s_n = \frac{1}{3}$ for all $n \in \mathbb{N}$. Then $K(s)$ will be ternary Cantor set and by Theorem 3.26 non-polar.

$$\frac{1}{9} \leq Cap(K(s)) \leq \frac{1}{3}. \tag{2.2}$$

Example 3.28. Take $s_n = 1 - (\frac{1}{2})^{2^n}$ for all $n \in \mathbb{N}$. Then $K(s)$ will be polar by Theorem3.23.

$$0 \leq Cap(K(s)) \leq \prod_{n=1}^{\infty}\frac{1}{2} = 0. \tag{2.3}$$

3.3 Transfinite Diameter and Chebyshev Constant

Definition 3.29. Let $K \subset \mathbb{C}$ be compact and $n \geq 2$. The *nth diameter of* K is defined as

$$\delta_n(K) := \sup\{\prod_{j<k}|\omega_j - \omega_k|^{\frac{2}{n(n-1)}} : \omega_1, \ldots, \omega_n \in K\}. \tag{3.1}$$

An $n-$*tuple* $\omega_1, \ldots, \omega_n \in K$ for which the supremum is attained is called a *Fekete n-tuple* for K.

By compactness of K a *Fekete n-tuple* always exists, but it may not be unique. For instance, for $\overline{\mathbb{D}}$ there exist infinitely many *Fekete n-tuples* for every $n \in \mathbb{N}$. It is easy to observe $\delta_2(K) = diam(K)$. In addition $\lim_{n\to\infty} \delta_n(K)$ exists and it is called *tranfinite diameter of* K usually denoted by $\tau(K)$. Fekete and Szegö showed that transfinite diameter and capacity of a compact subset of the complex plane are equivalent.

Fekete-Szegö Theorem. *Let $K \subset \mathbb{C}$ be compact. Then the sequence $(\delta_n(K))_{n=2}^{\infty}$ is decreasing and*

$$\lim_{n\to\infty} \delta_n(K) = \tau(K) = Cap(K).$$

Definition 3.30. Let $K \subset \mathbb{C}$ be compact and $n \geq 2$. A $Fekete\ polynomial$ for K of degree n is a polynomial of the form $q(z) = \displaystyle\prod_{k=1}^{n}(z - \omega_k)$, where $\omega_1, \ldots, \omega_n \in K$ is a $Fekete\ n-tuple$ for K.

Finally we will show equivalance of the chebyshev constant and the logarithmic capacity, which was first proved by Szegö. It will come as a corollary of following two theorems, so let us consider them with proofs.

Theorem 3.31. *If q is a monic polynomial of degree $n \geq 1$, then*

$$||q||_K^{\frac{1}{n}} \geq c(K). \tag{3.2}$$

Proof. Since $K \subset q^{-1}(\overline{B_{||q||_K}(0)})$, by Theorem 3.22,

$$c(K) \leq c(q^{-1}(\overline{B_{||q||_K}(0)})) = c(\overline{B_{||q||_K}(0)})^{\frac{1}{n}} = ||q||_K^{\frac{1}{n}} \tag{3.3}$$

□

Theorem 3.32. *If q is a Fekete polynomial of degree $n \geq 2$ for K, then*

$$||q||_K^{\frac{1}{n}} \leq \delta_n(K). \tag{3.4}$$

Proof. Let $q(z) = \displaystyle\prod_{k=1}^{n}(z - \omega_k)$, where $\omega_1, \ldots, \omega_n$ is a $Fekete\ n-tuple$ for K. If $z \in K$, then $z, \omega_1, \ldots, \omega_n$ is an $(n+1)-tuple$ in K. Therefore

$$\prod_{k=1}^{n} |z-\omega_k| \prod_{j<k} |\omega_j-\omega_k| \leq \delta_{n+1}(K)^{\frac{n(n+1)}{2}},$$

and it follows

$$|q(z)| \leq \frac{\delta_{n+1}(K)^{\frac{n(n+1)}{2}}}{\delta_n(K)^{\frac{n(n-1)}{2}}} \leq \delta_n(K)^n$$

Since z is arbitrary,

$$||q||_K \leq \delta_n(K)^n.$$

□

Szegö Theorem. *Let $K \subset \mathbb{C}$ be compact. Then*

$$cheb(K) := \lim_{n\to\infty} t_n^{\frac{1}{n}}(K) = Cap(K).$$

Proof. Let $2 \leq n \in \mathbb{N}$ be arbitrary. $Cap(K) \leq t_n^{\frac{1}{n}}(K)$ by Theorem 3.31. Let q be a Fekete polynomial of degree n for K. Then $t_n^{\frac{1}{n}}(K) \leq ||q||_K^{\frac{1}{n}}$ by definition of chebyshev polynomials. We also know $||q||_K^{\frac{1}{n}} \leq \delta_n(K)$ by Theorem 3.32. Combining these results implies

$$Cap(K) \leq t_n^{\frac{1}{n}}(K) \leq ||q||_K^{\frac{1}{n}} \leq \delta_n(K).$$

We know $\lim_{n\to\infty} t_n^{\frac{1}{n}}(K) = cheb(K)$ exsits by Theorem of Fekete and $\lim_{n\to\infty} \delta_n(K) = \tau(K) = Cap(K)$ by Theorem of Fekete-Szegö. Hence by letting n tends to infinity we get the result.

$$Cap(K) = cheb(K) = \tau(K)$$

□

Chapter 4

Widom Factors

4.1 Definition and Examples

Definition 4.1. Let K be a non-polar compact subset of $\mathbb{C}$. Then *nth Widom Factor of* K is defined as

$$W_n(K) := \frac{t_n(K)}{Cap^n(K)}.$$

Exact values of *Widom Factors* are known only for a few cases. We have calculated chebyshev numbers and logarithmic capacity of $[-1,1]$ and $\overline{\mathbb{D}}$ in previous chapters.

Example 4.2.

$$W_n([-1,1]) = \frac{t_n([-1,1])}{Cap^n([-1,1])} = \frac{2^{-n+1}}{2^n} = 2$$

Example 4.3.

$$W_n(\overline{\mathbb{D}}) = \frac{t_n(\overline{\mathbb{D}})}{Cap^n(\overline{\mathbb{D}})} = 1$$

Example 4.4. Let p_n be an algebraic polynomial of degree n with leading coefficient a_n and define $A := p_n^{-1}([-1,1])$. In [19], F. Peherstorfer proved that $t_{kn}(A) = \frac{2}{(2|a_n|)^k}$ for all $k \in \mathbb{N}$. Let $0<\alpha<1$ and $K := [-1,\alpha] \cup [\alpha,1]$. Observe that K is the inverse image of $[-1,1]$ under the polynomial $\frac{2z^2-\alpha^2-1}{1-\alpha^2}$. So

$t_{2k}(K) = \frac{(1-\alpha^2)^k}{2^{2k-1}}$. We have also showed that $Cap(K) = \frac{\sqrt{1-\alpha^2}}{2}$ in Example 3.23. Therefore

$$W_n([-1,\alpha] \cup [\alpha, 1]) = \frac{(1-\alpha^2)^{\frac{n}{2}}}{2^{n-1}} (\frac{2}{\sqrt{1-\alpha^2}})^n = 2$$

for all even $n \in \mathbb{N}$.

Widom's factors are invariant under dilation and translation:

Proposition 4.5. *For any non-polar compact subset of* $\mathbb{C}$

$$W_n(\lambda K + z) = W_n(K),$$

where $\lambda > 0$, $z \in \mathbb{C}$.

Proof. $t_n(\lambda K + z) = \lambda^n t_n(K)$ by Proposition 2.25 and $Cap(\lambda K + z) = \lambda Cap(K)$ by second part of Theorem 3.14. Therefore

$$W_n(\lambda K + z) = \frac{t_n(\lambda K + z)}{Cap^n(\lambda K + z)} = \frac{\lambda^n t_n(K)}{(\lambda Cap(K))^n} = W_n(K)$$

□

For any disc or interval sequence of Widom Factors $(W_n(K))_{n=1}^{\infty}$ is a constant sequence. However, limit of the sequence of Widom Factors does not exist except for a few cases. Even in simple cases the behavior of the sequence $(W_n(K))_{n=1}^{\infty}$ is rather irregular. N.I. Achieser showed in [3] and [4] that for the union of two intervals the sequence $(W_n(K))_{n=1}^{\infty}$ may have uncountably many accumulation points. Therefore we should consider lower and upper estimates of the sequence of Widom Factors.

4.2 Lower Estimates of Widom Factors

Generally, limit of the sequence of Widom Factors does not exist. However limit inferior of this sequence is finite for any non-polar compact subset of the complex plane as a corollary of Theorem 3.31.

Theorem 4.6. *Let K be a non-polar compact subset of $\mathbb{C}$. Then*

$$W_n(K) \geq 1$$

for all $n \in \mathbb{N}$ and this inequality is sharp.

Proof. By Theorem 3.31 we know $(t_n(K))^{\frac{1}{n}} \geq Cap(K)$ for any compact $K \subset \mathbb{C}$ and for any $n \in \mathbb{N}$. Taking exponential of both sides to degree n and dividing by $Cap^n(K)$ we get the result, since $Cap(K){>}0$. Since nth Widom Factor of any closed disc equals 1 for all $n \in \mathbb{N}$, inequality is sharp. □

This theorem shows limit inferior of Widom Factors always exist and $W(K) := \liminf_{n\to\infty} W_n(K) \geq 1$. We also know for any closed disc $W(K) = 1$ and for any interval $W(K) = 2$. N.I. Achieser showed in [3] and [4] that for the union of two intervals the sequence $W_n(K)$ may be irregular, but $W(K)$ is same with the case of interval.

Theorem 4.7. *[3][4] Let K be a union of two disjoint closed intervals.*

If there exists a polynomial P_n such that $P_n^{-1}([-1,1]) = K$, then $(W_n(K))_{n=1}^{\infty}$ has a finite number of accumulation points from which the smallest is 2.

Otherwise, if there is no P_n with $P_n^{-1}([-1,1]) = K$, then the accumulation points of $(W_n(K))_{n=1}^{\infty}$ fill out an entire interval of which the left endpoint is 2.

Corollary 4.8. *Let K be a union of two disjoint closed intervals. Then*

$$W(K) = 2$$

In 2008 K. Schiefermayr generalized Theorem 4.7 to any real compact set.

Theorem 4.9. *[1, Th.2] Let $K \subset \mathbb{R}$ be a non-polar compact set. Then*

$$W_n(K) \geq 2 \tag{2.1}$$

for each $n \in \mathbb{N}$, where $W(K) = 2$ if K is a polynomial preimage of $[-1,1]$.

Remark 4.10. Inequality 2.1 is sharp, since for any interval $W_n(K) = 2$ for all $n \in \mathbb{N}$.

Recently, V. Totik showed that the interval is the only real compact set for which $W_n(K)$ converges to 2.

Theorem 4.11. *[20, Th.3] If $K \subset \mathbb{R}$ is not an interval, then there is a $c>0$ and a subsequence $\mathcal{N}$ of the natural numbers such that $W_n(K) \geq (2+c)$ for $n \in \mathcal{N}$.*

Specifically, in the case when K is a finite union of disjoint intervals, V. Totik showed $W(K) = 2$ and found the best possible rate of convergence of subsequences from $(W_n(K))_{n=1}^{\infty}$.

Theorem 4.12. *[21, T.3] Let $K \subset \mathbb{R}$ be a compact set consisting of l intervals. Then there is a constant C such that for infinitely many n*

$$W_n(K) \leq 2(1 + \frac{C}{n^{1/(l-1)}}).$$

Corollary 4.13. *Let $K \subset \mathbb{R}$ be a union of finite disjoint intervals. Then $W(K) = 2$.*

The upper bound given in Theorem 4.12 is the best possible, because of the following result.

Theorem 4.14. *[21, Th.4] For every $l > 1$ there are a set K consisting of l intervals and a constant $c > 0$ such that for all n*

$$W_n(K) > 2(1 + \frac{c}{n^{1/(l-1)}}).$$

In the case when K is a finite union of disjoint intervals, V. Totik and F. Peherstorfer gave a characterization of $W_n(K) = 2$.

Theorem 4.15. *[21, Th.1] [22, Prop.1.1]*
Let $K = \cup_{j=1}^{l}[a_j, b_j]$. For a natural number $n \geq 1$ the following are pairwise equivalent.

a $W_n(K) = 2$.

b *$T_{n,K}$ has $n+l$ extreme points on K.*

c *$K = \{z \mid T_{n;K}(z) \in [-t_n(K), t_n(K)]\}$.*

d *If μ_K denotes the equilibrium measure of K, then each $\mu_K([a_j, b_j])$, $j = 1, 2, \ldots, l$, is of the form $\frac{q_j}{n}$ with integer $q_j's$ (q_j+1 is the number of extreme points on $[a_j, b_j]$).*

e *With $\pi(x) = \prod_{j=1}^{l}(x-a_j)(x-b_j)$ the equation*

$$P_n^2(x) - \pi(x)Q_{n-l}^2(x) = const>0$$

is solvable for the polynomials P_n and Q_{n-l} of degree n and $n-l$, respectively.

Definition 4.16. [21, p.739] A set $K = \cup_{j=1}^{l}[a_j, b_j]$ is called a $T-set$ if it satisfies properties (a)-(e) for some n.

Theorem 4.17. *[22, Th.2.1] Let $K = \cup_{j=1}^{l}[a_j, b_j]$. Then for every $\epsilon>0$ there is a polynomial inverse image $E = \cup_{j=1}^{l}[a_j, c_j]$ such that $b_j \le c_j \le b_j + \epsilon$ for $j = 1, 2, \ldots, l$, i.e. $T-sets$ are dense among all sets consisting of finitely many intervals.*

In complex case we showed $\lim_{n\to\infty} W_n(K) = 1$ for any disc. G. Faber in [23], using polynomials now named after him, showed this is also valid for a single analytic curve. Recall that a Jordan curve is a homemorphic image of $\partial\mathbb{D}$ and a Jordan arc is a homeomorphic image of $[-1, 1]$.

Theorem 4.18. *[23] [20, p.3] Let $K \subset \mathbb{C}$ be a single analytic curve. Then there exist $C>0$ and $0<q<1$ such that*

$$W_n(K) \le 1 + Cq^n$$

for all $n \in \mathbb{N}$.

In cases of disc, circle or single analytic curve $\lim_{n\to\infty} W_n(K) = 1$, but V. Totik proved this is not valid if the unbounded connected component of $\overline{\mathbb{C}}\backslash K$ is not simply connected. Here $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the extended complex plane.

Theorem 4.19. *[20, Th.2] Let $K \subset \mathbb{C}$ be compact and Ω denote the unbounded connected component of $\overline{\mathbb{C}}\backslash K$. If Ω is not simply connected, then there is a $C>0$ and a subsequence $\mathcal{N}$ of the natural numbers such that*

$$W_n(K) \geq (1 + C)$$

for all $n \in \mathcal{N}$.

H.Widom showed in the case when K consists of smooth Jordan curves $W(K) = 1$ which is explicitely stated by V. Totik in [20].

Theorem 4.20. *[20, p.2] Let $K \subset \mathbb{C}$ be union of finitely many disjoint smooth Jordan curves. Then $W(K) = 1$.*

4.3 Upper Estimates of Widom Factors

Contrary to the limit inferior there exist compact subsets of the complex plane of which $\limsup_{n\to\infty} W_n(K) = +\infty$. We will give examples in next chapter after defining weakly equilibrium Cantor-type sets. So in which cases Widom Factors are uniformly bounded from above? We know that this is valid for any disc, interval and single analytic curve.

V. Totik showed in [24] existence of a uniform upper bound for $(W_n(K))_{n=1}^{\infty}$ if $K \subset \mathbb{R}$ consists of finitely many closed intervals. He also noted that this result follows from Theorem 11.5 in [6], but he proves by another approach.

Theorem 4.21. *[6, Th.11.5] [24, Th.1] Let $K \subset \mathbb{R}$ consists of finitely many disjoint closed intervals. Then there exists a constant C independent of n such that*

$$W_n(K) \leq C$$

for all $n \in \mathbb{N}$.

K. Schiefermayr proved existence of uniform upper bound in the case K is a subset of unit circle such that its projection on the real line consists of finitely many disjoint intervals.

Theorem 4.22. *[1, Th.7] Let* $K = \{z \in \mathbb{C} \mid |z| = 1,\ Re z \in E\}$*, where* E *is union of finitely many disjoint closed intervals on* $\mathbb{R}$*. Then there exists a constant* C *independent of* n *such that*

$$W_n(K) \leq C$$

for all $n \in \mathbb{N}$*.*

Chapter 5

Weakly Equilibrium Cantor-Type Sets

Weakly equilibrium Cantor-type sets, denoted by $K(\gamma)$, were recently introduced by A. Goncharov[8]. This is a Cantor-type subset of $[0,1]$ that depends on a beforehand given sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ satisfying $0<\gamma_s<\frac{1}{4}$. In terms of $(\gamma_s)_{s=1}^{\infty}$ logarithmic capacity and 2^nth degree Chebyshev polynomials of $K(\gamma)$ are known. Therefore we can compute $W_{2^n}(K(\gamma))$ for non-polar $K(\gamma)$. Before giving these values firstly let us consider construction of $K(\gamma)$.

5.1 Construction of Weakly Equilibrium Cantor-Type Sets

Given sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s < 1/4$, let $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for $s \in \mathbb{N}$. Define inductively $P_2(x) = x(x-1)$ and $P_{2^{s+1}} = P_{2^s}(P_{2^s} + r_s)$ for $s \in \mathbb{N}$. The following lemma, which was proven in [8] is crucial in the construction.

Lemma 5.1. *[8, L.1] P'_{2^s} has $2^s - 1$ simple zeros. 2^{s-1} of them are minima of P_{2^s} with equal values $P_{2^s} = \frac{-r_{s-1}^2}{4}$ and remaining $2^{s-1} - 1$ extremas are local maxima of P_{2^s} with positive values.*

Consider the set $E_s := \{x \in \mathbb{R} : P_{2^{s+1}}(x) \leq 0\}$. Then $E_0 = [0,1]$ and $E_s = \{x \in \mathbb{R} : -r_s \leq P_{2^s}(x) \leq 0\}$ for all $s \in \mathbb{N}$. The restriction $0 < \gamma_s < 1/4$ on $(\gamma_s)_{s=1}^{\infty}$ and $r_s = \gamma_s r_{s-1}^2$ imply $r_s < r_{s-1}^2/4$. By this inequality and Lemma 5.1 we can conclude that $E_s = \cup_{j=1}^{2^s} I_{j,s}$ consists of 2^s disjoint closed intervals $I_{j,s}$ with length $l_{j,s}$. The s-th level intervals $I_{j,s}$ are disjoint and following lemma verifies $\max_{1\leq j\leq 2^s} |I_{j,s}| \to 0$ as $s \to \infty$.

Lemma 5.2. *[8, L.2] For all $s \in \mathbb{N}$, $\min_{1\leq j\leq 2^s} l_{j,s} = l_{1,s}$ and $\max_{1\leq j\leq 2^s} l_{j,s} \leq (1/\sqrt{2})^{s+1}$.*

Since $E_{s+1} \subset E_s$ by definition, we have a Cantor type set $K(\gamma) := \cap_{s=0}^{\infty} E_s$. In favor of this set, in comparison to usual Cantor-type sets, $K(\gamma)$ represents an intersection of polynomial inverse images of intervals. Indeed, the set E_s can also be presented as $(\frac{2}{r_s} P_{2^s} + 1)^{-1}([-1,1])$.

Let us consider P_{2^s} as a polynomial of complex variable and define $D_s := \{z \in \mathbb{C} : |P_{2^s} + r_s/2| < r_s/2\}$ for $s \in \mathbb{N}$. By Lemma 3 in [8], $\overline{D}_{s+1} \subset \overline{D}_s$ and Theorem 1 in [8] verifies

$$\cap_{s=0}^{\infty} \overline{D}_s = K(\gamma). \tag{1.1}$$

Using this representation of $K(\gamma)$ we can find its logarithmic capacity.

Theorem 5.3. *Logarithmic capacity of the set $K(\gamma)$ is given by*

$$Cap(K(\gamma)) = \exp(\sum_{n=1}^{\infty} 2^{-n} \log \gamma_n). \tag{1.2}$$

Proof. By definition $\overline{D}_s$ is the polynomial inverse image of $P_{2^s} + r_s/2$ on $\overline{B_{r_s/2}(0)}$ and hence by Theorem 3.22

$$Cap(P_{2^s} + r_s/2^{-1}(\overline{B_{r_s/2}(0)})) = (Cap(\overline{B_{r_s/2}(0)}))^{\frac{1}{2^s}} = (\frac{r_s}{2})^{2^{-s}}$$

for all $s \in \mathbb{N}$. Observe that $r_s = \gamma_s \gamma_{s-1}^2 \ldots \gamma_1^{2^{s-1}}$. So logaritmic capacity of $\overline{D}_s$

can be formulated as

$$
\begin{aligned}
Cap(\overline{D}_s) =& (\frac{r_s}{2})^{2^{-s}} \\
=& (\frac{1}{2}\prod_{k=1}^{s}\gamma_k^{2^{s-k}})^{2^{-s}} \\
=& \exp 2^{-s}(-\log 2 + \sum_{k=1}^{s} 2^{-k}\log\gamma_k) \\
=& 2^{-2^{-s}}\exp(\sum_{k=1}^{s} 2^{-k}\log\gamma_k).
\end{aligned}
$$

We know $\cap_{s=0}^{\infty}\overline{D}_s = K(\gamma)$. Therefore by first part of Theorem 3.17 we get the result.

$$
\begin{aligned}
Cap(K(\gamma)) &= \lim_{s\to\infty} Cap(\overline{D}_s) \\
&= \lim_{s\to\infty} 2^{-2^{-s}}\exp(\sum_{k=1}^{s} 2^{-k}\log\gamma_k) \\
&= \exp(\sum_{k=1}^{\infty} 2^{-k}\log\gamma_k).
\end{aligned}
$$

□

Logarithmic capacity of $K(\gamma)$ was calculated in [8] Corollary 1 by means of Green function and Harnack Principle.

Also, $2^s th$ Chebyshev polynomials and Chebyshev numbers of $K(\gamma)$ can be given in terms of $(\gamma_s)_{s=1}^{\infty}$ as a consequence of Theorem 2.16

Theorem 5.4. *[8, Prop.1] The polynomial $P_{2^s} + \frac{r_s}{2}$ is the $2^s th$ degree Chebyshev polynomial for $K(\gamma)$ and $2^s th$ Chebyshev number of $K(\gamma)$ is given by*

$$
t_{2^s}(K(\gamma)) = r_s/2 = \frac{1}{2}\exp(2^s\sum_{n=1}^{s} 2^{-n}\log\gamma_n) \tag{1.3}
$$

for all $s \in \mathbb{N}$.

Proof. By Theorem 2.27 $P_{2^s} + \frac{r_s}{2}$ is the Chebyshev polynomial of $K(\gamma)$ if and only if for each polynomial q of degree at most $2^s - 1$,

$$\max_{x \in K_0(\gamma)} Re\{P_{2^s} + \frac{r_s}{2}(x)\overline{q(x)}\} \geq 0,$$

where $K_0(\gamma) = \{x \in K(\gamma) : |P_{2^s} + \frac{r_s}{2}(x)| = ||P_{2^s} + \frac{r_s}{2}||_{K(\gamma)}\}$. Note that $K_0(\gamma)$ consists of endpoints of level intervals $I_{j,s}$ for $1 \leq j \leq 2^s$.

Let q be an arbitrary polynomial of degree at most $2^s - 1$. Observe that $P_{2^s} + \frac{r_s}{2} \subset \mathbb{R}$ on $K(\gamma)$ since its all coefficients are real. Therefore $Re\{(P_{2^s} + \frac{r_s}{2})\overline{q}\} = (P_{2^s} + \frac{r_s}{2})Req$ is product of two real polynomial on $K(\gamma)$. Assume that $(P_{2^s} + \frac{r_s}{2})Req<0$ on $K(\gamma)$. $|P_{2^s} + \frac{r_s}{2}| = \frac{r_s}{2}$ at endpoints of level intervals $I_{j,s}$ and $P_{2^s} + \frac{r_s}{2}$ has different signs at two endpoints of any level interval. Therefore Req has at least one zero on each level interval, i.e. Req has at least 2^s zero on $K(\gamma)$ which contradicts with the fact degree of the real polynomial Req is at most $2^s - 1$. So our assumption was wrong, $(P_{2^s} + \frac{r_s}{2})Req \geq 0$ on $K_0(\gamma)$ and we get the result by Theorem 2.27. □

5.2 Widom Factors for Weakly Equilibrium Cantor-Type Sets

Theorem 5.3 gives logarithmic capacity of the set $K(\gamma)$, so the set $K(\gamma)$ is non-polar if and only if

$$\sum_{n=1}^{\infty} 2^{-n} \log \frac{1}{\gamma_n} < \infty, \tag{2.1}$$

where the last sum gives the value of the Robin constant for the set $K(\gamma)$. Theorem 5.4 also gives 2^sth Chebyshev numbers of $K(\gamma)$. Therefore we can formulate 2^sth Widom Factors of non-polar $K(\gamma)$ in terms of $(\gamma_s)_{s=1}^{\infty}$.

Proposition 5.5. *Assume $(\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s < 1/4$ satisfies (2.1). Then for $s \in \mathbb{N}$*

$$W_{2^s}(K(\gamma)) = \frac{1}{2}\exp(2^s \sum_{n=s+1}^{\infty} 2^{-n} \log \frac{1}{\gamma_n}). \tag{2.2}$$

Now we can present a compact set with unbounded sequence of Widom's factors.

Example 5.6. For a fixed $M > 4$, let $\gamma_s = M^{-s}$ for $s \in \mathbb{N}$. Then $W_{2^s}(K(\gamma)) = M^{s+2}/2$.

By Theorem 3 in [8], in the case $\inf \gamma_s > 0$, the set $K(\gamma)$ is uniformly perfect. Recall that a compact set K is uniformly perfect if it has at least two points and the moduli of annuli in the complement of K which separate K are bounded.

Example 5.7. Assume $\gamma_0 \le \gamma_s < 1/4$ for $s \in \mathbb{N}$. Then $2 < W_{2^s}(K(\gamma)) \le 1/2\gamma_0$.

It is interesting that Proposition 5.5 and Example 5.7 are also valid in the limit case, when $\gamma_s = 1/4$ for all s. Here all local maxima of P_{2^s} are equal to 0. Therefore, $E_s = [0,1]$ for each s, $K(\gamma) = [0,1]$ and $W_n(K(\gamma)) = 2$ for all n. On the other hand, $T_{2^s, K(\gamma)}(x) = P_{2^s}(x) + r_s/2 = 2^{1-2^{s+1}}\, T_{2^s}(2x-1)$ for $s \in \mathbb{N}$, where T_n stands for the classical Chebyshev polynomial, that is $T_n(t) = \cos(n \arccos t)$ for $|t| \le 1$. Thus, in the limit case, $t_{2^s}(K(\gamma)) = 2^{1-2^{s+1}}$. Since $Cap[0,1] = 1/4$, we get $W_{2^s}(K(\gamma)) = 2$, which coincides with the value of the expression on the right in (2.2).

By Theorem 4.9, $W_n(K) \ge 2$ for any compact set on the line. By Theorem 4.12 and Theorem 4.14 in case of finite union of intervals there is a restriction on rate of convergence of subsequences of $W_n(K)$ converging to 2. Let us show that, for large Cantor sets $K(\gamma)$, the value 2 can be achieved by $W_{2^s}(K(\gamma))$ as fast as we wish.

Theorem 5.8. *For each monotone null sequence $(\sigma_s)_{s=0}^{\infty}$ there is a Cantor set K such that $W_{2^s}(K) = 2(1+\sigma_s)$ for all s.*

Proof. Let us take $\gamma_n = 1/4 \cdot (1+\delta_n)^{-1}$, where $(\delta_n)_{n=1}^{\infty}$ will be defined later. Then

$$W_{2^s}(K(\gamma)) = \frac{1}{2} \exp[\, 2^s \sum_{n=s+1}^{\infty} 2^{-n}(\log 4 + \log(1+\delta_n))] = 2\, \exp[\sum_{n=s+1}^{\infty} 2^{s-n} \log(1+\delta_n)].$$

This takes the desired value, if the system of equations

$$\sum_{n=s+1}^{\infty} 2^{s-n} \log(1+\delta_n) = \log(1+\sigma_s), \;\; s \in \mathbb{Z}_+$$

with unknowns $(\delta_n)_{n=1}^{\infty}$ is solvable. Multiplying $s-$th equation by 2 and subtracting $(s+1)-$th equation yields $\delta_{s+1} = (2\,\sigma_s + \sigma_s^2 - \sigma_{s+1})(1+\sigma_{s+1})^{-1}$ for $s \in \mathbb{Z}_+$. Since these values are positive, the set $K(\gamma)$ is well-defined. $\square$

Chapter 6

Widom Factors of Fast Growth

In Section 4.3 we noted that it is not possible to find a uniform bound of $W_n(K)$ valid for any non-polar compact $K \subseteq \mathbb{R}$ and Example 5.6 verifies this fact. In general there is no uniform bound, but it is possible to control rate of increase of Widom factors. As a consequence of Szegö theorem for any $K \subseteq \mathbb{C}$, increase of $W_n(K)$ is slower than exponential growth.

Theorem 6.1. *For every non-polar, compact $K \subseteq \mathbb{C}$, $\forall\ r>1\ \exists\ C>0$ such that*

$$W_n(K) \leq Cr^n \tag{0.1}$$

for all $n \in \mathbb{N}$.

Proof. Assume this statement is not true. Then $\exists\ K \subset \mathbb{C}$, $r>1$ such that $\forall\ N \in \mathbb{N}$ $\exists\ n>N$ satisfying $W_n(K)>r^n$, i.e. there exists a subsequence $(n_k)_{k=1}^{\infty}$ satisfying $W_{n_k}(K)>r^{n_k}$. Therefore

$$\frac{\ln W_{n_k}}{n_k} = \ln t_{n_k}^{\frac{1}{n_k}} - \ln cap(K) > \ln r.$$

By taking limit of both sides as $k \to \infty$ and using Szegő's theorem, we get

$$\lim_{k\to\infty} [\ln t_{n_k}^{\frac{1}{n_k}} - \ln cap(K)] = 0 > \ln r,$$

a contradiction. $\square$

We will show this is the best possible upper bound. First let us show how to construct $K \subset \mathbb{R}$ with preassigned values of a subsequence of the Widom factors. Recall that a sequence $(M_n)_{n=1}^{\infty}$ with $M_n \geq 1$ for $n \in \mathbb{N}$ has a subexponential growth if $\lim_{n\to\infty} \log M_n/n = 0$.

Lemma 6.2. *Suppose we are given a sequence $(M_n)_{n=1}^{\infty}$ of subexponential growth with $M_n > 1$ for all $n \in \mathbb{N}$ and a strictly monotone sequence $(\log M_n/n)_{n=1}^{\infty}$. Then there exists $K(\gamma)$ such that $W_{2^s}(K(\gamma)) = 2 \cdot M_{2^s}$ for $s \in \mathbb{Z}_+$.*

Proof. Let us define $\beta_n = \log M_n/n$. Then $\beta_n \searrow 0$ and the series $\sum_{n=s+1}^{\infty}(\beta_{2^{n-1}} - \beta_{2^n})$ converges to β_{2^s}. By assumption, $M_{2^s} < M_{2^{s-1}}^2$ for all $s \in \mathbb{N}$. Let us take $\gamma_s = 4^{-1}\exp[-2^s(\beta_{2^{s-1}} - \beta_{2^s})] = 4^{-1}M_{2^s}/M_{2^{s-1}}^2$. Then $\gamma_s < 1/4$ for all $s \in \mathbb{N}$ and the set $K(\gamma)$ is well-defined and is not polar. By Proposition 5.5,

$$W_{2^s}(K(\gamma)) = \frac{1}{2}\exp[2^s \sum_{n=s+1}^{\infty} 2^{-n}(\ln 4 + 2^n(\beta_{2^{n-1}} - \beta_{2^n}))] = 2\exp(2^s\, \beta_{2^s}) = 2\, M_{2^s}.$$

□

Corollary 6.3. *For every C with $2 \leq C < \infty$ there exists $K \subset \mathbb{R}$ and a subsequence $\mathcal{N} \subset \mathbb{N}$ such that $W_n(K) = C$ for all $n \in \mathcal{N}$.*

Proof. If $C = 2$ then K can be taken as any interval. If $C > 2$ then define $M_n = C/2$ for all n and apply the theorem. □

The condition of monotonicity of the sequence $(\log M_n/n)_{n=1}^{\infty}$ can be removed if we are not intending to get the exact values of $W_{2^s}(K(\gamma))$. In the next theorem we show that for small sets $K(\gamma)$ any subexponential growth of Widom's factors can be achieved on a subsequence.

Theorem 6.4. *For every $(M_n)_{n=1}^{\infty}$ of subexponential growth with $M_n > 1$ for all $n \in \mathbb{N}$ there exists $K \subset \mathbb{R}$ and a subsequence $\mathcal{N} \subseteq \mathbb{N}$ such that $W_n(K) > M_n$ for all $n \in \mathcal{N}$.*

Proof. Define $\beta_n = \log M_n/n$ and $\widetilde{\beta}_n = 1/n + \sup_{m\geq n} \beta_m$. Then $\widetilde{\beta}_n \searrow 0$ strictly and $\widetilde{\beta}_n > \beta_n$. Let us take $\widetilde{M}_n = \exp(n\widetilde{\beta}_n)$. Then the sequence $(\widetilde{M}_n)_{n=1}^{\infty}$ satisfies the conditions of Lemma 6.2. We apply it for $(\widetilde{M}_n)_{n=1}^\infty$ and construct $K(\gamma)$ such that $W_{2^s}(K(\gamma)) = 2\widetilde{M}_{2^s} > M_{2^s}$. $\square$

It is interesting to analyze the behavior of $W_n(K(\gamma))$ for the intermediate values of parameter. We guess that, at least for regular small sequences $(\gamma_s)_{s=1}^\infty$, the values of $W_n(K(\gamma))$ will exceed $W_{2^s}(K(\gamma))$ if $2^s < n < 2^{s+1}$. If this is true then a simple modification of the construction above will give a set K with $W_n(K) > M_n$ for all $n \in \mathbb{N}$, where $(M_n)_{n=1}^\infty$ is any sequence of subexponential growth given beforehand. But in the case of an irregular sequence $(\gamma_s)_{s=1}^\infty$ we have the set $K(\gamma)$ with highly irregular behavior of Widom's factors.

Chapter 7

Irregular Case

Even on simple sets Widom factors may be irregular. In Chapter 4 we have noticed Achieser's result Theorem 4.7 that Widom Factors on union of two disjoint intervals have uncountably many accumulation points, which fill out an interval, if the set is not preimage of $[-1,1]$ for any polynomial. In the case of an irregular sequence $(\gamma_s)_{s=1}^{\infty}$ we have the set $K(\gamma)$ with highly irregular behavior of Widom's factors. One subsequence of Widom Factors goes to infinity faster than beforehand given sequence of subexponential growth. However, another subsequence of Widom Factors on the same set goes to theoretical lower bound 2 faster than beforehand given sequence converging to 2 from above. Here we combine results of Theorem 5.8 and Theorem 6.4. The following example illustrates the construction in the last theorem.

Example 7.1. Suppose we are given an increasing sequence of natural numbers $(s_j)_{j=1}^{\infty}$ and a sequence $(\varepsilon_j)_{j=1}^{\infty}$ of positive numbers with $\varepsilon_1 \le 1$ and $\varepsilon_j \searrow 0$ as $j \to \infty$. Let us take $\gamma_s = \gamma_0 < 1/4$ for $s \neq s_k$ and $\gamma_{s_j} = \gamma_0 \, \varepsilon_j$ otherwise. By (2.1) of Chapter 5, the the set $K(\gamma)$ is not polar if and only if

$$\sum_{j=1}^{\infty} 2^{-s_j} \log \frac{1}{\varepsilon_j} < \infty. \tag{0.1}$$

By Proposition 5.5,

$$W_{2^{s_j}}(K(\gamma)) = 1/2\gamma_0 \cdot \exp(2^{s_j} \sum_{k=j+1}^{\infty} 2^{-s_k} \log \frac{1}{\varepsilon_k}).$$

If we take, for a given s_j, a large enough value of s_{j+1}, then $W_{2^{s_j}}(K(\gamma))$ can be obtained as closed to $1/2\gamma_0$ as we wish.

On the other hand,

$$W_{2^{s_j-1}}(K(\gamma)) = \frac{1}{2}\exp(2^{s_j-1}\sum_{n=s_j}^{\infty} 2^{-n}\log\frac{1}{\gamma_n}).$$

Taking into account only the first term in the series, we get

$$W_{2^{s_j-1}}(K(\gamma)) > \frac{1}{2}\,\frac{1}{\sqrt{\gamma_0\,\varepsilon_j}} > \frac{1}{\sqrt{\varepsilon_j}},$$

which may be large for small ε_j satisfying (0.1).

Let us construct a set $K(\gamma)$ for which both asymptotics (as in Theorem 5.8 and Theorem 6.4) are possible.

Theorem 7.2. *For any sequences $(\sigma_j)_{j=0}^{\infty}$ with $\sigma_j \searrow 0$ and $(M_n)_{n=1}^{\infty}$ of subexponential growth with $M_n \to \infty$ there exists a sequence $(\gamma_s)_{s=1}^{\infty}$ such that for the corresponding set $K(\gamma)$ there are two sequences $(s_j)_{j=1}^{\infty}$ and $(q_j)_{j=1}^{\infty}$ with $W_{2^{s_j}}(K(\gamma)) < 2(1+\sigma_j)$ and $W_{2^{q_j}}(K(\gamma)) > M_{2^{q_j}}$ for all $j \in \mathbb{N}$.*

Proof. Without loss of generality we can assume $\sigma_1 \le 1$ and $M_n \ge 1$ for all n. For the sequences $(s_j)_{j=1}^{\infty}$, $(\varepsilon_j)_{j=1}^{\infty}$ that will be specified later, we define $\gamma_s = (4\sqrt{1+\sigma_j})^{-1}$ for $s_j < s < s_{j+1}$ and $\gamma_{s_j} = \varepsilon_j\,(4\sqrt{1+\sigma_j})^{-1}$. Also we take $q_j = s_j - 1$. Then, as above,

$$W_{2^{q_j}}(K(\gamma)) > \frac{1}{2}\,\frac{1}{\sqrt{\gamma_{s_j}}} > \frac{1}{\sqrt{\varepsilon_j}},$$

so we can take $\varepsilon_j = M^{-2}_{2^{s_j-1}}$.

On the other hand, $W_{2^{s_j}}(K(\gamma)) = \frac{1}{2}\exp[\,2^{s_j}\sum_{n=s_j+1}^{\infty} 2^{-n}\log 1/\gamma_n]$ with

$$\sum_{n=s_j+1}^{\infty} 2^{-n}\log\frac{1}{\gamma_n} = \sum_{n=s_j+1, n\neq s_k}^{\infty} 2^{-n}\log\frac{1}{\gamma_n} + \sum_{k=j+1}^{\infty} 2^{-s_k}\log(4\sqrt{1+\sigma_k}) + \sum_{k=j+1}^{\infty} 2^{-s_k}\log\frac{1}{\varepsilon_k}.$$

We combine the first two sums on the right:

$$\sum_{k=j+1}^{\infty}\sum_{n=s_k+1}^{s_{k+1}} 2^{-n}\log(4\sqrt{1+\sigma_k}) < 2^{-s_j}\log(4\sqrt{1+\sigma_j}),$$

since $(\sigma_k)_{k=0}^\infty$ decreases. From here,

$$W_{2^{s_j}}(K(\gamma)) < 2\sqrt{1+\sigma_j}\,\exp[2^{s_j}\sum_{k=j+1}^\infty 2^{-s_k}\log\frac{1}{\varepsilon_k}]$$

and we have the desired result if the expression in square brackets does not exceed $\log(\sqrt{1+\sigma_j})$ or, by definition of ε_k,

$$\sum_{k=j+1}^\infty 2^{s_j-s_k}\log M_{2^{s_k-1}} < \frac{1}{4}\log(1+\sigma_j). \tag{0.2}$$

This can be achieved if we ensure for all k

$$2^{s_{k-1}-s_k}\log M_{2^{s_k-1}} < \frac{1}{8}\log(1+\sigma_k). \tag{0.3}$$

Indeed, provided (0.3), the $k-$th term in the series above is

$$2^{s_j-s_{k-1}}2^{s_{k-1}-s_k}\log M_{2^{s_k-1}} < 2^{s_j-s_{k-1}}\frac{1}{8}\log(1+\sigma_k) < 2^{s_j-s_{k-1}}\frac{1}{8}\log(1+\sigma_j),$$

by monotonicity of $(\sigma_k)_{k=0}^\infty$. Summing these terms, we get (0.2).

Thus it remains to choose $(s_k)_{k=1}^\infty$ satisfying (0.3). This can be done recursively since $(M_n)_{n=1}^\infty$ has subexponential growth and $2^{-s_k+1}\log M_{2^{s_k-1}}$ can be taken smaller than $2^{-s_{k-1}-2}\log(1+\sigma_k)$ for large enough s_k. Clearly, (0.2) implies (0.1). Hence the set $K(\gamma)$ is well-defined and is not polar. $\square$

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