WIENER DISORDER PROBLEM WITH OBSERVATION CONTROL

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By
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Suppose that a Wiener process gains a known drift rate at some unobservable disorder time with some zero-modified exponential distribution. The process is observed only at some intervals that we control. Beginning and end points and the lengths of the observation intervals are controlled optimally. We pay cost for observing the process and for switching on the observation. We show that Bayes optimal alarm times minimizing the expected total cost of false alarms, detection delay cost and observation costs exist. Optimal alarms may occur during the observations or between the observation times when the odds-ratio process hits a set. We derive the sufficient conditions for the existence of the optimal stopping and switching rules and describe the numerical methods to calculate optimal value function.

Keywords: Wiener Disorder Problem, Optimal Stopping Problems.

ÖZET

WİENER DÜZENSİZLİK PROBLEMİ VE GÖZLEM KONTROLÜ

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Bir Wiener sürecinin bilinmeyen ve gözlenemeyen sıfır-modifiye üstel dağılımlı bir zamanda bilinen bir sapma kazandığını varsayalım. Süreci sadece kontrol ettiğimiz zamanlarda gözlemleyelim. Gözlem aralıklarının başlangıç ve bitiş noktaları optimal olmak üzere kontrol altında olsun. Gözlem süreci ve gözlemi başlatmak için ayrıca fiyat ödeyelim. Beklenen toplam yanlış alarm fiyatı, geç tespit fiyatı ve gözlem fiyatlarını minimal yapan optimal Bayes zamanlarının varolması için yeterli koşulları göstereceğiz İlgili gözlem açma-kapama ve durma zamanları ve bunlara ait optimal değer fonksiyonlarını bulmak için sayısal yöntemler önereceğiz.

Anahtar sözcükler: En iyi durdurma zamanı problemleri, Wiener disorder problemi.

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Contents

Chapter 1

Introduction and Literature Review

1.1 Brief Literature Review

In this study we revisit the Wiener disorder problem with a different approach. Let us first state the classical Wiener disorder problem to emphasize the distinction between our problem and the classical problem. Shiryaev [10] studied classical Bayesian formulation of Wiener disorder problem, in which a Wiener process gains a constant nonzero know drift rate at some unknown unobserved random time with zero-modified exponential distribution. The aim is to detect the disorder time as soon as after it occurs by means of a stopping time of the continuously monitored Wiener process. More formally, he assumed that on a probability space $(\Omega, \mathcal{F}, \mathbb{P}^\pi)$, a nonnegative random variable θ and an independent standart Wiener process $W = (W_t,\ t \geq 0)$ are given such that

$$\mathbb{P}^\pi \{\theta = 0\} = \pi$$

$$\mathbb{P}^\pi \{\theta \geq t | \theta > 0\} = e^{-\lambda t}, \qquad t \geq 0,$$

where λ is a known constant, $\lambda \in (0, \infty)$ and $\pi \in [0, 1]$. He observed the random process $X = (X_t,\ t \geq 0)$ with stochastic differential

$$\mathrm{d}X_t = r\,(t - \theta)^+\,\mathrm{d}t + \sigma \mathrm{d}W_t, \qquad t \geq 0,$$

where $r \neq 0$ and $\sigma^2 > 0$. He considered the problem of earliest detection of θ in the Bayes formulation. Here the Bayes risk function is

$$R^\pi(\tau) = \inf \left\{ \mathbb{P}^\pi \left\{ \tau < \theta \right\} + c\mathbb{E}^\pi \left[(\tau - \theta)^+ \right] \right\},$$

where inf is taken over the class of all stopping times of natural filtration $\mathcal{F}_t = \sigma\,(X_s,\ 0 \leq s \leq t),\ t \geq 0$ of X. Shiryaev considered the posterior process $\Pi_t^\pi = \mathbb{P}^\pi \left\{ \theta \leq t | \mathcal{F}_t \right\}$ and this process admits the stochastic differential equation

$$\mathrm{d}\Pi_t^\pi = \lambda\,(1 - \Pi_t^\pi)\,\mathrm{d}t + \tfrac{r}{\sigma^2}\Pi_t^\pi\,(1 - \Pi_t^\pi)\,(\mathrm{d}X_t - r\Pi_t^\pi \mathrm{d}t)\,, \quad t \geq 0$$

with $\Pi_0^\pi = \pi$. The Bayesian risk takes the form

$$R^\pi(\tau) = \mathbb{E}^\pi \left[(1 - \Pi_t^\pi) + c \int_0^\tau \Pi_s^\pi \mathrm{d}s \right].$$

Shiryaev proved that the time $\tau^* = \inf \left\{ t \geq 0 : \quad R^\pi(\pi_t) = 1 - \pi_t \right\}$ is optimal for the risk function R^π by solving the Stefan problem:

$$\begin{aligned} \mathcal{D}f(\pi) &= -c\pi, \qquad 0 \leq \pi < A, \\ f(\pi) &= 1 - \pi, \qquad A \leq \pi \leq 1, \end{aligned}$$

where $\mathcal{D} = \lambda(1 - \pi)\frac{d}{d\pi} + \frac{r^2}{2\sigma^2}\left[\pi(1 - \pi)\right]^2 \frac{\mathrm{d}^2}{d^2\pi}$ is the infinitesimal generator of X, and A is an unknown constant in $[0, 1]$, and $f(\pi)$ is the unknown function from the class of convex, twice continuously differentiable functions.

In the classical framework, we have continuous, uninterrupted and zero-cost observations of the Wiener process. We have a more realistic approach as we introduce an observation cost. The formulation of the problem also requires an observation control where we can switch on/off the observation control and we have to pay a cost to switch on the observation. The problem is to how long observations last and when it's optimal to switch on/off or stop taking observations and raise an alarm. We'll show alarm can be raised during observation times or between the observation times. We describe optimal stopping, switching and continuation regions. Under suitable conditions we also describe how to numerically calculate the value function.

The Wiener disorder problem and its variations have been studied extensively. Shiryaev (1963; 1978) gave the classical Bayesian and variational formulations of Wiener disorder problem and solved it. Wiener disorder problem with finite horizon was solved by Gapeev and Peskir (2006). Sezer (2009) solved Bayesian and variational formulations of Wiener disorder problem when the disorder is caused by one of the shocks, which arrive according to an observable Poisson process independent of the Wiener process. Wiener disorder problem with observations at fixed discrete time epochs was solved by Dayanik (2010). Quickest change detection problems were reviewed in the monographs of Basseville and Nikiforov (1993), Peskir and Shiryaev (2006), and Poor and Hadjiliadis (2009).

Dayanik [1] formulated and solved Wiener disorder problem with observation at fixed discrete time epochs. Here the process is observed only at known fixed discrete time epochs, which may not always be spaced in equal distances. The problem is to detect the disorder time as quickly as possible by an alarm which depends only on the observations of Wiener process at those discrete time epochs. Formal description of the problem is as follows: On some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, suppose that $X = \{X_t;\ t \geq 0\}$ is a Wiener process whose zero drift changes to some known constant $\mu \neq 0$ at some unknown statistically independent time θ, which has zero-modified exponential distribution $\mathbb{P}\{\theta = 0\} = p$ and $\mathbb{P}\{\theta > t\} = (1-p)\, e^{-\lambda t}$ for every $t \geq 0$ for some known constants $p \in [0, 1)$ and $\lambda > 0$.
Let $0 = t_0 < t_1 < t_2 < \ldots < t_n < \ldots$ be an infinite sequence of fixed real numbers, along which the process X may be observed as long as it is desired before an alarm τ is raised to declare that the drift of process X has changed. The filtrations

$$\mathcal{F}_0 = \{0, \emptyset\} \quad \text{and} \quad \mathcal{F}_t = \sigma\{X_{t_n};\, t_n \le t,\, n \ge 0\}$$

with $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ are defined accordingly. The problem is to calculate the minimum Bayes risk

$$\begin{aligned} R(p) &= \inf_{\tau\in\mathcal{S}} R_\tau(p) \\ &= \inf_{\tau\in\mathcal{S}} \mathbb{P}\{\tau < \theta\} + c\mathbb{E}\left[(\tau-\theta)^+\right], \quad p \in [0,1), \end{aligned}$$

where the infimum is taken over the collection $\mathcal{S}$ of all stopping times of the filtration $\mathbb{F}$, and to find a stopping time in $\mathcal{S}$ which attains the infimum, if such a stopping time exists. He defined the conditional odds-ratio process

$$\Phi_t = \frac{\Pi_t}{1-\Pi_t} = \frac{\mathbb{P}(\theta \le t|\mathcal{F}_t)}{\mathbb{P}(\theta > t|\mathcal{F}_t)}, \quad t \ge 0,$$

and calculated its dynamics. He also proved that the minimum Bayes risk equals $R(p) = 1 - p + (1-p)cV(p/1-p)$ for every $p \in [0,1)$, where $V(.)$ is the value function of the optimal stopping problem

$$V(\phi) = \inf_{\tau\in S} \mathbb{E}^{\phi}_{\infty}\left[\int_0^\tau \left(\phi_t - \frac{\lambda}{c}\right) \mathrm{d}t\right], \quad t \ge 0,$$

where $\mathbb{P}_\infty$ is defined in Dayanik [1] (page 5, 6), i.e. he reduced the original problem to an optimal stopping problem of the process Φ. In the remainder, he solved this optimal stopping problem. The solution method reduces the continuous-time optimal stopping problem to a discrete-time optimal stopping problem by means of suitable single-jump operators, which take advantage of the special structure of admissible stopping times. He separate the solution into two cases, the solution at observation times and the solution between the observation times. For the solution at observation times, he introduced the dynamic programming operator J_0 in (4.1) [1] and used this operator to define successive approximations of the cost function $V(.)$. He also showed that convergence of the successive approximations to the cost function is uniform. The solution method for the second case is the similar. Next, he defined and characterized ϵ-optimal stopping time $\sigma_\epsilon^{(m)}(t)$ ([1], (5.4)) and optimal stopping boundaries $\phi_0^{(m)}(s)$, $s \ge 0$. He showed that if an alarm has not yet been raised until time $t \ge 0$, then an optimal alarm time

$$\sigma_0(t) = \inf\left\{ s \ge t;\ \sum_{n=0}^{\infty} 1_{[t_n,t_{n+1})}(s)\phi_{t_n} \ge \phi_0(s) \right\}, \quad t \ge 0 \tag{1.1}$$

is the first time $s \geq t$, when the conditional odds-ratio Φ_{t_n} calculated at the last observation time t_n ($n \geq 0$ such that $t_n \leq s < t_{n+1}$) exceeds the optimal stopping boundary $\phi_0(s)$. For every $n \geq 0$, the optimal stopping boundary $\phi_0(s)$, $s \in [t_n, t_{n+1})$ between the nth and $(n+1)$st observation times is continuous and increases to infinity as $s \nearrow t_{n+1}$. If the boundary is not strictly increasing, then it firsly decreases and then increases. It is strictly monotone wherever it doesn't vanish. Therefore, it is never optimal to stop as the next observation time nears. If the optimal stopping boundary is strictly increasing and it is not optimal to raise alarm at the last observation, then the same remains true at least until the next observation time. Otherwise an alarm may sound at some time strictly between the last and next observations.

Dayanık's problem and ours are closely related, both problems don't assume continuous observation of the corresponding Wiener processes. Both studies reduce the original problem to an optimal stopping problems of odds-ratio process Φ(see Chapter 3 for the details). Both Dayanık and we introduced successive approximations to approximate the corresponding value functions and properties of the value functions are similar. Both $V(.)$ of [1] and our value functions (4.1) are concave, nondecreasing and bounded with the successive approximations having those properties. The major & important difference is that here the observation times are dynamically and optimally controlled whereas in Dayanık [1], they are fixed a priori. Although idea of our problem and Dayanık's problem is very similar, optimal policies differ in some ways and structure of alarm regions are also different. We have two different cost functions due to starting observation at time 0, therefore two stopping regions which are closed half-rays, these alarm regions have a simpler structure comparing to the Dayanik's problem. We found two thresholds determining the stopping regions, whereas Dayanik introduced the optimal stopping boundary $\phi_0(s)$ which has characteristics described as above. We choose to switch on/off the observation, contrary to observing at fixed discrete time epochs, hence we also have two switching regions. Above stopping policy (1.1) is different than our stopping policy which is simply raising an alarm when Φ hits the stopping region. Our optimal policy also involves switching on/off the observation and these switching on/off times are also first hitting times of the process Φ to the switching regions. We have one more region (namely switching region) besides stopping and continuation regions, which have simpler structures comparing to Dayanik's, both switching and continuation regions are finite number of disjoint intervals. Overall, we have a simpler region structure and policy.

In this thesis we provide a solution to the problem of efficiently deciding when to observe an Wiener process in order to detect a change in its probability law. Our formulation of the problem brings a realistic approach to the classical Wiener disorder problem. The solution of Wiener disorder problem with observation control may help reduce the risks and costs associated with the atmospheric science and earthquake observations where observations are often taken when the risk of natural disasters and earthquake increase and continuosly monitoring is not applicable.

1.2 Introduction

In this thesis we study Wiener disorder problem by assuming that we control when to observe the Wiener process and how long observations last. Although we don't observe the whole process, we may raise an alarm at observation times or between the observation times. Our goal is to solve the continuous-time Bayesian quickest detection problem while also controlling when and how long to take observations.

We firstly reduce the original problem to an optimal stopping for the process Φ, see Lemma 3.1 for its dynamics. The value function of the optimal stopping is expected to satisfy certain variational inequalities, but they involve a difficult second order integro-differential equation. We overcome the anticipated difficulties of solving the variational inequalities by means of successive approximations. We show the limit v_∞ of (v_n) is the optimal cost function of our optimal stopping problem under suitable conditions.

The thesis is organized as follows. We describe the problem in Chapter 2.

We define switching on and off times (O_i and C_i's) as stopping times of suitably defined filtrations. In Chapter 3, we examine the conditional odds-ratio process Φ and compute its dynamics. In Chapter 4, we firstly reduce the original problem to an optimal stopping problem for the process Φ. Then we heuristically derive the variational inequalities and the dynamic programming equation that the value function of the optimal stopping problem is expected to satisfy. Later, we prove verification theorems for this variational inequalities. In that chapter we define successive approximations of the optimal stopping problem's value function and identify its important properties. We show that successive approximations converge to the original value function under suitable conditions. Therefore, we built them into an approximation algorithm explained in Chapter 5 and illustrated on several numerical examples.

Chapter 2

Problem Description

Suppose that on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $X = \{X_t;\ t \geq 0\}$ is a Wiener process with zero drift. At some unobservable statistically independent time θ, drift changes to some known constant $\mu \neq 0$. θ has zero-modified exponential distribution $\mathbb{P}(\theta = 0) = p$ and $\mathbb{P}(\theta > t) = (1-p)e^{-\lambda t}$ for some known constants $p \in [0, 1)$ and $\lambda > 0$. We decide to take observations dynamically; i.e, we decide when to switch on/off taking observations due to observation cost. We pay c per time unit for delayed detection, we pay a per time unit during observation and we pay b for switching on the observation. We denote the optimal observation decision with $\delta = (O_1, C_1, O_2, C_2, ...)$ where O_i is a random variable representing the time of the i^{th} switching on and C_i representing the time of the i^{th} switching off. We'll assume at time 0 the observation process is on. The problem is to calculate the minimum Bayes risk

$$\begin{aligned} R(p) &= \inf_{(\tau,\delta)\in\Delta} R_{\delta,\tau}(p) \\ &= \inf_{(\tau,\delta)\in\Delta} \mathbb{E}\left[c(\tau-\theta)^+ + \mathbf{1}_{\{\tau<\theta\}} + \int_0^\tau \sum_{i=1}^{\infty} a\mathbf{1}_{\{O_i \leq s \leq C_i\}} \mathrm{d}s + \sum_{i=1}^{\infty} b\mathbf{1}_{\{O_i \leq \tau\}}\right] \end{aligned} \tag{2.1}$$

over

$$\Delta = \Big\{(\tau,\delta); \delta = (O_1, C_1, O_2, C_2, \dots), \quad O_i \in \mathbb{F}^{(\delta,2i-3)}, i \geq 2, \quad C_j \in \mathbb{F}^{(\delta,2j-2)},\ j \geq 1, \\ \tau \in \mathbb{F}^{\delta}\Big\},$$

$\mathbb{F}^{(\delta,i)} = \left\{\mathcal{F}_t^{(\delta,i)}\right\}_{t\geq 0}$, and $\mathbb{F}^{\delta} = \left\{\mathcal{F}_t^{\delta}\right\}_{t\geq 0}$. We define the filtrations $\mathbb{F}^{(\delta,i)}$ and $\mathbb{F}^{\delta}$ as below.

Let $\delta = (O_1, C_1, O_2, C_2, \dots)$ consists of stopping times $0 = O_1 < C_1 < O_2 < C_2 < \dots$. We define

$$\mathcal{F}_t^{(\delta,0)} = \{\Omega, \emptyset\}, \quad t \geq 0.$$

Let C_1 is a stopping time of $\mathcal{F}_t^{(\delta,0)}$, $t \geq 0$.Define

$$\mathcal{F}_t^{(\delta,1)} = \sigma\left(X_s;\ s \leq t \wedge C_1\right), \quad t \geq 0.$$

Let O_2 be a stopping time of $(\mathcal{F}_t^{(\delta,1)})_{t\geq 0}$.Define

$$\mathcal{F}_t^{(\delta,2)} = \left(\mathcal{F}_t^{(\delta,1)}\right) \vee \sigma\left(X_s;\ O_2 \leq s \leq t\right).$$

and let C_2 be a stopping time of $\mathcal{F}_t^{(\delta,2)}$. Let O_3 be a stopping time of $\mathcal{F}_t^{(\delta,3)}$ where

$$\mathcal{F}_t^{(\delta,3)} = \sigma\left(X_s;\ O_1 \leq s \leq C_1,\ O_2 \leq s \leq C_2 \wedge t\right).$$

Continuing this fashion, O_i is a stopping time of

$$\mathcal{F}_t^{(\delta,2i-3)} = \sigma\left(X_s;\ O_1 \leq s \leq C_1,\ O_2 \leq s \leq C_2, \dots,\ O_{i-1} \leq s \leq C_{i-1} \wedge t\right)$$

and C_i is a stopping time of

$$\mathcal{F}_t^{(\delta,2i-2)} = \left(\mathcal{F}_t^{(\delta,2i-3)}\right) \vee \sigma\left(X_s;\ O_i \leq s \leq t\right),$$

and we define $\mathcal{F}_t^{\delta}$ as

$$\mathcal{F}_t^{\delta} = \bigcup_i \mathbf{1}_{[O_i,C_i]}(t)\mathcal{F}_t^{(\delta,i)}.$$

Remark 2.1. *$\mathcal{F}_s^\delta \subseteq \mathcal{F}_t^\delta$ for $s \le t$.*

PROOF. Let $s \le t$, we show $\mathcal{F}_s^{\delta,n} \subseteq \mathcal{F}_t^{\delta,n}$ for $n \ge 1$. Since $s \wedge C_1 \le t \wedge C_1$, $\mathcal{F}_s^{\delta,1} \subseteq \mathcal{F}_t^{\delta,1}$. Previous inclusion together with $\sigma\left(X_r;\ O_2 \le r \le s\right) \subseteq \sigma\left(X_r;\ O_2 \le r \le t\right)$ implies $\mathcal{F}_s^{\delta,2} \subseteq \mathcal{F}_t^{\delta,2}$. Similarly, since $s \wedge C_{i-1} \le t \wedge C_{i-1}$, $\mathcal{F}_s^{\delta,2i-3} \subseteq \mathcal{F}_t^{\delta,2i-3}$. This inclusion together with $\sigma\left(X_r;\ O_i \le r \le s\right) \subseteq \sigma\left(X_r;\ O_i \le r \le t\right)$ implies $\mathcal{F}_s^{\delta,2i-2} \subseteq \mathcal{F}_t^{\delta,2i-2}$. Therefore $\mathcal{F}_s^{\delta,n} \subseteq \mathcal{F}_t^{\delta,n}$, $n \ge 1$, hence $\mathcal{F}_s^\delta \subseteq \mathcal{F}_t^\delta$.

Let us define a reference probability measure $\mathbb{P}_0$ on $(\Omega, \mathcal{F})$ which hosts the following two independent stochastic elements:
(1) a random variable θ with distribution $\mathbb{P}_0(\theta = 0) = p$ and $\mathbb{P}_0(\theta > t) = (1-p)e^{-\lambda t}$, $t \ge 0$
and
(2) a standart Brownian motion $X = \{X_t;\ t \ge 0\}$.
We'll enlarge our filtration generated by observed process, by including the sigma-algebra generated by θ . First, we define

$$\mathcal{H}_t^\delta = \sigma(\theta) \vee \mathcal{F}_t^\delta,\ t \ge 0.$$

We also define $\mathcal{F}$ as $\sigma\left(\cup_{t\ge 0}\mathcal{H}_t^\delta\right)$, thus we also have the information about θ . Now we'l retrieve the probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, that we started with. We define Radon-Nikodym derivative of $\mathbb{P}$ with respect to $\mathbb{P}_0$ on $\mathcal{H}_t^\delta$ as

$$Z_t^\delta = \left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_0}\right|_{\mathcal{H}_t^\delta} \triangleq \exp\left\{\int_0^t \mu \mathbf{1}_{\{s>\theta\}} \mathrm{d}X_s - \frac{1}{2}\int_0^t \mu^2 \mathbf{1}_{\{s>\theta\}} \mathrm{d}s\right\}.$$

Since $g(s) = \mu\mathbf{1}_{\{s>\theta\}}(s)$ is a deterministic function of s, the Novikov condition holds trivially therefore Z_t is a martingale and X_t is a Brownian motion with respect to $\mathbb{P}$.

Define

$$L_t = \exp\left\{\int_0^t \mu \mathrm{d}X_s - \frac{1}{2}\int_0^t \mu^2 \mathrm{d}s\right\} = e^{\mu X_t - \frac{\mu^2}{2}t}.$$

Then

$$
\begin{aligned}
Z_t &= \exp\left\{\int_{t\wedge\theta}^{t}\mu \mathrm{d}X_s - \frac{1}{2}\int_{t\wedge\theta}^{t}\mu^2\mathrm{d}s\right\} \\
&= \exp\left\{\mu\left(X_t - X_{t\wedge\theta}\right) - \frac{\mu^2}{2}\left(t - t\wedge\theta\right)\right\} \\
&= \frac{\exp\left\{\mu X_t - \frac{\mu^2}{2}t\right\}}{\exp\left\{\mu X_{t\wedge\theta} - \frac{\mu^2}{2}t\wedge\theta\right\}} \\
&= \frac{L_t}{L_{t\wedge\theta}} \\
&= \mathbf{1}_{\{t\leq\theta\}} + \mathbf{1}_{\{t>\theta\}}\frac{L_t}{L_\theta}.
\end{aligned}
$$

We immediatley see that $\mathbb{P}$ and $\mathbb{P}_0$ agree on $\mathcal{H}_0^\delta = \sigma(\theta)$, therefore θ has the same probability law under both measures. As we explained above, X_t is a Brownian motion with respect to $\mathbb{P}$. This verifies the probability laws that θ and X_t were supposed to follow under measure $\mathbb{P}$.

Chapter 3

Dynamics of the Odds-Ratio Process

We define Π^{δ} , the posterior probability process as $\Pi^{\delta}_t = \mathbb{P}(\theta \leq t|\mathcal{F}^{\delta}_t)$. Our aim is to compute the dynamics of the odds-ratio process

$$\Phi^{\delta}_t = \frac{\Pi^{\delta}_t}{1-\Pi^{\delta}_t} = \frac{\mathbb{P}(\theta \leq t|\mathcal{F}^{\delta}_t)}{\mathbb{P}(\theta > t|\mathcal{F}^{\delta}_t)}, \quad t \geq 0.$$

Here there are two cases; in the first case t is in the some observation interval. In the second case t is not in any of the observation intervals. We start with the first case:

Case 1: t is in the n^{th} observation interval, for $n \geq 0$.

We pick points t_{ij} such that

$$\begin{gathered}
0 = t_{11} \leq t_{12}........t_{1m_1} \leq A_2, \\
A_3 \leq t_{21} \leq t_{22}........t_{2m_2} \leq A_4, \\
. \\
. \\
. \\
A_{2n-1} \leq t_{n1} \leq t_{n2}.........t_{nm_n} \leq t.
\end{gathered}$$

Our aim is to compute the odds-ratio process

$$\Phi_t^\delta = \frac{\Pi_t^\delta}{1-\Pi_t^\delta} = \frac{\mathbb{P}(\theta \le t|\mathcal{F}_t^\delta)}{\mathbb{P}(\theta > t|\mathcal{F}_t^\delta)}, \quad t \ge 0.$$

Here we see the numerator is

$$\mathbb{P}(\theta \le t|\mathcal{F}_t^\delta) = \frac{\mathbb{E}_0\left[\mathbf{1}_{\{\theta\le t\}} Z_t|\mathcal{F}_t^\delta\right]}{\mathbb{E}_0\left[Z_t|\mathcal{F}_t^\delta\right]} = \frac{\mathbb{E}_0\left[\mathbf{1}_{\{\theta\le t\}}\frac{L_t}{L_\theta}\middle|\mathcal{F}_t^\delta\right]}{\mathbb{E}_0\left[Z_t|\mathcal{F}_t^\delta\right]}.$$

Similarly the denominator is

$$\mathbb{P}(\theta > t|\mathcal{F}_t^\delta) = \frac{\mathbb{E}_0\left[\mathbf{1}_{\{\theta> t\}} Z_t|\mathcal{F}_t^\delta\right]}{\mathbb{E}_0\left[Z_t|\mathcal{F}_t^\delta\right]} = \frac{\mathbb{E}_0\left[\mathbf{1}_{\{\theta> t\}}|\mathcal{F}_t^\delta\right]}{\mathbb{E}_0\left[Z_t|\mathcal{F}_t^\delta\right]}.$$

Therefore,

$$\Phi_t^\delta = \frac{\mathbb{E}_0\left[\mathbf{1}_{\{\theta\le t\}}\frac{L_t}{L_\theta}\middle|\mathcal{F}_t^\delta\right]}{\mathbb{E}_0\left[\mathbf{1}_{\{\theta> t\}}|\mathcal{F}_t^\delta\right]}.$$

Since $\mathbf{1}_{\{\theta>t\}}$ is independent of $\mathcal{F}_t^\delta$ under $\mathbb{P}_0$, the denominator is

$$\mathbb{E}_0\left[\mathbf{1}_{\{\theta> t\}}|\mathcal{F}_t^\delta\right] = (1-p)\, e^{-\lambda t}, \quad t \ge 0.$$

We compute the numerator as

$$\mathbb{E}_0\left[\mathbf{1}_{\{\theta\le t\}}\frac{L_t}{L_\theta}\middle|\mathcal{F}_t^\delta\right] = p\,\mathbb{E}_0\left[L_t|\mathcal{F}_t^\delta\right] + (1-p)\int_0^t \lambda e^{-\lambda s}\,\mathbb{E}_0\left[\frac{L_t}{L_s}\middle|\theta \in \mathrm{d}s, \mathcal{F}_t^\delta\right]\mathrm{d}s.$$

Now we need to calculate $\int_0^t \lambda e^{-\lambda s}\mathrm{d}s\,\mathbb{E}_0\left[\frac{L_t}{L_s}\middle|\theta \in \mathrm{d}s, \mathcal{F}_t^\delta\right]$.We seperate this computation into two, here s can fall into one of observation intervals or one of the intervals in which no observations are taken.

Case 1.1: s is in the j^{th} observation interval i.e $\exists j \le n,\ O_j \le s \le C_j$.
In this case

$$\mathbb{E}_0\left[\frac{L_t}{L_s}\middle|\theta\in \mathrm{d}s,\mathcal{F}_t^\delta\right]=\frac{L_t}{L_s}.$$

Case 1.2: s is not in any of the observation intervals i.e $\exists j\le n-1,\ C_j\le s\le O_{j+1}$.
In this case

$$\begin{aligned}
\mathbb{E}_0\left[\frac{L_t}{L_s}\middle|\theta\in \mathrm{d}s,\mathcal{F}_t^\delta\right] &= e^{\mu\left(X_t-X_{t_{j+1}}\right)-\frac{\mu^2}{2}(t-s)}\,\mathbb{E}_0\left[e^{\mu\left(X_{t_{j+1}}-X_s\right)}\right]\\
&= e^{\mu\left(X_t-X_{t_{j+1}}\right)-\frac{\mu^2}{2}(t-s)}\,e^{\frac{\mu^2}{2}(t_{j+1}-s)}\\
&= \frac{e^{\mu X_t-\frac{\mu^2}{2}t}}{e^{\mu X_{t_{j+1}}-\frac{\mu^2}{2}t_{j+1}}}\\
&= \frac{L_t}{L_{O_{j+1}}}\quad\text{for}\quad C_j<s<O_{j+1}.
\end{aligned}$$

Combining two cases we compute the numerator as

$$\begin{aligned}
\mathbb{E}_0\left[\mathbf{1}_{\{\theta\le t\}}\frac{L_t}{L_\theta}\middle|\mathcal{F}_t^\delta\right] &= p\,L_t+(1-p)\,L_t\int_0^t\lambda e^{-\lambda s}\mathrm{d}s\sum_{i=1}^{N}\left[\frac{1}{L_s}\mathbf{1}_{[O_i,C_i]}(s)\right]\\
&+\sum_{i=1}^{N-1}\left[\frac{1}{L_{O_{i+1}}}\mathbf{1}_{[C_i,O_{i+1}]}(s)\right].
\end{aligned}$$

Hence we compute Φ_t^δ as

$$\Phi_t^\delta=\frac{p}{1-p}e^{\lambda t}L_t+e^{\lambda t}L_t\int_0^t\lambda e^{-\lambda s}\mathrm{d}s\sum_{i=1}^{N}\left[\frac{1}{L_s}\mathbf{1}_{[O_i,C_i]}(s)\right]+\sum_{i=1}^{N-1}\left[\frac{1}{L_{O_{i+1}}}\mathbf{1}_{[C_i,O_{i+1}]}(s)\right].$$

This concludes the first case where t is in the n^{th} observation interval, we can find stochastic differential equations satisfied by Φ_t^δ and Π_t^δ. Applying Ito's formula, we get that L_t solves the stochastic differential equation

$$\mathrm{d}L_t=\mathrm{d}\left[e^{\mu X_t-\frac{\mu^2}{2}t}\right]=\mu L_t\mathrm{d}X_t,$$

where we know X_t is a Brownion motion under $\mathbb{P}_0$. Then using Itô's formula again, we get the processes Φ^δ and Π^δ solve the differential equations

$$\mathrm{d}\Phi_t^\delta = \lambda\left(1+\Phi_t^\delta\right)\mathrm{d}t + \mu\Phi_t^\delta \mathrm{d}X_t, \quad t \geq 0$$

with $\Phi_0 = \dfrac{p}{1-p}$ and

$$\mathrm{d}\Pi_t^\delta = \left(\lambda\left(1-\Pi_t^\delta\right) - \frac{\mu^2}{2}\frac{(\Pi_t^\delta)^2}{1-\Pi_t^\delta}\right)\mathrm{d}t + \mu\Pi_t^\delta\left(1-\Pi_t^\delta\right)\mathrm{d}X_t, \quad t \geq 0; \quad \Pi_0 = p.$$

Case 2: t is not in any of observation intervals, $t \in (C_n, O_{n+1})$.
We use the same framework as in Case 1, here as $t \in (C_n, O_{n+1})$, $\mathcal{F}_t^\delta = \mathcal{F}_{C_n}^\delta$.Again we'll compute the odds-ratio process

$$\Phi_t^\delta = \frac{\Pi_t^\delta}{1-\Pi_t^\delta} = \frac{\mathbb{P}(\theta \leq t|\mathcal{F}_{C_n}^\delta)}{\mathbb{P}(\theta > t|\mathcal{F}_{C_n}^\delta)}, \quad t \geq 0.$$

The numerator and denominator are the same as in Case 1

$$\mathbb{E}_0\left[\mathbf{1}_{\{\theta>t\}}|\mathcal{F}_{C_n}^\delta\right] = (1-p)\,e^{-\lambda t}.$$

$$\mathbb{E}_0\left[\mathbf{1}_{\{\theta\leq t\}}\frac{L_t}{L_\theta}\middle|\mathcal{F}_{C_n}^\delta\right] = p\,\mathbb{E}_0\left[L_t|\mathcal{F}_{C_n}^\delta\right] + (1-p)\int_0^t \lambda e^{-\lambda s}\mathrm{d}s\,\mathbb{E}_0\left[\frac{L_t}{L_s}\middle|\theta \in \mathrm{d}s, \mathcal{F}_{C_n}^\delta\right].$$

Again, as in the Case 1, we divide the computation of the term $\int_0^t \lambda e^{-\lambda s}\mathrm{d}s\,\mathbb{E}_0\left[\frac{L_t}{L_s}\middle|\theta \in \mathrm{d}s, \mathcal{F}_{C_n}^\delta\right]$ into three parts:

Case 2.1: s is in the j^{th} observation interval i.e $\exists j \leq n,\ O_j \leq s \leq C_j$.

In this case

$$
\begin{aligned}
\mathbb{E}_0\left[\frac{L_t}{L_s}\middle|\theta\in \mathrm{d}s,\mathcal{F}^{\delta}_{C_n}\right] &= \frac{1}{L_s}\,\mathbb{E}_0\left[e^{\mu X_{C_n}-\frac{\mu^2}{2}t}\,e^{\mu(X_t-X_{C_n})}\middle|\theta\in \mathrm{d}s,\mathcal{F}^{\delta}_{C_n}\right]\\
&= \frac{1}{L_s}\,e^{\mu X_{C_n}-\frac{\mu^2}{2}t}\,\mathbb{E}_0\left[e^{\mu(X_t-X_{C_n})}\middle|\theta\in \mathrm{d}s,\mathcal{F}^{\delta}_{C_n}\right]\\
&= \frac{1}{L_s}\,e^{\mu X_{C_n}-\frac{\mu^2}{2}t}\,e^{\frac{\mu^2}{2}(t-C_n)}\\
&= \frac{1}{L_s}\,e^{\mu X_{C_n}-\frac{\mu^2}{2}C_n}\\
&= \frac{L_{C_n}}{L_s}.
\end{aligned}
$$

Case 2.2: s is not in any of the observation intervals i.e $\exists j\le n-1,\ C_j\le s\le O_{j+1}$.

In this case

$$
\begin{aligned}
\mathbb{E}_0\left[\frac{L_t}{L_s}\bigg|\theta\in \mathrm{d}s,\mathcal{F}^{\delta}_{C_n}\right] &= e^{-\dfrac{\mu^2}{2}(t-s)}\ \mathbb{E}_0\left[e^{\mu(X_t-X_s)}|\theta\in \mathrm{d}s,\mathcal{F}^{\delta}_{C_n}\right] \\
&= e^{-\dfrac{\mu^2}{2}(t-s)}\ e^{-\mu\left(X_{C_n}-X_{O_{j+1}}\right)} \\
&\mathbb{E}_0\ \left[e^{\mu(X_t-X_{C_n})}\ e^{\mu(X_{O_{j+1}}-X_s)}|\theta\in \mathrm{d}s,\mathcal{F}^{\delta}_{C_n}\right] \\
&= e^{-\dfrac{\mu^2}{2}(t-s)}\ e^{-\mu\left(X_{C_n}-X_{O_{j+1}}\right)}\ \mathbb{E}_0\left[e^{\mu(X_t-X_{C_n})}\right] \\
&\mathbb{E}_0\ \left[e^{\mu(X_{O_{j+1}}-X_s)}\right] \\
&= e^{-\dfrac{\mu^2}{2}(t-s)}\ e^{-\mu\left(X_{C_n}-X_{O_{j+1}}\right)}\ e^{-\dfrac{\mu^2}{2}(t-C_n)}\ e^{-\dfrac{\mu^2}{2}(O_{j+1}-s)} \\
&= e^{-\dfrac{\mu^2}{2}(C_n-O_{j+1})+\mu(X_{C_n}-X_{O_{j+1}})} \\
&= \frac{L_{C_n}}{L_{O_{j+1}}}.
\end{aligned}
$$

Case 2.3: s is in $(C_n,t]$. In this case

$$
\begin{aligned}
\mathbb{E}_0\left[\frac{L_t}{L_s}\bigg|\theta\in \mathrm{d}s,\mathcal{F}^{\delta}_{C_n}\right] &= e^{-\dfrac{\mu^2}{2}(t-s)}\ \mathbb{E}_0\left[e^{\mu(X_t-X_s)}|\theta\in \mathrm{d}s,\mathcal{F}^{\delta}_{C_n}\right] \\
&= e^{-\dfrac{\mu^2}{2}(t-s)}\ \mathbb{E}_0\left[e^{\mu(X_t-X_s)}\right] \\
&= e^{-\dfrac{\mu^2}{2}(t-s)}\ e^{\dfrac{\mu^2}{2}(t-s)}=1.
\end{aligned}
$$

Combining three cases we compute the numerator as

$$
\begin{aligned}
\mathbb{E}_0\left[\mathbf{1}_{\{\theta\le t\}}\frac{L_t}{L_\theta}\middle|\mathcal{F}^\delta_{C_n}\right] &= p\,L_{C_n}+(1-p)\int_0^t \lambda e^{-\lambda s}\mathrm{d}s\sum_{i=1}^{N}\left[\frac{L_{C_n}}{L_s}\mathbf{1}_{[O_i,C_i]}(s)\right]\\
&+\sum_{i=1}^{N-1}\left[\frac{L_{C_n}}{L_{O_{i+1}}}\mathbf{1}_{[C_i,O_{i+1}]}(s)+\mathbf{1}_{(C_n,t]}(s)\right]\\
&= p\,L_{C_n}+(1-p)\int_0^{C_n}\lambda e^{-\lambda s}\mathrm{d}s\sum_{i=1}^{N}\left[\frac{L_{C_n}}{L_s}\mathbf{1}_{[O_i,C_i]}(s)\right]\\
&+\sum_{i=1}^{N-1}\left[\frac{L_{C_n}}{L_{O_{i+1}}}\mathbf{1}_{[C_i,O_{i+1}]}(s)\right]+(1-p)\left(e^{-\lambda C_n}-e^{-\lambda t}\right).
\end{aligned}
$$

Hence we compute Φ^δ_t as

$$
\begin{aligned}
\Phi^\delta_t &= \frac{p}{1-p}\,e^{\lambda t}\,L_{C_n}\\
&+ e^{\lambda t}\,L_{C_n}\int_0^{C_n}\lambda e^{-\lambda s}\mathrm{d}s\sum_{i=1}^{N}\frac{1}{L_s}\mathbf{1}_{[O_i,C_i]}(s)+e^{\lambda t}L_{C_n}\sum_{i=1}^{N-1}\frac{1}{L_{O_{i+1}}}\left(e^{-\lambda C_i}-e^{-\lambda O_{i+1}}\right)\\
&+ e^{\lambda t}\,L_{C_n}\left(e^{-\lambda C_n}-e^{-\lambda t}\right).
\end{aligned}
$$

Again applying Ito's formula, we get Φ^δ_t and solve the stochastic differential equations

$$\mathrm{d}\Phi^\delta_t=\lambda\left(\Phi^\delta_t+1\right)\mathrm{d}t$$

$$\mathrm{d}\Pi^\delta_t=\lambda\left(1-\Pi^\delta_t\right)\mathrm{d}t.$$

As a result, we proved the following Lemma.

Lemma 3.1. *The dynamics of odds-ratio and the posterior probability processes are*

$$
\mathrm{d}\Phi_t^\delta = \left\{ \begin{matrix} \lambda\left(1+\Phi_t^\delta\right)\mathrm{d}t + \mu\Phi_t^\delta \mathrm{d}X_t, \quad t \in [O_n, C_n) \\ \lambda\left(\Phi_t^\delta + 1\right)\mathrm{d}t, \quad t \in [C_n, O_{n+1}) \end{matrix} \right\} \quad \textit{and}
$$

$$
\mathrm{d}\Pi_t^\delta = \left\{ \begin{matrix} \left(\lambda\left(1-\Pi_t^\delta\right) - \frac{\mu^2}{2}\frac{(\Pi_t^\delta)^2}{1-\Pi_t^\delta}\right)\mathrm{d}t + \mu\Pi_t^\delta\left(1-\Pi_t^\delta\right)\mathrm{d}X_t, \quad t \in [O_n, C_n) \\ \lambda\left(1-\Pi_t^\delta\right)\mathrm{d}t, \quad t \in [C_n, O_{n+1}) \end{matrix} \right\}.
$$

Chapter 4

The HJB and Solution

4.1 Dynamic Programming Equation

We have the dynamics of the process Φ^δ and Π^δ. Now we can go back to the Bayesian risk function

$$R_{\delta,\tau}(p) = \mathbb{E}_{(\tau,\delta)\in\Delta}\left[c(\tau-\theta)^+ + \mathbf{1}_{\{\tau<\theta\}} + \int_0^\tau \sum_{i=1}^{\infty} a\mathbf{1}_{\{O_i\leq s\leq C_i\}}\mathrm{d}s + \sum_{i=1}^{\infty} b\mathbf{1}_{\{O_i\leq\tau\}}\right]$$

for an admissable decison rule $(\tau,\delta)\in\Delta$.

We can compute the terms of risk function individually:

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{\tau<\theta\}}\right] &= \mathbb{E}_0\left[Z_{\tau\wedge\theta}\mathbf{1}_{\{\tau<\theta\}}\right] = \mathbb{E}_0\left[Z_\tau\mathbf{1}_{\{\tau<\theta\}}\right] = \mathbb{E}_0\left[\mathbf{1}_{\{\tau<\theta\}}\right] \\
&= \mathbb{P}_0\left\{\theta>\tau\right\} = 1-\mathbb{P}_0\left\{\theta\le\tau\right\} \\
&= 1-\left[p+(1-p)\int_0^\tau \lambda e^{-\lambda t}\mathrm{d}t\right],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[(\tau-\theta)^+\right] &= \mathbb{E}_0\left[\int_0^\infty Z_t\mathbf{1}_{\{\theta\le t\}}\mathbf{1}_{\{t<\tau\}}\mathrm{d}t\right] \\
&= \mathbb{E}_0\left[\int_0^\infty \mathbb{E}_0\left[Z_t\mathbf{1}_{\{\theta>t\}}|\mathcal{F}_t^\delta\right]\frac{\mathbb{E}_0\left[Z_t\mathbf{1}_{\{\theta\le t\}}|\mathcal{F}_t^\delta\right]}{\mathbb{E}_0\left[Z_t\mathbf{1}_{\{\theta>t\}}|\mathcal{F}_t^\delta\right]}\mathbf{1}_{\{t<\tau\}}\mathrm{d}t\right] \\
&= \mathbb{E}_0\left[\int_0^\infty \mathbb{P}_0\left\{\theta>t\right\}\frac{\mathbb{P}_0\left[\mathbf{1}_{\{\theta\le t\}}|\mathcal{F}_t^\delta\right]}{\mathbb{P}_0\left[\mathbf{1}_{\{\theta>t\}}|\mathcal{F}_t^\delta\right]}\mathbf{1}_{\{t<\tau\}}\mathrm{d}t\right] \\
&= (1-p)\mathbb{E}_0\left[\int_0^\infty e^{-\lambda t}\Phi_t^\delta\mathrm{d}t\right],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{O_i\le\tau\}}\right] &= \mathbb{E}_0\left[Z_{O_i}\mathbf{1}_{\{O_i\le\tau\}}\right] = \mathbb{E}_0\left[\mathbf{1}_{\{O_i<\theta\}}\mathbf{1}_{\{O_i\le\tau\}}\right] + \mathbb{E}_0\left[Z_{O_i}\mathbf{1}_{\{O_i\ge\theta\}}\mathbf{1}_{\{O_i\le\tau\}}\right] \\
&= (1-p)\mathbb{E}_0\left[e^{-\lambda O_i}\mathbf{1}_{\{O_i\le\tau\}}\right] + \mathbb{E}_0\left[\Phi_{O_i}^\delta(1-p)e^{-\lambda O_i}\mathbf{1}_{\{O_i\le\tau\}}\right] \\
&= (1-p)\mathbb{E}_0\left[(1+\Phi_{O_i}^\delta)e^{-\lambda O_i}\mathbf{1}_{\{O_i\le\tau\}}\right],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_0\left[Z_s\mathbf{1}_{[O_i,C_i]}(s)\mathbf{1}_{\{s<\tau\}}\right] &= \mathbb{E}_0\left[\mathbf{1}_{\{s\le\theta\}}\mathbf{1}_{[O_i,C_i]}(s)\mathbf{1}_{\{s<\tau\}}\right] \\
&+ \mathbb{E}_0\left[\frac{\mathbb{E}_0\left[Z_s\mathbf{1}_{\{s\ge\theta\}}|\mathcal{F}_s^\delta\right]}{\mathbb{E}_0\left[Z_s\mathbf{1}_{\{s<\theta\}}|\mathcal{F}_t^\delta\right]}\mathbb{E}_0\left[Z_s\mathbf{1}_{\{s<\theta\}}|\mathcal{F}_t^\delta\right]\mathbf{1}_{[O_i,C_i]}(s)\mathbf{1}_{\{s<\tau\}}\right] \\
&= \mathbb{E}_0\left[(1-p)\,e^{-\lambda s}\,\mathbf{1}_{[O_i,C_i]}(s)\,\mathbf{1}_{\{s<\tau\}}\right] \\
&+ \mathbb{E}_0\left[(1-p)\,e^{-\lambda s}\,\Phi_s^\delta\,\mathbf{1}_{[O_i,C_i]}(s)\,\mathbf{1}_{\{s<\tau\}}\right] \\
&= (1-p)\mathbb{E}_0\left[e^{-\lambda s}\,(1+\Phi_s^\delta)\,\mathbf{1}_{[O_i,C_i]}(s)\,\mathbf{1}_{\{s<\tau\}}\right].
\end{aligned}$$

Therefore, we see

$$\begin{aligned}
\mathbb{E}_0\left[Z_s\,\mathbf{1}_{[O_i,C_i]}(s)\,\mathbf{1}_{\{s<\tau\}}\right] &= \mathbb{E}_0\left[\mathbf{1}_{\{s\le\theta\}}\,\mathbf{1}_{[O_i,C_i]}(s)\,\mathbf{1}_{\{s<\tau\}}\right] \\
&+ \mathbb{E}_0\left[\frac{\mathbb{E}_0\left[Z_s\mathbf{1}_{\{s\ge\theta\}}|\mathcal{F}_s^\delta\right]}{\mathbb{E}_0\left[Z_s\mathbf{1}_{\{s<\theta\}}|\mathcal{F}_t^\delta\right]}\,\mathbb{E}_0\left[Z_s\mathbf{1}_{\{s<\theta\}}|\mathcal{F}_t^\delta\right]\,\mathbf{1}_{[O_i,C_i]}(s)\mathbf{1}_{\{s<\tau\}}\right] \\
&= \mathbb{E}_0\left[(1-p)\,e^{-\lambda s}\,\mathbf{1}_{[O_i,C_i]}(s)\,\mathbf{1}_{\{s<\tau\}}\right] \\
&+ \mathbb{E}_0\left[(1-p)\,e^{-\lambda s}\,\Phi_s^\delta\,\mathbf{1}_{[O_i,C_i]}(s)\,\mathbf{1}_{\{s<\tau\}}\right] \\
&= (1-p)\mathbb{E}_0\left[e^{-\lambda s}\,(1+\Phi_s^\delta)\,\mathbf{1}_{[O_i,C_i]}(s)\,\mathbf{1}_{\{s<\tau\}}\right].
\end{aligned}$$

Hence the minimum Bayes risk can be written as

$$\begin{aligned}
R_{\delta,\tau}(p) &= 1-p+c(1-p)\mathbb{E}_0\left[\int_0^\tau e^{-\lambda t}\left(\Phi_t^\delta-\frac{\lambda}{c}\right)\mathrm{d}t\right. \\
&+ \left.\frac{a}{c}\int_0^\tau e^{-\lambda s}(1+\Phi_s^\delta)\sum_{i=1}^\infty \mathbf{1}_{[O_i,C_i]}(s)\mathrm{d}s+\frac{b}{c}\sum_{i=1}^\infty(1+\Phi_{O_i^-}^\delta)e^{-\lambda O_i}\mathbf{1}_{\{O_i\le\tau\}}\right]
\end{aligned}$$

in terms of the conditional odds-ratio process Φ.

We define $\delta(t)$ and $\delta_{\text{on}}(t)$ to be

$$\delta(t)=\sum_{i=1}^\infty \mathbf{1}_{[O_i,C_i]}(t)\quad\text{and}\quad \delta_{\text{on}}(t)=\sum_{i=1}^\infty \mathbf{1}_{[O_i,\infty)}(t).$$

The risk function takes the form

$$R_{\delta,\tau}(\phi) = 1 - p \quad + \quad c(1-p)\mathbb{E}_0^{\phi}\left[\int_0^{\tau} e^{-\lambda t}\left\{\Phi_t^{\delta}\left(1 + \frac{a}{c}\delta(t)\right) + \delta(t)\frac{a}{c} - \frac{\lambda}{c}\right\}\mathrm{d}t \right.$$
$$\left. + \quad \int_0^{\tau} e^{-\lambda t}\,\frac{b}{c}\,(1+\Phi_t^{\delta})\mathrm{d}\delta_{\mathrm{on}}(t)\right].$$

Lemma 4.1. *Optimal solution is found by minimizing*

$$\overline{R}_{\delta,\tau}(\phi) = \mathbb{E}_0^{\phi}\left[\int_0^{\tau} e^{-\lambda t}\left\{\Phi_t^{\delta}\left(1 + \frac{a}{c}\delta(t)\right) + \delta(t)\frac{a}{c} - \frac{\lambda}{c}\right\}\mathrm{d}t + \int_0^{\tau} e^{-\lambda t}\,\frac{b}{c}\,(1+\Phi_t^{\delta})\mathrm{d}\delta_{on}(t)\right].$$

with respect to τ and δ.

We proceed by studying the optimal cost function $V : \mathbb{R}_+ \to \mathbb{R}$ given by

$$V(\phi) \triangleq \inf_{(\delta,\tau)\in\Delta} \overline{R}_{\delta,\tau}(\phi). \tag{4.1}$$

We also define

$$\alpha = \left\{\begin{array}{ll} 1, & \text{if the observation process is on at time 0} \\ 0, & \text{if the observation process is off at time 0} \end{array}\right\} = \delta(0) = \mathbf{1}_{[0,\infty)}(0).$$

Bellman's principle of optimality states that choosing an optimal control in some infinitesmally small time interval $[0,h]$ yields an optimal control if we continue optimally at h. From Lemma 4.1, we have three choices: we may take observations in $[0,h]$ if $\alpha = 1$ or take no action if $\alpha = 0$ and continue optimally from the point h with $V(\phi_h, \alpha)$, or we can immediately switch to $V(\phi, 1-\alpha)$ by paying $(1-\alpha)\frac{b}{c}(1+\phi)$ cost, or we can stop and raise an alarm which costs 0. This tells us that V satisfies the dynamic programming equation

$$V(\phi,\alpha) \quad = \min\left\{\mathbb{E}_0\left[\int_0^h \left(e^{-\lambda t}\phi_t^{\delta}\left(1+\delta(t)\tfrac{a}{c}\right) + \tfrac{\delta a - \lambda}{c}\right)\mathrm{d}t + e^{-\lambda h}V(\Phi_h,\alpha)\right],\right.$$
$$\left. V(\phi, 1-\alpha) + (1-\alpha)\tfrac{b}{c}(1+\phi), 0\right\}$$

for sufficiently small h. When h is small, using a Taylor expansion

$$\int_0^h e^{-\lambda t}\left\{\phi_t^{\delta}\left(1+\delta(t)\frac{a}{c}\right) + \frac{\delta a - \lambda}{c}\right\}\mathrm{d}t \cong \left(\phi\left(1+\frac{a}{c}\alpha\right) + \frac{a}{c} - \frac{\lambda}{c}\right)h + o(h^2).$$

Hence

$$0 = \min\left\{\mathbb{E}_0\left[\phi\left(1+\alpha\frac{a}{c}\right)h + \frac{\alpha a-\lambda}{c}h + e^{-\lambda h}V(\Phi_h,\alpha) - V(\phi,\alpha)\right],\right.$$
$$\left.(1-\alpha)\frac{b}{c}(1+\phi) + V(\phi,1-\alpha) - V(\phi,\alpha), -V(\phi,\alpha)\right\}.$$

Applying Itô's formula, we get

$$\begin{aligned} e^{-\lambda h}V(\Phi_h^\delta,\alpha) &= V(\Phi_0^\delta,\alpha) + \int_0^h e^{-\lambda t}\left(\frac{\partial V}{\partial \phi}\lambda(\Phi_t+1) + \frac{1}{2}\frac{\partial^2 V}{\partial \phi^2}\alpha^2\mu^2\Phi_t^2 - \lambda V(\Phi_t,\alpha)\right)\mathrm{d}t \\ &+ \int_0^h e^{-\lambda t}\frac{\partial V}{\partial \phi}\alpha\mu\Phi_t \mathrm{d}W_t. \end{aligned}$$

Here we assumed V is twice-continuously differentiable, we'll relax this condition in Lemma 3.7 and show that V is smooth enough to apply a generalization of Itô's formula in Propositon 4.2. If we let h approach to 0, we get

$$\lim_{h\to 0}\mathbb{E}_0\left[\frac{e^{-\lambda h}V(\Phi_h,\alpha) - V(\phi,\alpha)}{h}\right] = \lambda(\phi+1)\frac{\partial V}{\partial \phi} + \frac{1}{2}\alpha^2\mu^2\phi^2\frac{\partial^2 V}{\partial \phi^2} - \lambda V(\phi,\alpha).$$

Let us define $L\colon \mathcal{C}^2(\mathbb{R}_+\times\{0,1\}) \to \mathcal{C}(\mathbb{R}_+\times\{0,1\})$ and $H\colon \{f \mid f:\mathbb{R}_+\times\{0,1\}\to\mathbb{R}\} \to \{f \mid f:\mathbb{R}_+\times\{0,1\}\to\mathbb{R}\}$ to be the operators

$$(LV)(\phi,\alpha) = \lambda(\phi+1)\frac{\partial V}{\partial \phi} + \frac{1}{2}\alpha^2\mu^2\phi^2\frac{\partial^2 V}{\partial \phi^2}$$
$$(HV)(\phi,\alpha) = V(\phi,1-\alpha) - V(\phi,\alpha).$$

Also let $f(\phi,\alpha)$ and $g(\phi,\alpha)$ be the functions

$$f(\phi,\alpha) = (1-\alpha)\frac{b}{c}(1+\phi)$$
$$g(\phi,\alpha) = \phi\left(1+\alpha\frac{a}{c}\right) + \frac{\alpha a-\lambda}{c}.$$

Lemma 4.2. *Rearranging the above, the value function V is expected to satisfy the HJB equation*

$$0 = \min\{LV - \lambda V + g, HV + f, -V\}. \tag{4.2}$$

We define the continuation region ($\mathcal{C}(\alpha)$), where V satisfies the inequalities

$$\begin{aligned} I. \qquad & HV + f \geq 0, \\ & LV - \lambda V + g = 0, \\ & V(\phi,\alpha) < 0. \end{aligned}$$

The switching region ($\mathcal{S}(\alpha)$) *is defined by the inequalities*

$$\begin{aligned} II. \qquad HV + f &= 0, \\ LV - \lambda V + g &\geq 0, \\ V(\phi,\alpha) &< 0. \end{aligned}$$

We define the stopping region ($\mathcal{A}(\alpha)$) *by*

$$\begin{aligned} III. \qquad HV + f &\geq 0, \\ LV - \lambda V + g &\geq 0, \\ V(\phi,\alpha) &= 0. \end{aligned}$$

Here $\mathcal{C}(\alpha)$, $\mathcal{S}(\alpha)$, $\mathcal{A}(\alpha) \subseteq \mathbb{R}_+$ *for fixed* α *and they're pairwise disjoint.*

We observe that the nature of the problem brings the operator H to the scene as it denotes the switching cost. We also observe that the variational inequalities are similar with the classical Wiener disorder problem in the sense that both problems involve continuation and stopping regions.

4.2 Verification Theorems

In the previous section we saw that finding an optimal stopping time and switching times amounts to obtaining a characterization of the regions $\mathcal{S}(\alpha)$, $\mathcal{A}(\alpha)$ and $\mathcal{C}(\alpha)$. The HJB equation (4.2) provides the main tool for finding such characterization, we'll first characterize the optimal stopping time. We'll establish optimality of stopping time $\tau_{\mathcal{D}(\alpha)} = \inf\{t \geq 0 : \phi_t \notin \mathcal{D}(\alpha)\}$ with $\mathcal{D}(\alpha) = \mathbb{R}_+ \setminus \mathcal{A}(\alpha)$ by a verification argument using Itô's rule. The main difficulty in verification is that we don't know whether the value function is a $\mathcal{C}^2(\mathbb{R}_+)$ function. The problem occurs on the boundaries of $\mathcal{S}(\alpha)$ and $\mathcal{C}(\alpha)$. In general, the value function is at best $\mathcal{C}^1$ across the boundary. For this reason, the classical Itô's rule can not be applied in verification arguments. Luckily, there are generalizations of Itô's rule that do not require functions to be $\mathcal{C}^2$ eveywhere, as long as they are sufficiently smooth. Our such generalized Itô's rule for one-dimensional diffusions is contained in the following lemma, for proof see Lemma VI.45.9 of [8]. For now, we'll assume V is smooth enough and prove it in the next section.

Lemma 4.3. *Suppose that X is a solution of*

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t), \qquad X(0) = x$$

with $x \in \mathbb{R}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be of class $\mathcal{C}^1(\mathbb{R})$. Suppose that there exists a measurable function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that: for all $l > 0$, φ is Lebesgue integrable on $[-l, l]$, and for all $y \in \mathbb{R}$,

$$f'(y) - f'(0) = \textstyle\int_0^y \varphi(z)\mathrm{d}z.$$

Then for all $t \in [0, \infty)$, almost surely

$$f(X_t) = f(x) + \int_0^t f'(X_s)\sigma(X_s)\mathrm{d}W_s + \int_0^t \left[f'(X_s)b(X_s) + (1/2)\varphi(X_s)\sigma^2(X_s)\right]\mathrm{d}s.$$

Clearly, if f is $\mathcal{C}^2$, then $\varphi = f''$. In general, the function φ is a second derivative in a weak or generalized sense. Note that φ is not necessarily continuous, but the assumptions of the lemma imply that f is "$\mathcal{C}^2$ almost everywhere". The proof of the lemma involves approximating f with $\mathcal{C}^2$ functions, applying the classical Itô's rule to the approximations, and taking limits. One key step in the proof involves showing that the diffusion process does not spend too much time at the points where φ is discontinuous.
Now we're ready to state the the verification theorem for the optimal stopping time.

Theorem 4.1. *Suppose we know the optimal switching times i.e. $\delta^* = (O_1^*, C_1^*, O_2^*, C_2^*, \ldots, O_m^*, C_m^*)$ (possibly finite or infinite) attains the infimum $\inf\limits_{(\delta,\tau)\in\Delta} \overline{R}_{\delta,\tau}(\phi) = \inf\limits_{\tau\in\mathbb{F}^{\delta^*}} \overline{R}_{\delta^*,\tau}(\phi)$, so we fix α. Under this switching times the risk function becomes*

$$\overline{R}_{\delta^*,\tau}(\phi) = \mathbb{E}_0^{\phi}\left[\int_0^{\tau} e^{-\lambda t}\left\{\Phi_t^{\delta}\left(1 + \frac{a}{c}\alpha\right) + \alpha\frac{a}{c} - \frac{\lambda}{c}\right\}\mathrm{d}t + \sum_{i=1}^{m}\frac{b}{c}\left(1 + \phi_{O_{i-}^*}\right)e^{-\lambda O_i^*}\right].$$

Suppose there is a function $U : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}$ *which satisfies:*
(a) U *is of class* $\mathcal{C}^1(\mathbb{R}_+)$ *in the first argument.*
(b) There exists a measurable function $\varphi : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}$ *such that: for all* $l > 0$, φ *is Lebesgue measurable on* $[0,l]$, *and for all* $y \in \mathbb{R}$, $U'(y,\alpha) - U'(0,\alpha) = \int_0^y \varphi(z,\alpha)\mathrm{d}z$. *for fixed* α.
(c) For all $\phi \in \mathbb{R}_+$,

$$\min\left\{\lambda(\phi+1)U' + \frac{1}{2}\alpha^2\mu^2\phi^2\varphi - \lambda U + g, HU + f, -U\right\} = 0 \qquad (4.3)$$

(From now on I'll write $LU(\phi,\alpha)$ *instead of* $\lambda(\phi+1)U'(\phi,\alpha) + \frac{1}{2}\alpha^2\mu^2\phi^2\varphi(\phi,\alpha)$ *for simplicity).*
We previously defined the operators H *and* L *as*

$$(LU)(\phi,\alpha) = \lambda(\phi+1)\frac{\partial U}{\partial \phi} + \frac{1}{2}\alpha^2\mu^2\phi^2\frac{\partial^2 U}{\partial \phi^2}$$
$$(HU)(\phi,\alpha) = U(\phi, 1-\alpha) - U(\phi,\alpha).$$

(d) U *is bounded in the first argument.*
Then for all $\phi \in \mathbb{R}_+$ *we have* $U(\phi,\alpha) \le \overline{R}_{(\delta^*,\tau)}(\phi)$ *for any* $\tau \in \mathbb{F}^{\delta^*}$ *and thus* $U(\phi,\alpha) \le V(\phi,\alpha)$. *Furthermore, let* $\mathcal{D}(\alpha) = \{\phi \in \mathbb{R}_+ : V(\phi,\alpha) < 0\}$ *and define* $\tau_{\mathcal{D}(\alpha)} = \inf\{t \ge 0 : \phi_t \notin \mathcal{D}(\alpha)\}$. *Then* $U(\phi,\alpha) = \overline{R}_{\delta^*,\tau_{\mathcal{D}(\alpha)}}(\phi)$. *Hence* $V(\phi,\alpha) = \overline{R}_{\delta^*,\tau_{\mathcal{D}(\alpha)}}(\phi)$, *that is* $\tau_{\mathcal{D}(\alpha)}$ *is an optimal stopping time.*

PROOF. Fix $\phi \in \mathbb{R}_+$. Applying Lemma 4.3 to $e^{-\lambda t}U(\phi_t,\alpha) + \sum_{i=1}^m \frac{b}{c}\left(1+\phi_{O^*_{i-}}\right)e^{-\lambda O^*_i}$ for $t \ge 0$ yields

$$\begin{aligned} e^{-\lambda t}U(\phi_t,\alpha) + \sum_{i=1}^m \frac{b}{c}\left(1+\phi_{O^*_{i-}}\right)e^{-\lambda O^*_i} &= U(\phi,\alpha) + \int_0^t \alpha\mu e^{-\lambda u}U'(\phi_u^{\delta^*},\alpha)\phi_u^{\delta^*}\mathrm{d}W_u \\ &+ \int_0^t e^{-\lambda u}\left[LU(\phi_u,\alpha) - \lambda U(\phi_u,\alpha)\right]\mathrm{d}u. \end{aligned}$$

For $r \in (0,\infty)$ define the stopping time $T_r = \inf\{t \ge 0 : |\phi_t - \phi| \ge r\}$. Note that $T_r \to \infty$ a.s. as $r \to \infty$. Let $M_t(\alpha) = \int_0^{t\wedge T_r} \mu e^{-\lambda u}U'(\phi_u^{\delta^*},\alpha)\phi_u^{\delta^*}\mathrm{d}W_u$. Since Φ is bounded on $[0,T_r]$ and integrands are continuous, the integrand in the stochastic integral bounded, thus $\{M_t(\alpha)\}_{t\ge 0}$ is a martingale and in particular

$\mathbb{E}_0\left[M_t(\alpha)\right] = 0$. Let $\tau \in \mathbb{F}^{\delta^*}$ be an arbitrary stopping time and let $N \in (0, \infty)$. Then $\tau \wedge N$ is a bounded stopping time and so by the optional sampling theorem $\mathbb{E}_0\left[M_{\tau\wedge N}(\alpha)\right] = 0$. Let $S = \tau \wedge N \wedge T_r$. Then taking expected values in the above Itô expansion we have

$$\begin{aligned}
U(\phi,\alpha) &= \mathbb{E}_0^{\phi}\left[e^{-\lambda S}U(\phi_S,\alpha)\right] - \mathbb{E}_0^{\phi}\left[\int_0^S e^{-\lambda u}\left(LU(\phi_u,\alpha) - \lambda U(\phi_u,\alpha)\right)\mathrm{d}u\right. \\
&+ \left.\sum_{i=1}^{m}\frac{b}{c}\left(1+\phi_{O_{i-}^*}\right)e^{-\lambda O_i^*}\right] \\
&\leq \mathbb{E}_0^{\phi}\left[e^{-\lambda S}U(\phi_S,\alpha)\right] + \mathbb{E}_0^{\phi}\left[\int_0^S e^{-\lambda u}g(\phi_u,\alpha)\mathrm{d}u + \sum_{i=1}^{m}\frac{b}{c}\left(1+\phi_{O_{i-}^*}\right)e^{-\lambda O_i^*}\right]
\end{aligned}$$

where the second line follows since $U(\phi,\alpha)$ satisfies the variational inequalities in (4.3). Note that as $r \to \infty$, $N \to \infty$, we have $S \to \tau$ a.s. Now g is a polynomial therefore we have

$$\mathbb{E}_0^{\phi}\left[\int_0^{\infty} e^{-\lambda u}|g(\phi_u^{\delta^*},\alpha)|\mathrm{d}u\right] < \infty.$$

Combining above and the dominated convergence theorem

$$\mathbb{E}_0^{\phi}\left[\int_0^{S} e^{-\lambda u}g(\phi_u^{\delta^*},\alpha)\mathrm{d}u\right] \to \mathbb{E}_0^{\phi}\left[\int_0^{\tau} e^{-\lambda u}g(\phi_u^{\delta^*},\alpha)\mathrm{d}u\right] \text{ as } r\to\infty,\ N\to\infty.$$

Again by variational inequalities we have $U(\phi,\alpha) \leq 0$. Then since $S \leq N < \infty$ we have

$$\mathbb{E}_0^{\phi}\left[e^{-\lambda S}U(\phi_S,\alpha)\right] \leq 0.$$

Using this bound together with the facts that Φ has continuous paths, U is continuous and bounded, and $S \to \tau_{\mathcal{D}(\alpha)}$ a.s., an application of the dominated convergence theorem yields

$$\mathbb{E}_0^{\phi}\left[e^{-\lambda S}U(\phi_S,\alpha)\right] \to \mathbb{E}_0^{\phi}\left[e^{-\lambda\tau}U(\phi_\tau,\alpha)\right] \text{ as } r\to\infty,\ N\to\infty.$$

Thus taking limits as $N \to \infty$, $r \to \infty$ yields

$$\begin{aligned}
U(\phi,\alpha) &\le \mathbb{E}_0^{\phi}\left[e^{-\lambda\tau}U(\phi_\tau,\alpha)\right] + \mathbb{E}_0^{\phi}\left[\int_0^\tau e^{-\lambda u} g(\phi_u^{\delta^*},\alpha)\mathrm{d}u + \sum_{i=1}^m \frac{b}{c}\left(1+\phi_{O_{i-}^*}\right)e^{-\lambda O_i^*}\right] \\
&\le \mathbb{E}_0^{\phi}\left[\int_0^\tau e^{-\lambda u} g(\phi_u^{\delta^*},\alpha)\mathrm{d}u + \sum_{i=1}^m \frac{b}{c}\left(1+\phi_{O_{i-}^*}\right)e^{-\lambda O_i^*}\right] \\
&= \overline{R}_{(\delta^*,\tau)}(\phi).
\end{aligned}$$

where the second line follows since $U \le 0$.Thus $U(\phi,\alpha) \le \overline{R}_{(\delta^*,\tau)}(\phi)$ for any $\tau \in \mathbb{F}^{\delta^*}$. Now consider $\tau_{\mathcal{D}(\alpha)}$ as defined in the theorem. Letting $S^* = \tau_{\mathcal{D}(\alpha)} \wedge N \wedge T_r$, by a calculation similar to that above we have

$$U(\phi,\alpha) = \mathbb{E}_0^{\phi}\left[e^{-\lambda S^*}U(\phi_{S^*},\alpha)\right] + \mathbb{E}_0^{\phi}\left[\int_0^{S^*} e^{-\lambda u} g(\phi_u^{\delta^*},\alpha)\mathrm{d}u + \sum_{i=1}^m \frac{b}{c}\left(1+\phi_{O_{i-}^*}\right)e^{-\lambda O_i^*}\right]$$

where the equality follows since $\phi_t \in \mathcal{D}(\alpha)$ for $u < S^*$ and since U solves variational inequalities. Since $S^* \to \tau_{\mathcal{D}(\alpha)}$ as $r \to \infty$, $N \to \infty$, taking limits as in previous paragraph yields

$$\begin{aligned}
U(\phi,\alpha) &= \mathbb{E}_0^{\phi}\left[e^{-\lambda\tau_{\mathcal{D}(\alpha)}}U(\phi_{\tau_{\mathcal{D}(\alpha)}},\alpha) + \int_0^{\tau_{\mathcal{D}(\alpha)}} e^{-\lambda u} g(\phi_u^{\delta^*},\alpha)\mathrm{d}u + \sum_{i=1}^m \frac{b}{c}\left(1+\phi_{O_{i_}^*}\right)e^{-\lambda O_i^*}\right] \\
&= \mathbb{E}_0^{\phi}\left[\int_0^{\tau_{\mathcal{D}(\alpha)}} e^{-\lambda u} g(\phi_u^{\delta^*},\alpha)\mathrm{d}u + \sum_{i=1}^m \frac{b}{c}\left(1+\phi_{O_{i-}^*}\right)e^{-\lambda O_i^*}\right] \\
&= \overline{R}_{\delta^*,\tau_{\mathcal{D}(\alpha)}}(\phi)
\end{aligned}$$

where the second line follows since $\phi_{\tau_{\mathcal{D}(\alpha)}} \notin \mathcal{D}(\alpha)$, so $U(\phi_{\tau_{\mathcal{D}(\alpha)}},\alpha) = 0$.

Here we assumed optimal switching times are attained by δ^*. Now we prove this assumption by adopting impulse control approach of Øksendal and Sulem [7], Chapter 6. Typically, in an impulse control problem an optimal control can be described in terms of a continuation region. The controller takes no action when the state process is within the continuation region and acts only when the state exists this region. The impulse control for our system is $\delta = (O_1, C_1, O_2, C_2, \ldots, O_j, C_j, \ldots)_{j \le M}$.; $M \le \infty$ where O_j and C_j's are stopping times defined as in Chapter 2 and $\tau_{\mathcal{D}(\alpha)}$ defined as above. Since $\mathbb{R}_+ \setminus \mathcal{D}(\alpha)$ is the optimal stopping region, we know $\mathcal{D}(\alpha) = \mathcal{C}(\alpha) \cup \mathcal{S}(\alpha)$. Here $\mathcal{C}(\alpha)$ is the continuation region and the value function is

$$V(\phi,\alpha) = \inf_{\delta} \mathbb{E}_0^{\phi}\left[\int_0^{\tau_{\mathcal{D}(\alpha)}} e^{-\lambda t}\left\{\Phi_t^{\delta}\left(1+\alpha\frac{a}{c}\right)+\alpha\frac{a}{c}-\frac{\lambda}{c}\right\}\mathrm{d}t + \sum_{O_i\le\tau_{\mathcal{D}(\alpha)}}\frac{b}{c}\left(1+\phi_{O_{i^-}}\right)e^{-\lambda O_i}\right],$$

where infimum is taken over the set of admissable switching time controls $\delta = (O_1, C_1, O_2, C_2, \ldots) = (\tau_1, \tau_2, \ldots)$ such that $\lim_{j\to\infty} \tau_j = \tau_{\mathcal{D}(\alpha)}$ (if $M > \infty$ we assume $\tau_M^{\alpha} = \tau_{\mathcal{D}(\alpha)}$ a.s.) for fixed α. Due to the variational inequalities, $\mathcal{C}(\alpha)$ is our continuation region and our candidate for an optimal impulse control $\delta^* = (O_1^*, C_1^*, O_2^*, C_2^*, \ldots) = (\tau_1^*, \tau_2^*, \ldots)$ can be described as follows. First suppose that $O_1^* > 0$ i.e $\alpha = 0$. Let $C_0^* = 0$ define δ^* for $j = 1, 2, \ldots$ as inductively by

$$\begin{aligned} O_j^* &= \inf\left\{t > C_{j-1}^* : \phi_t \in \mathcal{S}(0)\right\} \wedge \tau_{D(0)}, \\ C_j^* &= \inf\left\{t > O_j^* : \phi_t \in \mathcal{S}(1)\right\} \wedge \tau_{D(1)}. \end{aligned} \tag{4.4}$$

If instead $O_1^* = 0$ i.e $\alpha = 1$, proceed constructing δ^* as above. In our case, when a switching occurs $V(\phi,\alpha)$ moves to state $V(\phi, 1-\alpha)$. We can now state a verification theorem for the optimal switching times.

Theorem 4.2. *Suppose U is a solution of the variatonal inequalities (4.3) and δ^* is constructed as above. Suppose further U is bounded and smooth enough to apply the generalized Itô's rule. Then $U(\phi,\alpha) = V(\phi,\alpha)$ and δ^* is an optimal switching time control.*

PROOF. (a) Apply Lemma 4.3. to the function $e^{-\lambda t}U(\phi,\alpha)$ over the time interval $\left[\tau_j^*, \tau_{j+1-}^*\right]$, which yields

$$\begin{aligned} \mathbb{E}_0^{\phi}\left[e^{-\lambda\tau_j^*}U\left(\phi_{\tau_j^*},\alpha\right)\right] &- \mathbb{E}_0^{\phi}\left[e^{-\lambda\tau_{j+1}^*}U\left(\phi_{\tau_{j+1-}^*},\alpha\right)\right] \\ &= -\mathbb{E}_0^{\phi}\left[\int_{\tau_j^*}^{\tau_{j+1}^*} e^{-\lambda s}\left(LU\left(\phi_s,\alpha\right)-\lambda U\left(\phi_s,\alpha\right)\right)\mathrm{d}s\right]. \end{aligned}$$

Summing over $j = 0, 1, \ldots, m$ and then letting $m \to M$ we get

$$\begin{aligned} U(\phi,\alpha) &+ \sum_{j=1}^{M}\mathbb{E}_0^{\phi}\left[e^{-\lambda\tau_j^*}\left(U(\phi_{\tau_j^*},\alpha)-U(\phi_{\tau_{j-}^*},\alpha)\right)\right] \\ &= -\mathbb{E}_0^{\phi}\left[\int_0^{\tau_{\mathcal{D}(\alpha)}} e^{-\lambda s}\left(LU\left(\phi_s,\alpha\right)-\lambda U\left(\phi_s,\alpha\right)\right)\mathrm{d}s\right] \end{aligned}$$

and after some algebra we have

$$\begin{aligned} U(\phi,\alpha) &= \mathbb{E}_0^\phi\left[\int_0^{\tau_{\mathcal{D}(\alpha)}} e^{-\lambda s} g(\phi_s,\alpha)\mathrm{d}s\right] + \mathbb{E}_0^\phi\left[\sum_{j=1}^M e^{-\lambda O_j^*}\frac{b}{c}\left(1+\phi_{O_{j-}^*}\right)\right] \\ &- \mathbb{E}_0^\phi\left[\int_0^{\tau_{\mathcal{D}(\alpha)}} e^{-\lambda s}\left(LU(\phi_s,\alpha)-\lambda U(\phi_s,\alpha)+g(\phi_s,\alpha)\right)\mathrm{d}s\right] \\ &- \sum_{j=1}^M \mathbb{E}_0^\phi\left[e^{-\lambda\tau_j^*}\left(U\left(\phi_{\tau_j^*},1-\alpha\right)-U\left(\phi_{\tau_j^*},\alpha\right)+(1-\alpha)\frac{b}{c}\left(1+\phi_{\tau_{j-}^*}\right)\right)\right]. \end{aligned}$$

It follows from the construction of U and δ^* that $\phi_{O_{j-}^*} \notin \mathcal{C}(\alpha)$, therefore

$$U\left(\phi_{\tau_j^*},1-\alpha\right)-U\left(\phi_{\tau_j^*},\alpha\right)+(1-\alpha)\tfrac{b}{c}\left(1+\phi_{\tau_{j-}^*}\right)=0,$$

and the fourth term on the RHS is 0. It also follows from construction of U and δ^* that if $\phi_s \in \mathcal{C}(\alpha)$ then $LU(\phi_s,\alpha)-\lambda U(\phi_s,\alpha)+g(\phi_s,\alpha)=0$. Furthermore, suppose that the state process Φ is always moved instantaneously back to $\mathcal{C}(\alpha)$ whenever it exits the region $\mathcal{C}(\alpha)$. Thus Φ "spends 0 time" outside of $\mathcal{C}(\alpha)$ in the sense that $\mathbb{P}_0(\{s : \phi_s \notin \mathcal{C}(\alpha)\}) = 0$, therefore the third term on the RHS is 0. This shows that $U(\phi,\alpha) = \overline{R}_{\delta^*,\tau_{\mathcal{D}(\alpha)}}(\phi)$.

If $\delta = (O_1, C_1, O_2, C_2, \ldots)$ is an arbitrary admissable switching time control, the same calculations as above yield

$$\begin{aligned} U(\phi,\alpha) &= \mathbb{E}_0^\phi\left[\int_0^{\tau_{\mathcal{D}(\alpha)}} e^{-\lambda s} g(\phi_s,\alpha)\mathrm{d}s\right] + \mathbb{E}_0^\phi\left[\sum_{j=1}^M e^{-\lambda O_j}\frac{b}{c}\left(1+\phi_{O_{j-}}\right)\right] \\ &- \mathbb{E}_0^\phi\left[\int_0^{\tau_{\mathcal{D}(\alpha)}} e^{-\lambda s}\left(LU(\phi_s,\alpha)-\lambda U(\phi_s,\alpha)+g(\phi_s,\alpha)\right)\mathrm{d}s\right] \\ &- \sum_{j=1}^M \mathbb{E}_0^\phi\left[e^{-\lambda\tau_j}\left(U\left(\phi_{\tau_j},1-\alpha\right)-U\left(\phi_{\tau_j},\alpha\right)+(1-\alpha)\frac{b}{c}\left(1+\phi_{\tau_{j-}}\right)\right)\right]. \end{aligned}$$

Since U solves (4.3) we have $LU(\phi_s,\alpha)-\lambda U(\phi_s,\alpha)+g(\phi_s,\alpha)\geq 0$ for all $s \geq 0$. Also note that if U solves (4.3) then $U(\phi,\alpha) \leq U(\phi,1-\alpha)+(1-\alpha)\frac{b}{c}(1+\phi)$ for any ϕ. Since δ is arbitrary it follows from above that $U(\phi,\alpha) \leq V(\phi,\alpha)$. Finally, since $\overline{R}_{\delta^*,\tau_{\mathcal{D}(\alpha)}}(\phi) \geq V(\phi,\alpha)$ we have $U(\phi,\alpha) = V(\phi,\alpha) = \overline{R}_{\delta^*,\tau_{\mathcal{D}(\alpha)}}(\phi)$.

Lemma 4.4. *Suppose that $U : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}$ is a bounded and continuous function. Suppose also there are finite number of disjoint intervals*
(1) $\mathcal{C}^1(\alpha), \ldots, \mathcal{C}^n(\alpha)$, $\alpha = 0, 1$ (open)
(2) $\mathcal{S}^1(\alpha), \ldots, \mathcal{S}^m(\alpha)$, $\alpha = 0, 1$ (closed)
(3) $\mathcal{A}(0)$ and $\mathcal{A}(1)$ (closed)
such that U respectively satisfies the inequalities I, II, III on page 21 and $\mathcal{C}^2$ everywhere except possibly at the boundaries of $\mathcal{C}^{\cdot}$,$\mathcal{S}^{\cdot}$ and $\mathcal{A}^{\cdot}$, and $\mathcal{C}^1$ everywhere. Then $U = V$ on $\mathbb{R}_+$.

PROOF. Follows from Theorem 4.1 and Theorem 4.2.

4.3 Successive Approximations

Now, for $\alpha = 1$, $LV - \lambda V + g = 0$ is a nonhomogenous second order ODE with the forcing function g which is difficult to solve as boundary conditions are unknown. Instead of solving it directly, we define an operator M acting on the bounded Borel functions $w : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}$ according to

$$(M\omega)(\phi, \alpha) \triangleq \min\left\{(K\omega)(\phi, \alpha), (G\omega)(\phi, \alpha), 0\right\}, \quad \phi \geq 0$$

where K and G are defined by

$$(K\omega)(\phi, \alpha) \triangleq \inf_{\tau} \mathbb{E}_0^{\phi,\alpha}\left[\int_0^{\tau} e^{-\lambda t} g(\phi_t^{\delta}, \alpha)\mathrm{d}t + e^{-\lambda\tau}\omega(\phi_{\tau}^{\delta}, \alpha)\right], \quad \phi \geq 0$$

$$(G\omega)(\phi, \alpha) \triangleq \omega(\phi, 1-\alpha) + (1-\alpha)\frac{b}{c}(1+\phi), \quad \phi \geq 0.$$

Now, we define $v_n : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}$ successively by

$$\begin{aligned} v_0(\cdot, \alpha) = 0 \quad \text{and} \quad v_{n+1}(\cdot, \alpha) &\triangleq Mv_n(\cdot, \alpha), \quad n \geq 0 \\ v_{\infty}(\cdot, \cdot) &\triangleq \lim_{n\to\infty} v_n(\cdot, \cdot) \end{aligned} \tag{4.5}$$

we'll show then $\{v_n(\cdot,\alpha)\}$ converges for fixed α and it coincide with the value function V of the optimal stopping problem in (4.1). By using this result we'll describe a numerical algorithm in the next section.

Lemma 4.5. *For every bounded $w : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}$, the function $Mw : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}$ is bounded. If w is bounded in ϕ and $w(\cdot,\alpha) \geq -1/c$, then $0 \geq (Mw)(\cdot,\alpha) \geq -1/c$. Moreover, if $w : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}$ is concave in ϕ for both α's, then so is $Mw : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}$ concave in ϕ. The mapping $w \to Mw$ on the collection of bounded functions is monotone for $\forall\alpha \in \{0,1\}$.*

PROOF. Fix α. Suppose that w is bounded in ϕ. First we show Kw is bounded in ϕ. Since $\tau = 0$ is an $\mathbb{F}^\delta$ stopping time, we have $(Kw)(\cdot,\alpha) \leq 0$ and $g(\phi,\alpha) = \phi\left(1+\alpha\frac{a}{c}\right) + \frac{\alpha a-\lambda}{c} \geq -\lambda/c$ for every $t \geq 0$, so we have

$$\begin{aligned}
0 \geq Kw(\phi,\alpha) &\geq \inf_\tau \mathbb{E}_0^{\phi,\alpha}\left[\int_0^\tau e^{-\lambda t} g(\phi,\alpha)\mathrm{d}t + e^{-\lambda\tau}\omega(\phi,\alpha)\right] \\
&\geq \inf_\tau \mathbb{E}_0^{\phi,\alpha}\left[\int_0^\tau e^{-\lambda t}\frac{\lambda}{c}\mathrm{d}t - \frac{1}{c}e^{-\lambda\tau}\right] \\
&= \inf_\tau \mathbb{E}_0^{\phi,\alpha}\left[-\frac{1}{c}\left(1-e^{-\lambda\tau}\right) - \frac{1}{c}e^{-\lambda\tau}\right] = -\frac{1}{c}.
\end{aligned}$$

As Mw is minimum of two functions and 0, Mw is bounded above by 0. If w is bounded in ϕ, Gw is also bounded from below and Kw is bounded from below by $-1/c$ in ϕ, therefore Mw is bounded in ϕ. If $w(\cdot,\alpha) \geq -1/c$, then

$$(Gw)(\phi,\alpha) = w(\phi,\alpha) + (1-\alpha)b/c(1+\phi) \geq -1/c,$$

hence, Gw is bounded below by -1/c and $0 \geq (Mw)(\cdot,\alpha) \geq -1/c$. Suppose now $w(\cdot,\cdot)$ is concave for both $\alpha = 0$ and $\alpha = 1$. Kw is concave for fixed α, see Dayanik, Poor and Sezer[1, Proof of Remark 3.1]. Since G is affine in ϕ, Gw is also concave in ϕ. Being minimum of three concave functions, Mw is also concave in ϕ. The monotonicity of $w \to Kw$ and $w \to Gw$ are evident, so Mw is clearly monotone in ϕ.

Lemma 4.6. *The sequence $\{v_n(\cdot,\alpha)\}_{n\geq 0}$ is decreasing, and the limit $v_\infty(\phi,\alpha) \triangleq \lim_{n\to\infty} v_n(\phi,\alpha)$ exists for $\forall\alpha \in \{0,1\}$. The functions $\phi \to v_n(\phi,\alpha)$ are concave, nondecreasing and bounded between $-1/c$ and zero for $\forall\alpha \in \{0,1\}$.*

PROOF. Fix α. We see $v_1(\phi,\alpha) = (Mv_0)(\phi,\alpha) \le 0 \equiv v_0(\phi,\alpha)$. Suppose $v_n(\cdot,\alpha) \le v_{n-1}(\cdot,\alpha)$ for some $n \ge 1$. Then $v_{n+1}(\cdot,\alpha) = Mv_n(\cdot,\alpha) \le Mv_{n-1}(\cdot,\alpha) = v_n(\cdot,\alpha)$ by Lemma 4.5 and $\{v_n(\cdot,\alpha)\}_{n\ge 0}$ is a decreasing sequence by induction. Since $v_0(\cdot,1) = v_0(\cdot,0) \equiv 0$ are both concave and bounded between 0 and $-1/c$, Lemma 4.5 together with another induction imply that every $v_n(\cdot,\cdot)$ is concave and bounded between $-1/c$ and 0. Finally, we remark that every concave bounded function on $\mathbb{R}_+$ must be nondecreasing.

Corollary 4.1. *Every $v_n(\cdot,\alpha)$, $n \in \mathbb{N}$ is bounded, concave, nondecreasing for fixed α and $-1/c \le \ldots \le v_n \le v_{n-1} \le \ldots \le v_1 \le v_0 \equiv 0$. The limit*

$$v_\infty(\cdot,\alpha) \triangleq \lim_{n\to\infty} v_n(\cdot,\alpha)$$

exists, and is bounded, concave and nondecreasing. Both $v_n(\cdot,\alpha) : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}$, $n \in \mathbb{N}$ and $v_\infty(\cdot,\alpha) : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}$ are continuous in their first arguments. Their left and right derivatives with respect to the first argument are bounded on every compact subset of $\mathbb{R}_+$.

PROOF. The conclusions follow from Lemma 4.5, 4.6 and properties of concave functions (see e.g., Protter and Morrey (1991)).

Lemma 4.7. *The functions $v_\infty(\cdot,\alpha) = \lim_{n\to\infty} v_n(\cdot,\alpha)$ are bounded solutions of the equation $w(\cdot,\alpha) = (Mw)(\cdot,\alpha)$ for $\alpha = 0$ and $\alpha = 1$.*

PROOF. Fix α. By Lemma 4.6 $\{v_n(\cdot,\alpha)\}_{n\geq 0}$ is a decreasing sequence of bounded functions therefore the dominated convergence theorem implies that

$$\begin{aligned}
v_\infty(\phi,\alpha) &= \inf_{n\geq 1} v_{n+1}(\phi,\alpha) \\
&= \inf_{n\geq 1} \min\Bigg\{ \inf_\tau \mathbb{E}_0^{\phi,\alpha}\left[\int_0^\tau e^{-\lambda t} g(\phi_t^\delta,\alpha)\mathrm{d}t + e^{-\lambda\tau} v_n(\phi,\alpha)\right], \\
&\qquad v_n(\phi,1-\alpha) + (1-\alpha)\frac{b}{c}(1+\phi), 0 \Bigg\} \\
&= \min\Bigg\{ \inf_{n\geq 1}\inf_\tau \mathbb{E}_0^{\phi,\alpha}\left[\int_0^\tau e^{-\lambda t} g(\phi_t^\delta,\alpha)\mathrm{d}t + e^{-\lambda\tau} v_n(\phi,\alpha)\right], \\
&\qquad \inf_{n\geq 1} v_n(\phi,1-\alpha) + (1-\alpha)\frac{b}{c}(1+\phi), 0 \Bigg\} \\
&= \min\Bigg\{ \inf_\tau\inf_{n\geq 1} \mathbb{E}_0^{\phi,\alpha}\left[\int_0^\tau e^{-\lambda t} g(\phi_t^\delta,\alpha)\mathrm{d}t + e^{-\lambda\tau} v_n(\phi,\alpha)\right], \\
&\qquad v_\infty(\phi,1-\alpha) + (1-\alpha)\frac{b}{c}(1+\phi), 0 \Bigg\} \\
&= \min\Bigg\{ \inf_\tau \mathbb{E}_0^{\phi,\alpha}\left[\int_0^\tau e^{-\lambda t} g(\phi_t^\delta,\alpha)\mathrm{d}t + e^{-\lambda\tau} (\inf_{n\geq 1} v_n(\phi,\alpha))\right], \\
&\qquad v_\infty(\phi,1-\alpha) + (1-\alpha)\frac{b}{c}(1+\phi), 0 \Bigg\} \\
&= (Mv_\infty)(\phi,\alpha).
\end{aligned}$$

4.4 The Solution

In this section, we'll show that v_∞ and value function V of the optimal stopping problem in (4.1) coincides. We'll also discuss the structure of the optimal stopping regions for both α. In previous chapter we defined $v_n(\cdot,\alpha)$ recursively by (4.5). We took $v_0(\cdot,\alpha)\equiv 0$, hence

$$v_1(\phi,1) = \min\left\{ \inf_\tau \mathbb{E}_0^{\phi}\left[\int_0^\tau e^{-\lambda t} g(\phi_t^\delta,1)\mathrm{d}t\right], 0, 0\right\} \tag{4.6}$$

and

$$v_1(\phi,0) = \min\left\{ \inf_\tau \mathbb{E}_0^{\phi}\left[\int_0^\tau e^{-\lambda t} g(\phi_t^\delta,0)\mathrm{d}t\right], \frac{b}{c}(1+\phi), 0\right\} \tag{4.7}$$

We know the structure of these solutions of the optimal stopping problems on the RHS, which are studied extensively in [6], Chapter 10. Both $v_1(\phi,1)$ and $v_1(\phi,0)$ are 0 on their optimal stopping regions $\mathcal{A}_1(\alpha) \triangleq [\xi_1^\alpha,\infty)$ where ξ_1^α is uniquely determined by $g(\phi,\alpha)$. The functions $v_1(\phi,\alpha)$ are continuously differentiable on $[0,\infty)$ and twice continuously differentiable on $[0,\infty)\setminus\{\xi_1^\alpha\}$. Here we introduce the alarm regions

$$\begin{aligned} \mathcal{A}_{n+1}(\alpha) &\triangleq \{\phi\in\mathbb{R}_+ : v_n(\phi,\alpha)=0\}, \quad n\geq 1 \\ \mathcal{A}_\infty(\alpha) &\triangleq \{\phi\in\mathbb{R}_+ : v_\infty(\phi,\alpha)=0\} \end{aligned} \tag{4.8}$$

and switching regions

$$\begin{aligned} \mathcal{S}_{n+1}(\alpha) &\triangleq \left\{\phi\in\mathbb{R}_+ : v_{n+1}(\phi,\alpha)=v_n(\phi,1-\alpha)+(1-\alpha)\frac{b}{c}(1+\phi),\ v_n(\phi,\alpha)\leq 0\right\},\ n\geq 1 \\ \mathcal{S}_\infty(\alpha) &\triangleq \left\{\phi\in\mathbb{R}_+ : v_\infty(\phi,\alpha)=v_\infty(\phi,1-\alpha)+(1-\alpha)\frac{b}{c}(1+\phi),\ v_\infty(\phi,\alpha)\leq 0\right\} \end{aligned} \tag{4.9}$$

with

$$\begin{aligned} \mathcal{D}_n(\alpha) &= \mathbb{R}_+\setminus\mathcal{A}_n(\alpha) \\ \mathcal{D}_\infty(\alpha) &= \mathbb{R}_+\setminus\mathcal{A}_\infty(\alpha). \end{aligned}$$

Clearly, a concrete characterization of the stopping regions $\mathcal{A}_n(\alpha)$, $n\geq 0$ will help us to understand the stopping region $\mathcal{A}_\infty(\alpha)$. We know $[\lambda/c,\infty)\supseteq\mathcal{A}_1(1)$ and $[\lambda-a/c+a,\infty)\supseteq\mathcal{A}_1(0)$ by Proposition 5.1 of Dayanik [2] i.e. $[\lambda-\alpha a/c+\alpha a,\infty)\supseteq\mathcal{A}_1(\alpha)$. Since the sequence of nonpositive functions $\{v_n(\cdot,\alpha)\}_{n\geq 0}$ decreases to $v_\infty(\cdot,\alpha)$, we see

$$[\lambda-\alpha a/c+\alpha a,\infty)\supseteq\mathcal{A}_1(\alpha)\supseteq\mathcal{A}_2(\alpha)\supseteq\ldots\supseteq\mathcal{A}_{n+1}(\alpha)\supseteq\ldots\mathcal{A}_\infty(\alpha),$$

$$[0,\lambda-\alpha a/c+\alpha a)\subseteq\mathcal{D}_1(\alpha)\subseteq\mathcal{D}_2(\alpha)\subseteq\ldots\subseteq\mathcal{D}_{n+1}(\alpha)\subseteq\ldots\mathcal{D}_\infty(\alpha). \tag{4.10}$$

Let us define

$$\xi_n^\alpha\triangleq\inf\{\phi\in\mathbb{R}_+ : v_n(\phi,\alpha)=0\},\quad n\geq 2 \qquad\text{and}\qquad \xi^\alpha\triangleq\inf\{\phi\in\mathbb{R}_+ : v_\infty(\phi,\alpha)=0\}.$$

Proposition 4.1. *For fixed α, we have $\lambda-\alpha a/c+\alpha a\leq\xi_1^\alpha\leq\xi_2^\alpha\leq\ldots\xi_n^\alpha\leq\ldots\leq\xi^\alpha$ and*

$$\mathcal{A}_n(\alpha)=[\xi_n^\alpha,\infty),\quad n\geq 1 \qquad\textit{and}\qquad \mathcal{A}_\infty(\alpha)=[\xi^\alpha,\infty). \tag{4.11}$$

Moreover, $\xi_n^\alpha \nearrow \xi^\alpha$ as $n \to \infty$. The functions $v_n(\cdot,\alpha)$, $n \geq 1$ and $v_\infty(\cdot,\alpha)$ are strictly increasing on $\mathcal{D}(\alpha) = [0, \xi_n^\alpha)$, $n \geq 1$ and $\mathcal{D}_\infty(\alpha) = [0, \xi^\alpha)$, respectively.

PROOF. Fix α. By (4.9) we have $\lambda - \alpha a/c + \alpha a \leq \xi_n^\alpha \leq \xi^\alpha$ for every $n \geq 1$, and the sequence $(\xi_n^\alpha)_{n\geq 1}$ is increasing. Since the nonpositive functions $v_n(\cdot,\alpha)$, $n \geq 1$ and $v_\infty(\cdot,\alpha)$ are nondecreasing and continuous by Corollary 4.1, the identities in (4.10) follow. Because the functions are also concave, they are strictly increasing on $\mathcal{D}_n(\alpha)$, $n \geq 1$ and $\mathcal{D}_\infty(\alpha)$, respectively.
Because $(\xi_n)_{n\geq 1}(\alpha)$ is increasing, we have $\xi^\alpha \geq \xi_*^\alpha \triangleq \lim_{n\to\infty} \xi_n^\alpha \in \mathcal{A}_k(\alpha)$ and $v_k(\xi_*^\alpha, \alpha) = 0$ for every $k \geq 1$. Therefore, $v_\infty(\xi_*^\alpha, \alpha) = \lim_{k\to\infty} v_k(\xi_*^\alpha, \phi) = 0$ and $\xi_*^\alpha \in \mathcal{A}_\infty(\alpha)$, i.e., $\xi_*^\alpha \geq \xi^\alpha$. Hence $\xi^\alpha = \xi_*^\alpha \equiv \lim_{n\to\infty} \xi_n^\alpha$.

Now, we're ready show v_∞ and the value function V coincide, therefore $\mathcal{A}(\alpha)$ and $\mathcal{A}_\infty(\alpha)$ coincides.

Proposition 4.2. *The pointwise limit $v_\infty(\cdot,\alpha)$ of the sequence $\{v_n(\phi,\alpha)\}_{n\geq 0}$ in (4.5) and the value function $V(\cdot,\alpha)$ of the optimal stopping problem in (4.1) coincide. The first entrance time $\tau_{[\xi^\alpha,\infty)}$ of the process Φ into the half interval $[\xi^\alpha,\infty)$ is optimal for the Bayesian sequential change detection problem in (2.1) and $\delta^* = (O_1^*, C_1^*, \ldots)$ is the optimal switching time control where O_i^* and C_i^* defined inductively by*

$$\begin{aligned} O_j^* &= \inf\left\{t > C_{j-1}^* : \phi_t \in \mathcal{S}_\infty(0)\right\} \wedge \tau_{\mathcal{D}_\infty(0)}, \\ C_j^* &= \inf\left\{t > O_j^* : \phi_t \in \mathcal{S}_\infty(1)\right\} \wedge \tau_{\mathcal{D}_\infty(1)}. \end{aligned} \tag{4.12}$$

for $\alpha = 0$ with $C_0^* = 0$ (if instead $\alpha = 1$ i.e. $O_1^* = 0$, proceed constructing as above). Switching and continuation regions together with $\mathcal{D}_\infty(\alpha)$ are defined by

$$\begin{aligned} \mathcal{S}_\infty(\alpha) &= \left\{\phi \in \mathbb{R}_+ \colon v_\infty(\phi,\alpha) = v_\infty(\phi,\alpha) + (1-\alpha)\frac{b}{c}(1+\phi)\right\}, \\ \mathcal{C}_\infty(\alpha) &= \mathbb{R}_+ \setminus (\mathcal{S}_\infty(\alpha) \cup \mathcal{A}_\infty(\alpha)) \\ \mathcal{D}_\infty(\alpha) &= \mathbb{R}_+ \setminus \mathcal{A}_\infty(\alpha). \end{aligned}$$

where we previously defined the alarm region $\mathcal{A}_\infty(\alpha)$ as the half interval $[\xi^\alpha,\infty)$.

PROOF. We'll show that $v_\infty(\phi,\alpha)$ satisfies conditions of Lemma 4.4. Fix α. We know $v_\infty(\cdot,\alpha)$ is bounded and continuous. Now we need to show $v_\infty(\cdot,\alpha)$ is smooth enough to satisfy Lemma 4.4 and it also satisfies (4.3). By Corollary 4.1 of Sezer [9], $v_1(\cdot,\alpha)$ is continuously differentiable on $\mathbb{R}_+$ for both $\alpha = 0$ and $\alpha = 1$ so it has continuous right derivative on $\mathbb{R}_+$. $v_2(\cdot,\alpha)$ is defined as

$$v_2(\phi,\alpha) = \min\left\{\inf_{\tau} \mathbb{E}_0^{\phi,\alpha}\left[\int_0^{\tau} e^{-\lambda t} g(\phi_t^{\delta},\alpha)\mathrm{d}t + e^{-\lambda\tau} v_1(\phi_\tau,\alpha)\right],\right.$$
$$\left. v_1(\phi,1-\alpha) + (1-\alpha)\frac{b}{c}(1+\phi), 0\right\}$$

and solution of the first term on the RHS is is continuously differentiable on $\mathbb{R}_+$ again by Corollary 4.1 of Sezer [9]. All terms on the RHS has continuous right derivatives. Being minimum of three concave, nondecreasing and continuously differentiable functions, $v_2(\phi,\alpha)$ has continuous right derivative, together with concaveness this implies the differentiablity of $v_2(\phi,\alpha)$ on $\mathbb{R}_+$. By an induction argument on n, v_∞ is continuously differentiable on $\mathbb{R}_+$ i.e. $v_\infty(\phi,\alpha)$ is $\mathcal{C}^1$ everywhere.

Now we examine structures of the regions $\mathcal{S}_\infty(\alpha)$, $\mathcal{C}_\infty(\alpha)$ and $\mathcal{A}_\infty(\alpha)$. The alarm region $\mathcal{A}_\infty(\alpha) = [\xi^\alpha,\infty)$ is closed in $\mathbb{R}_+$. $\mathcal{S}_\infty(\alpha)$ is also closed (4.9), hence $\mathcal{C}_\infty(\alpha)$ and $\mathcal{D}_\infty(\alpha)$ are open. Next, we have a closer look at boundaries of $\mathcal{S}_\infty(\alpha)$, $\mathcal{C}_\infty(\alpha)$ and $\mathcal{A}_\infty(\alpha)$. For $\mathcal{A}_\infty(\alpha)$, it's obvious: only boundary point is ξ^α. By Lemma 4.7, $v_\infty(\phi,\alpha) = Mv_\infty(\phi,\alpha)$ i.e.,

$$v_\infty(\phi,\alpha) = \min\left\{(Kv_\infty)(\phi,\alpha), (Gv_\infty)(\phi,\alpha), 0\right\},$$

with K and G defined as in (4.5). Here we compare $(Kv_\infty)(\phi,\alpha)$ with $(Gv_\infty)(\phi,\alpha)$ on $[0,\xi^\alpha)$ and decide in which regions it's optimal to continue or switch. The points where $v_\infty(\phi,\alpha) = (Gv_\infty)(\phi,\alpha)$ belong to the switching region and $v_\infty(\phi,\alpha) = (Kv_\infty)(\phi,\alpha)$ belong to the continuation region. Therefore boundaries of $\mathcal{S}_\infty(\alpha)$ and $\mathcal{C}_\infty(\alpha)$ are the points for which $(Kv_\infty)(\phi,\alpha) = (Gv_\infty)(\phi,\alpha)$ hold in $[0,\xi^\alpha)$. Both (Kv_∞), (Gv_∞) are nondecreasing, bounded and concave with bounded derivatives on the compact interval $[0,\xi^\alpha]$ by Corollary 4.1 of Sezer [9]. Hence they can at most intersect at finitely many points on $[0,\xi^\alpha]$, say $\rho^1,\rho^2,\ldots,\rho^n$ with $n<\infty$. We include these points in $\mathcal{S}_\infty$ by (4.9). Therefore $\mathcal{S}_\infty(\alpha)$ and $\mathcal{C}_\infty(\alpha)$ consist of closed and open intervals with endpoints ρ^i i.e., $\mathcal{S}_\infty(\alpha) = \bigcup_{i=1}^{s}\mathcal{S}^i(\alpha)$ and $\mathcal{C}_\infty(\alpha) = \bigcup_{i=1}^{r}\mathcal{C}^i(\alpha)$ for $s,r\le n$, where $\mathcal{S}^i(\alpha)$'s are closed and $\mathcal{C}^i(\alpha)$'s are open intervals. Obviously $\mathcal{S}^i(\alpha) \cap \mathcal{C}^j(\alpha) = \emptyset$ for every $i \leq s$, $j \leq r$. Moreover, $\mathcal{S}^i(0)$ can't overlap with $\mathcal{S}^j(1)$ for any $i, j \leq s$. If two switching regions overlap, the process Φ gets trapped between these two switching regions forever which means infinitely many switch on and off's hence, infinite cost.

Next, we show that $v_\infty(\phi, \alpha)$ is $\mathcal{C}^2$ everywhere except possibly at the boundary points $\xi^\alpha, \rho^1, \rho^2, \ldots, \rho^n$. We'll use an induction argument:
(i) $v_1(\phi, 1)$ (4.6) and $v_1(\phi, 0)$ (4.7) are twice continuously differentiable on $[0, \infty)$ except two boundary points ξ_1^α, as we showed in page 32 and 33. They're 0 on their optimal stopping regions $[\xi_1^\alpha, \infty)$ and they're $\mathcal{C}^2$ in $[0, \xi_1^\alpha)$.
(ii) $v_2(\phi, \alpha)$ are 0 on $[\xi_2^\alpha, \infty)$. On $\mathcal{S}_2(\alpha)$, $v_2(\phi, \alpha) = v_1(\phi, \alpha) + (1-\alpha)\frac{b}{c}(1+\phi)$, therefore twice continuously differentiable on $\mathcal{S}_2(\alpha) \setminus \partial\mathcal{S}_2(\alpha)$ by (i). On $\mathcal{C}_2(\alpha)$, $v_2(\phi, \alpha) = (Kv_1)(\phi, \alpha)$, therefore twice continuously differentiable on $\mathcal{C}_2(\alpha) \setminus \partial\mathcal{C}_2(\alpha)$ by Corollary 4.1 of Sezer[9]. Here we excluded finitely many boundary points of $\mathcal{C}_2(\alpha)$, $\mathcal{S}_2(\alpha)$ and $\mathcal{A}_2(\alpha)$.
$\vdots$
$v_\infty(\phi, \alpha) \triangleq 0$ on $[\xi^\alpha, \infty)$. On $\mathcal{S}_\infty(\alpha) \setminus \partial\mathcal{S}_\infty(\alpha)$ and $\mathcal{C}_\infty(\alpha) \setminus \partial\mathcal{S}_\infty(\alpha)$, it's twice continuously differentiable by induction. Here we excluded finitely many boundary points (we also exclude the point ξ^α), therefore $v_\infty(\phi, \alpha)$ is C^2 everywhere except possibly at finitely many points.
Now we show $v_\infty(\cdot, \alpha)$ satisfies the variational inequalities in (4.2) everywhere except possibly at finitely many points (precisely boundary points of $\mathcal{A}_\infty(\alpha)$, $\mathcal{C}_\infty(\alpha)$ and $\mathcal{S}_\infty(\alpha)$). By Lemma 4.7, $v_\infty(\phi, \alpha) = Mv_\infty(\phi, \alpha)$ i.e.,

$$v_\infty(\phi, \alpha) = \min\left\{ \inf_\tau \mathbb{E}_\alpha^{\phi,\alpha}\left[\int_0^\tau e^{-\lambda t} g(\phi_t, \alpha)\mathrm{d}t + e^{-\lambda\tau} v_\infty(\phi_\tau, \alpha)\right], \right.$$
$$\left. v_\infty(\phi, 1-\alpha) + (1-\alpha)\frac{b}{c}(1+\phi), 0 \right\}$$

Since we showed $v_\infty(\cdot, \alpha)$ is smooth enough for a generalized Itô's rule, we can use the standart arguments in [6], Chapter 10 for the optimal stopping problem on the RHS and substracting $v_\infty(\phi, \alpha)$ from both sides, we get

$$0 = \min\left\{ Lv_\infty(\phi, \alpha) - \lambda v_\infty(\phi, \alpha) + g, v_\infty(\phi, 1-\alpha) - v_\infty(\phi, \alpha) + (1-\alpha)\frac{b}{c}(1+\phi), \right.$$
$$\left. -v_\infty(\phi, \alpha) \right\},$$

hence $v_\infty(\phi,\alpha)$ satisfies (4.3) and by Lemma 4.4, $v_\infty(\phi,\alpha)$ and $V(\phi,\alpha)$ coincide. Therefore the stopping regions $\{\phi \in \mathbb{R}_+ : v_\infty(\phi,\alpha) = 0\} = \{\phi \in \mathbb{R}_+ : V(\phi,\alpha) = 0\}$, i.e. $\mathcal{A}(\alpha)$ of Lemma 4.2 and $\mathcal{A}_\infty(\alpha)$ in (4.) coincide too. By Proposition 4.1, it's immediate that $\mathcal{A}_\infty(\alpha) = [\xi^\alpha, \infty) = \mathcal{A}(\alpha)$, $\mathcal{S}(\alpha) = \mathcal{S}_\infty(\alpha)$, $\mathcal{C}(\alpha) = \mathcal{C}_\infty(\alpha)$ and $\mathcal{D}(\alpha) = \mathcal{D}_\infty(\alpha)$.

We showed the optimal stopping region of (4.1) is $\mathcal{A}(\alpha) = [\xi^\alpha, \infty)$. Now we'll make some remarks about the optimal threshold ξ^α. Again, we'll start with

$$v_\infty(\phi,0) = \min\left\{\inf_\tau \mathbb{E}_0^\phi\left[\int_0^\tau e^{-\lambda t} g(\phi_t,0)\mathrm{d}t + e^{-\lambda\tau} v_\infty(\phi_\tau,0)\right],\right.$$
$$\left. v_\infty(\phi,1) + \frac{b}{c}(1+\phi), 0\right\}. \tag{4.13}$$

By Proposition 4.1 of Sezer [9], solution of first term on the RHS is strictly increasing, negative and continuous on $[0, r_\infty^0)$ and 0 on $[r_\infty^0, \infty)$ where r_∞^0 is the threshold uniquely determined by this problem. By Corollary 4.1, $v_\infty(\phi,1)$ is a nondecreasing, concave and continuous function and $-1/c \le v_\infty(\phi,1) \le 0$, so

$$\frac{-1}{c} + \frac{b}{c}(\phi+1) \le v_\infty(\phi,1) + \frac{b}{c}(\phi+1) \le \frac{b}{c}(\phi+1), \quad \phi \ge 0. \tag{4.14}$$

We have two cases here:
1. If $b > 1$, $v_\infty(\phi,1) + \frac{b}{c}(\phi+1)$ is always positive, so the equation $v_\infty(\phi,1) + \frac{b}{c}(\phi+1) = 0$ has no roots on $\mathbb{R}_+$. Hences second term on the RHS of (4.11) is always positive, so $\xi^0 = r_\infty^0$.
2. If $b \le 1$, $v_\infty(\phi,1) + \frac{b}{c}(\phi+1) = 0$ has a unique root by continuity and monotonicity of $v_\infty(\phi,1)$, we'll this root d_∞^0. In this case $\xi^0 = r_\infty^0 \vee d_\infty^0$.
We continue with $v_\infty(\phi,1)$, it's defined as

$$v_\infty(\phi,1) = \min\left\{\inf_\tau \mathbb{E}_0^\phi\left[\int_0^\tau e^{-\lambda t} g(\phi_t,1)\mathrm{d}t + e^{-\lambda\tau} v_\infty(\phi_\tau,1)\right],\right.$$
$$\left. v_\infty(\phi,0), 0\right\}. \tag{4.15}$$

Again, we'l call optimal threshold of the optimal stopping problem on RHS as r_∞^1. By above arguments, $v_\infty(\phi,0)$ is 0 on the optimal stopping region $[\xi_0, \infty)$. Therefore, $\xi^1 = \xi^0 \vee r_\infty^1$.

Chapter 5

Numerical Examples

We'll describe the numerical computation of successive approximations $v_n(\cdot,\cdot)$ of the value function $V(\cdot,\cdot)$. Here we compute v_n by

$$v_0(\cdot,0)=0, \quad v_0(\cdot,1)=0.$$

$$v_{n+1}(\phi,\alpha)=\min\left\{\mathbb{E}_0\left[\int_0^h e^{-\lambda t}g(\phi_t^\delta,\alpha)\mathrm{d}t+e^{-\lambda h}v_n(\phi_h,\alpha)\right],\right.$$
$$\left. v_n(\phi,1-\alpha)+(1-\alpha)\frac{b}{c}(1+\phi),0\right\}.$$

These computations are based on Kushner and Dupuis's [4] Markov chain approximation method for the expectations of functions of Markov chains. We'll illustrate the method on several examples. We use Markov chain approximation for approximating the term $\mathbb{E}_0\left[\int_0^h e^{-\lambda t}g(\phi_t^\delta,\alpha)\mathrm{d}t+e^{-\lambda h}v_n(\phi_h,\alpha)\right]$. For $\alpha=1$, as we've seen previously this term derives the second degree ODE

$$\lambda(\phi+1)\frac{\partial V}{\partial\phi}+\frac{1}{2}\mu^2\phi^2\frac{\partial^2 V}{\partial\phi^2}-\lambda V+g(\phi,1)=0.$$

If we replace $V''(\phi,1)$ and $V'(\phi,1)$ with their finite-difference approximations

$$\frac{V^h(\phi+h,1)+V^h(\phi-h,1)-2V^h(\phi,1)}{h^2} \quad \text{and} \quad \frac{V^h(\phi+h,1)-V^h(\phi,1)}{h}$$

respectively, then we obtain

$$\frac{V^h(\phi+h,1)+V^h(\phi-h,1)-2V^h(\phi,1)}{h^2}\frac{\sigma^2(\phi)}{2}+\frac{V^h(\phi+h,1)-V^h(\phi,1)}{h}b(\phi)$$
$$-\lambda V^h(\phi,1)+g(\phi,1)=0.$$

Rearranging the terms we get

$$\begin{aligned} V^h(\phi,1) &= \frac{\sigma^2(x)/2}{b(x)h+\sigma^2(x)+\lambda h^2}V^h(\phi-h,1)+\frac{b(x)h+\sigma^2(x)/2}{b(x)h+\sigma^2(x)+\lambda h^2}V^h(\phi+h,1) \\ &+ \frac{h^2}{b(x)h+\sigma^2(x)+\lambda h^2}g(\phi,1) \end{aligned}$$

which can be written as
$V^h(\phi,1)=V^h(\phi-h,1)p^{h,1}(\phi,\phi-h)+V^h(\phi+h,1)p^{h,1}(\phi,\phi+h)+\Delta t^{h,1}(\phi)g(\phi,1)$.
We define transition probabilities as

$$p^{h,1}(\phi,\phi+h)=\frac{b(\phi)h+\sigma^2(\phi)/2}{b(\phi)h+\sigma^2(\phi)+\lambda h^2}$$

$$p^{h,1}(\phi,\phi-h)=\frac{\sigma^2(\phi)/2}{b(\phi)h+\sigma^2(\phi)+\lambda h^2}$$

$$p^{h,1}(\phi,\phi)=\frac{\lambda h^2}{b(\phi)h+\sigma^2(\phi)+\lambda h^2}$$

and $p^{h,1}(\phi,\varsigma)=0$ for $\varsigma\neq\phi,\phi-h,\phi+h$ with

$$\Delta t^{h,1}(\phi)=\frac{h^2}{b(\phi)h+\sigma^2(\phi)+\lambda h^2}$$

where $b(\phi)=\lambda(\phi+1)$ and $\sigma(\phi)=\mu\phi$.
For $\alpha=0$, we have

$$\lambda(\phi+1)\frac{\partial V}{\partial \phi} - \lambda V + g(\phi) = 0.$$

Again replacing $V'(\phi, 0)$ with its finite-difference approximation and performing a similar calculation we have

$$p^{h,0}(\phi, \phi+h) = \frac{b(\phi)h}{b(\phi)+\lambda h}$$

$$p^{h,0}(\phi, \phi) = \frac{\lambda h}{b(\phi)+\lambda h}$$

and $p^{h,0}(\phi, \varsigma) = 0$ for $\varsigma \neq \phi, \phi+h$ with

$$\Delta t^{h,0}(\phi) = \frac{h}{b(\phi)+\lambda h}$$

where $b(\phi) = \lambda(\phi+1)$. Let $\left\{\xi_n^{h,\alpha}; n \geq 0\right\}$ be two discrete-time Markov chains with transition probabilites $p^{h,\alpha}$ defined as above and define the continuous-time processes $\left\{\xi^{h,\alpha}(t); t \geq 0\right\}$ on the same space by adding the interpolation interval $\Delta t^{h,\alpha}(\xi_n^{h,\alpha})$. The processes $\left\{\xi^{h,\alpha}(t); t \geq 0\right\}$ is locally consistent with $\{\Phi_t; t \geq 0\}$, therefore these processes and functions $V^h(., \alpha)$ well approximate $\{\Phi_t; t \geq 0\}$ and $V(., \alpha)$ respectively; see Kushner and Dupuis [4] for the details. With this setup,

$$\mathbb{E}_0\left[\int_0^h e^{-\lambda t} g(\phi_t^\delta, \alpha)\mathrm{d}t + e^{-\lambda h} V_n(\phi_h, \alpha)\right] \approx g(\phi, \alpha) + (1-\lambda h)\mathbb{E}_0^\phi s\left[V_n^h(\phi_h, \alpha)\right].$$

More explicitly the iterations take the form

$$V_n(\phi, 0) = 0, \quad V_n(\phi, 1) = 0$$

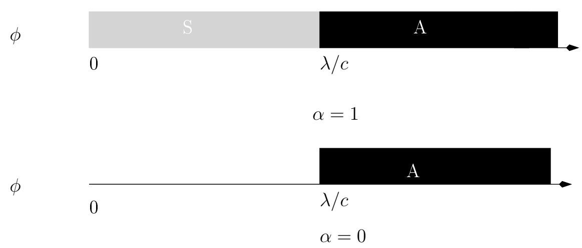
$$\begin{aligned} V_{n+1}(\phi, 0) &= \min\left\{\phi - \frac{\lambda}{c} + (1-\lambda h)\mathbb{E}_0^\phi\left[V_n^h(\phi, 0)\right], V_n(\phi, 1) + \frac{b}{c}(\phi+1), 0\right\} \\ V_{n+1}(\phi, 1) &= \min\left\{\phi\left(1+\frac{a}{c}\right) + \frac{a-\lambda}{c} + (1-\lambda h)\mathbb{E}_0^\phi\left[V_n^h(\phi, 1)\right], V_n(\phi, 0), 0\right\}. \end{aligned} \tag{5.1}$$

We calculate the expectations by the help of the approximating Markov chains for both $\alpha = 0$, $\alpha = 1$.

5.1 Examples

In the previous section, we introduced an approximation scheme for the succesive approximations (4.5). (5.1) describes an algorithm for approximating the value function V, we illustrate the solution of the optimal stopping problem in (4.1) on several numerical examples by this algorithm. As seen from (5.1), the solution is directly related to a/c, b/c and λ/c. We'll investigate how these parameters affect the stopping, switching and continuation regions.

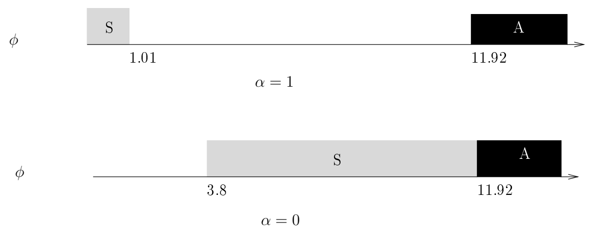
1. If $a > \lambda$, the solution is trivial. Independent from the value of b/c, it has the following structure:



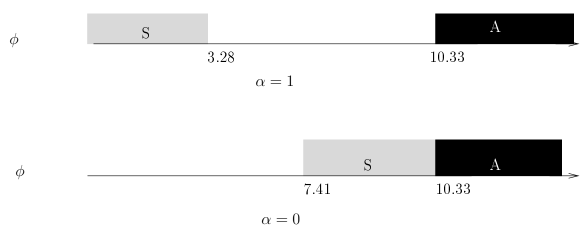
Here, grey shaded area is the switching region and black area is the alarm region . The running cost $g(\phi, 1) = \phi\left(1 + \frac{a}{c}\right) + \frac{a-\lambda}{c}$ is always positive, so $(Kv_i)(\phi, 1)$ in (4.5) is never negative for any $\phi \in \mathbb{R}_+$ and for any i by induction. Hence, if we start with $\alpha = 1$, we immediately switch off the observation and never turn on again (so $\mathcal{S}(0) = \emptyset$). If we start with $\alpha = 0$, we wait until Φ reaches to the level λ/c and raise the alarm. If we do the first and second iterations, we see $v_2(\phi, 1) = v_1(\phi, 0) = \phi - \frac{\lambda}{c}$ on $\left[0, \frac{\lambda}{c}\right)$ and 0 on $\left[\frac{\lambda}{c}, \infty\right)$. By mutual induction, we can show that after $n = 2$, optimal threshold for both $\alpha = 0$ and $\alpha = 1$ is $\frac{\lambda}{c}$. In this case we take no observations, the problem has a deterministic structure.

From now on we'll consider nontrivial the cases where $a < \lambda$. In all examples, the approximation parameter h is 0.01 and $\mu = 1$.

2. We consider a special case, we take $b = 0$; i.e., cost of switching on the observation is 0 and $a > 0$. In the first case (a), we take $a/c = 0.1$ and compare it to case $a/c = 1$ (b). In the fist case, a/c is relatively small, therefore most of the time the observation is on and $\alpha = 0$ has a nontrivial switching region. If we increase a/c, we expect shorter observation intervals. We also expect switching region of $\alpha = 0$ to shrink. We know if we increase a/c up to λ/c (as we remarked a can be at most λ), we get no switching region for $\alpha = 0$ and we take no observations for $\alpha = 1$. Obviously, in this case a/c dominates b/c.



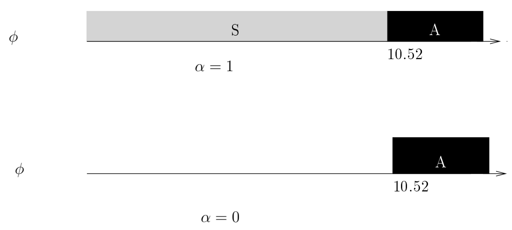
(a)



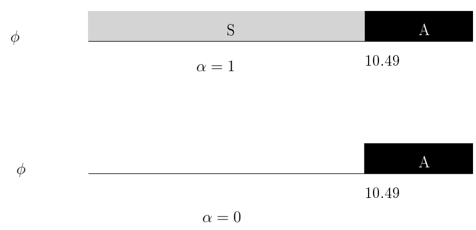
(b)

Next, we'll try to observe how a/c dominates b/c in determining the regions in extreme cases.

3. Now we take a/c relatively high and vary b/c between high and low. If both a/c and b/c are high, optimal strategy is to never turn on the observation. Even if b/c is also low, it doesn't affect the structures of the regions much, since we can't afford to take observations. We take $\lambda/c = 10$, $a/c = 9$ and $b/c = 0.1, 1$ in the following examples.



(c)

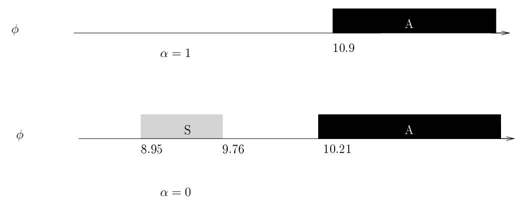


(d)

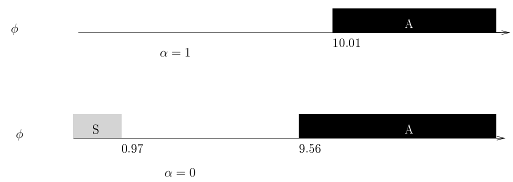
4. We fix a/c low and vary b/c between low and high. We take $\lambda/c = 10$, $a/c = 0.01$ and $b/c = 0.01, 0.1, 1$ in (e), (f) and (g). In this case, since a/c

low, continuation region for $\alpha = 1$ is wider than high a/c values. Once we start observation on, we never switch it off as observation cost is low. If cost of switching on is high (refer to (g)), even though observation cost is low we don't switch on the observation. In this case $\mathcal{C}(1)$ is determined by a/c and b/c seems to determine $\mathcal{S}(0)$.

(e)

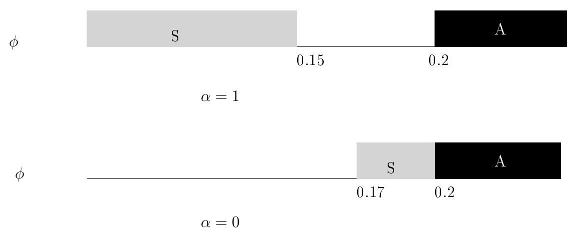


(f)

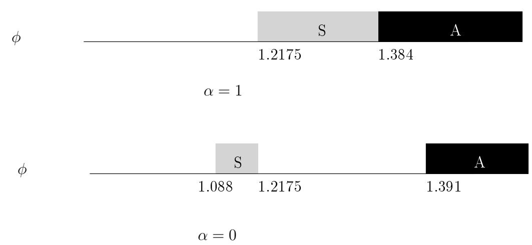


(g)

5. In other cases (where a/c and b/c are not ver high or low) it's rather hard to understand how a and b affect the structures of regions. For instance, we take $\lambda/c = 0.5$, $a/c = 0.02$ and $b/c = 0.02$ in (h) and $\lambda/c = 1.387$, $a/c = 0.099$ and $b/c = 0.009$ in (i). We changed λ/c, decreased b/a and the structure of the regions changed completely.



(h)



(i)

Appendix A

Code

We described the numerical computation of the successive approximations $v_n(.,.)$ in the previous section. The below code calculates the approximations v_n, $n \geq 0$, of the value function $V(.)$, until the maximum difference between two successive functions is reduced to an acceptable level and waiting, switching & stopping regions for $\alpha = 0$ and $\alpha = 1$.

CalcSuccessiveApprox.m

```
function [breakpoints0,breakpoints1,labels0 , labels1 ]  = CalcSuccessiveApprox(a,b,c,lambda,
mu,epsilon,h,V0prev,V1prev,label0,label1,brkpts0,brkpts1)
totalbreakpoints  = merge(brkpts, brkpts1);
numofinterpolatingpts = totalbreakpoints / h +1;
oldlabels0  =  label0;
oldlabels1  =  label1;
oldbreakpts0 = brkpts0;
oldbreakpts1 = brkpts1;

numberofiters = 0;

interpolatingpts1  = zeros(1,  numofinterpolatingpts+1);
interpolatingpts0  = zeros(1,  numofinterpolatingpts+1);
for  i=0:numofinterpolatingpts
```

```
    interpolatingpts0(i) = ((lambda/c)/ numofinterpolatingpts) *i;
    interpolatingpts1(i) = ((lambda/c)/ numofinterpolatingpts) *i;
end

interpolatingpts0 = V0prev(interpolatingpts0);
interpolatingpts1 = V1prev(interpolatingpts1);

while(1)
  newlabels0 = [ ];
  newlabels1 = [ ];
  newbreakpts0 = [0];
  newbreakpts1 = [0];
  newinterpolatingpts0 = [ ];
  newinterpolatingpts1 = [ ];
  a = length(totalbreakpoints);
    for i=1:a-1
    [piece0add bpts0add labels0add] = CalcExpect0(oldbreakpts0,oldbreakpts1,
    oldlabels0, oldlabels1, totalbreakpoints(i), totalbreakpoints(i+1),a-i,mu,h,
    lambda,a,b,c,interpolatingpts0, interpolatingpts1, numofinterpolatingpts)
    newlabels0 = [newlabels0,labels0add]
    newbreakpts0 = [newbreakpts0,bpts0add]
    newinterpolatingpts0 = [newinterpolatingpts0,pieceadd0]

    [piece1add bpts1add labels1add] = CalcExpect1(oldbreakpts0,oldbreakpts1,
    oldlabels0, oldlabels1, totalbreakpoints(i), totalbreakpoints(i+1),a-i,mu,h
    lambda,a,b,c,interpolatingpts0, interpolatingpts1, numofinterpolatingpts);
    newlabels1 = [newlabels1,labels1add]
    newbreakpts1 = [newbreakpts1,bpts1add]
    newinterpolatingpts1 = [newinterpolatingpts0,pieceadd1]
  end

  oldlabels0 = newlabels0;
  oldbreakpts0 = newbreakpts0;

  oldlabels1 = newlabels1;
  oldbreakpts1 = newbreakpts1;

  if((abs(interpolatingpts0(1)-newinterpolatingpts(1)) < epsilon) &&
     (abs(interpolatingpts1(1) - newinterpolatingpts(1))< epsilon))
      break;
```

```
  end

interpolatingpts0  = newinterpolatingpts0;
interpolatingpts1  = newinterpolatingpts1;

totalbreakpoints  = merge(oldbreakpts0,oldbreakpts1);
totalbreakpoints  = unique(totalbreakpoints);
numberofiters = numberofiters+1;

end

breakpoints0 = newbreakpts0;
breakpoints1 = newbreakpts1;
labels0         = newlabela0;
labels1         = newlabels1;

end
```

Here, merge is standart merge function of two sorted arrays. In each iteration we divide $[0, lastbreakpoint]$ into three regions and hold the corresponding thresholds and labels of the regions(i.e 'wait', 'switch', 'stop') for both $\alpha = 1$ and $\alpha = 0$. We put these values into the arrays newlabels0, newbreakpts0, newlabels1, newbreakpts1 for $\alpha = 0$ and $\alpha = 1$, respectively. Also, we interpolate the points of $v_n(., \alpha)$ for corresponding α again in the interval $[0, lastbreakpoint]$. We hold the interpolated points in the arrays interpolatingpts0 and interpolatingpts1 for $\alpha = 0$ and $\alpha = 1$, respectively. This function returns the critical thresholds and corresponding region labels.

In CalcSuccessiveApprox, we calculate the $v_n(., \alpha)$ and critical thresholds at each iteration by the functions CalcExpect0 and CalcExpect1 for $\alpha = 0$ and $\alpha = 1$. CalcExpext0 and CalcExpect1 compares $(Kv_n)(., \alpha)$, $(Gv_n)(, .\phi)$ defined as in Chapter 4 and 0 by calculating the interpolated points of these functions from interpolated points of $v_n(, .\alpha)$ to find $v_{n+1}(., \alpha)$. We approximate $(Kv_n)(, .\alpha)$ by Markov chain approximation we described in Chapter 5. At the following code segments, we denoted (Kv_n) by waitfunc, (Gv_n) by switchfunc and zero function by zerofun. We interpolate waitfunc & switchfunc, compare them with each other and 0 in the interval $[0, lastbreakpoint]$, calculate new breakpoints and label this interval with corresponding labels 'wait', 'switch', 'stop' for $\alpha = 0$ and $\alpha = 1$, respectively.

CalcExpect0.m

```
function [piece0add bptsadd0 labelsadd0 ] = CalcExpect0( breakpts0, breakpts1, labels0,
labels1 , bp1pos, bp2pos, last , mu, h, lambda,a,b,c, interpolatingpts0 , interpolatingpts1 ,
numofinterpolatingpts)
tolval = c * h^2/lambda;
waitfunc = zeros(1, bp2pos-bp1pos+2);

for i=1:bp2pos-bp1pos+2
        waitfunc(i) = ((lambda/c)/ numofinterpolatingpts) *(i-1);
end

switchfunc = zeros(1, bp2pos-bp1pos+2);

for i=1:bp2pos-bp1pos+2
        switchfunc(i) = ((lambda/c)/ numofinterpolatingpts) *(i-1);
end

ptsintheinterval = bp2pos-bp1pos+1;
maptothesmallinterval = @(i) round(((i-1)*ptsintheinterval+numofinterpolatingpts-i)
                        /numofinterpolatingpts-1);

i=1;
for j=bp1pos:bp2pos+1
  phi = waitfunc(i);
  waitfunc(i) = phi - lambda/c + (1- lambda*h)* ( ((phi+1)* interpolatingpts0(j +1)
            /(phi+2) + interpolatingpts0(j)) /(phi+1) );
  i =i+1;
end

i=1;
for j = bp1pos:bp2pos+1
  phi = switchfunc(i);
  switchfunc(i) = (1+phi)*b/c+ interpolatingpts1(j);
  i = i+1;
```

```
end

dif = waitfunc−switchfunc;
diff1 = waitfunc(1) − switchfunc(1);
diff2 = waitfunc(ptsintheinterval)−switchfunc(ptsintheinterval );

if ( diff1 ∗ diff2 >= 0 )
  if ( diff1 <= 0)
    if ( last ==1)
      if (waitfunc( ptsintheinterval ) <= 0)
        labelsadd0= [ 'wait', 'stop' ];
        bptsadd0 = [bp1pos, bp2pos];
        piece0add = waitfunc(1: ptsininterval +1);
      else
        bpnewpos = binsearch(waitfunc,tolval);

        if (bpnewpos > bp1pos)
          labelsadd0 = ['wait', 'stop' ];
          bptsadd0 = [bp1pos, bpnewpos];
          piece0add = [waitfunc(1:maptothesmallinterval(bpnewpos)), zeros(1,ptsintheinterval−
          maptothesmallinterval(bpnewpos)+1)];
        elseif (bpnewpos==bp1pos)
          labelsadd0 = ['stop' ];
          bptsadd0 = [bp1pos];
          piece0add = zeros (1, ptsintheinterval +1);
        end
      end
    else
      labelsadd0 = ['wait' ];
      bptsadd0 = [bp1pos, bp2pos];
      piece0add = waitfunc(1:ptsintheinterval−1);
    end
  elseif ( diff1 > 0)
    if ( last ==1)
      if (switchfunc( ptsintheinterval ) <= 0)
        labelsadd0 =[ 'switch', 'stop' ];
        bptsadd0 = [bp1pos, bp2pos];
        piece0add = switchfunc(1:ptsintheinterval+1);
      else
        bpnewpos = binsearch(switchfunc, tolval);
```

```
        if (bpnewpos > bp1pos)
          labelsadd0 = ['switch', 'stop'];
          bptsadd0 = [bp1pos, bpnewpos]
          piece0add = [switchfunc(1:maptothesmallinterval(bpnewpos)),
          zeros(1, ptsintheinterval -maptothesmallinterval(bpnewpos)+1)];
        elseif (bpnewpos==bp1pos)
          labelsadd0 = ['stop'];
          bptsadd0 = [bp1pos];
          piece0add = zeros(1, ptsintheinterval+1);
        end
      end
    else
      labelsadd0 = ['switch'];
      bptsadd0 = [bp1pos, bp2pos];
      piece0add = switchfunc(1:ptsintheinterval-1)
    end
  end
elseif ( diff1 * diff2 < 0)
  bpnewpos = binsearch(dif,tolval);
  if ( diff1 <= 0)
    if (last==1)
        if (switchfunc( ptsintheinterval ) <= 0)
                labelsadd0= ['wait','switch', 'stop'];
                bptsadd0 = [bp1pos,bpnewpos, bp2pos];
                piece0add = [waitfunc(1:maptothesmallinterval(bpnewpos)),
                switchfunc(maptothesmallinterval(bpnewpos)+1:ptsintheinterval+1)];
              else
                bpnew2pos = binsearch(switchfunc,tolval)
                if (bpnew2pos > bpnewpos)
                  labelsadd0 = ['wait','switch', 'stop'];
                  bptsadd0 = [bp1pos, bpnewpos, bpnew2pos];
                  piece0add = [waitfunc(1:maptothesmallinterval(bpnewpos))
                  switchfunc(maptothesmallinterval(bpnewpos)
                  +1:maptothesmallinterval(bpnew2pos)), zeros(1, ptsintheinterval-
                  maptothesmallinterval(bpnewpos2)+1 )];
                elseif (bpnewpos==bpnew2pos)
                  labelsadd0 = ['wait','stop'];
                  bptsadd0 = [bp1pos, bpnewpos];
                  piece0add = [waitfunc(1: maptothesmallinterval(bpnewpos)),
                  zeros(1, ptsintheinterval -maptothesmallinterval(bpnewpos)+1)];
```

```
                end
              end
      else
        labelsadd0 = ['wait', 'switch']
        bptsadd0 = [bp1pos, bpnewpos, bp2pos]
        piece0add = [waitfunc(1:maptothesmallinterval(bpnewpos)),
        switchfunc(maptothesmallinterval(bpnewpos)+1:ptsintheinterval)];
      end
    elseif ( diff1  > 0)
        if ( last==1)
              if (waitfunc( ptsintheinterval )  <= 0)
                  labelsadd0= ['switch','wait', 'stop'];
                  bptsadd0 = [bp1pos,bpnewpos, bp2pos];
                  piece0add = [switchfunc(1:maptothesmallinterval(bpnewpos)),
                  waitfunc(maptothesmallinterval(bpnewpos)+1:ptsintheinterval+1)];
                else
                  bpnew2pos = binsearch(waitfunc,tolval)
                  if (bpnew2pos > bpnewpos)
                      labelsadd0 = ['switch','wait', 'stop'];
                      bptsadd0 = [bp1pos, bpnewpos, bpnew2pos];
                      piece0add = [switchfunc(1:maptothesmallinterval(bpnewpos)),
                      waitfunc(maptothesmallinterval(bpnewpos)
                      +1:maptothesmallinterval(bpnew2pos)), zeros(1,ptsintheinterval-
                      maptothesmallinterval(bpnewpos2)+1 )];
                    elseif (bpnewpos==bpnew2pos)
                      labelsadd0 = ['switch','stop'];
                      bptsadd0 = [bp1pos, bpnewpos];
                      piece0add = [switchfunc(1: maptothesmallinterval(bpnewpos)),
                      zeros(1,  ptsintheinterval - maptothesmallinterval(bpnewpos)+1)];
                  end
                end
            else
              labelsadd0 = ['switch', 'wait'];
              bptsadd0 = [bp1pos, bpnewpos, bp2pos];
              piece0add = [switchfunc(1:maptothesmallinterval(bpnewpos)),
              waitfunc(maptothesmallinterval(bpnewpos)+1:ptsintheinterval)];
      end
   end

end
```

CalcExpect1.m

```
function [piece1add bptsadd1 labelsadd1 ] = CalcExpect1( breakpts0, breakpts1, labels0,
labels1 , bp1pos, bp2pos, last ,mu,h,lambda,a,b,c, interpolatingpts0 ,
interpolatingpts1 , numofinterpolatingpts )
tolval = c * h^2/lambda;
waitfunc = zeros(1, bp2pos-bp1pos+2);

for i=1:bp2pos-bp1pos+2
        waitfunc(i) = ((lambda/c)/ numofinterpolatingpts) *(i-1);
end

switchfunc = zeros(1, bp2pos-bp1pos+2);

for i=1:bp2pos-bp1pos+2
        switcfunc(i) = ((lambda/c)/ numofinterpolatingpts) *(i-1);
end



i=1;
for j= bp1pos:bp2pos+1
        phi = waitfunc(i);
        waitfunc(i) = phi *( 1 + a/c) + (a-lambda)/c + (1-lambda * h) *
        ( ((lambda * h^2 * interpolatingpts1(j)) / (lambda * (phi+1))) +
        ( (lambda*(phi+1)*h+((mu^2*phi^2)/2)* interpolatingpts1(j+1))/( lambda *
        (phi+1)*h_mu^2*phi^2+lambda*h^2)));
    if( i >1)
      waitfunc(i) = waitfunc(i) + ( ((mu^2*phi^2)/2) * interpolatingpts1(j-1)
                    / ( lambda * (phi+1)*h +mu^2*phi^2 + lambda * h^2));
  end
end
i=1;
for j=bp1pos:bp2pos
  switchfunc(i) = interpolatingpts0(j);
end

dif = waitfunc-switchfunc;
diff1 = waitfunc(1) - switchfunc(1);
diff2 = waitfunc(ptsintheinterval)-switchfunc(ptsintheinterval);

if( diff1*diff2 >= 0 )
```

```
if ( diff1  <= 0)
  if ( last==1)
    if (waitfunc(bp2) <= 0)
      labelsadd1= [ 'wait', 'stop' ];
      bptsadd1 = [bp1, bp2];
      piece1add = waitfunc(1: ptsininterval +1);
  else
    bpnew = binsearch(waitfunc,tolval)
    if (bpnew > bp1)
      labelsadd1 = ['wait', 'stop' ];
      bptsadd1 = [bp1, bpnew];
      piece1add = [waitfunc(1:maptothesmallinterval(bpnewpos)),zeros(1
       ptsintheinterval -maptothesmallinterval(bpnewpos)+1)];
     elseif (bpnew==bp1)
      labelsadd1 = ['stop' ];
      bptsadd1 = [bp1];
      piece1add = zeros (1,  ptsintheinterval +1);
    end
  end
else
    labelsadd1 = ['wait' ];
    bptsadd1 = [bp1, bp2];
    piece1add = waitfunc(1:ptsintheinterval-1);
end
 elseif ( diff1  > 0)
  if ( last==1)
    if (switchfunc(bp2) <= 0)
      labelsadd1 =[ 'switch', 'stop' ];
      bptsadd1 = [bp1, bp2];
      piec10add = switchfunc(1:ptsintheinterval+1);
    else
      bpnew = binsearch(switchfunc,tolval);
      if (bpnew > bp1)
        labelsadd1 = ['switch', 'stop' ];
        bptsadd1 = [bp1, bpnew];
        piece1add = [switchfunc(1:maptothesmallinterval(bpnewpos)),
        zeros (1, ptsintheinterval - maptothesmallinterval(bpnewpos)+1)];
       elseif (bpnew==bp1)
        labelsadd1 = ['stop' ];
        bptsadd1 = [bp1];
```

```
          piece1add = zeros(1, ptsintheinterval+1);
        end
      end
    else
      labelsadd1 = ['switch'];
      bptsadd1 = [bp1, bp2];
      piece1add = switchfunc(1:ptsintheinterval-1);
    end
  end
elseif ( diff1 * diff2 < 0)
  bpnew = binsearch(dif,tolval);
  if ( diff1 <= 0)
    if ( last==1)
        if (switchfunc(bp2) <= 0)
                labelsadd1= ['wait','switch', 'stop'];
                bptsadd1 = [bp1,bpnew, bp2]
                piece1add = [waitfunc(1:maptothesmallinterval(bpnewpos))
                switchfunc(maptothesmallinterval(bpnewpos)+1:ptsintheinterval+1)];
              else
                bpnew2 = binsearch(switchfunc,tolval)
                if (bpnew2 > bpnew)
                  labelsadd1 = ['wait','switch', 'stop'];
                  bptsadd1 = [bp1, bpnew, bpnew2];
                  piece1add = [waitfunc(1:maptothesmallinterval(bpnewpos)),
                  switchfunc(maptothesmallinterval(bpnewpos)
                  +1:maptothesmallinterval(bpnew2pos)), zeros(1,
                  ptsintheintervamaptothesmallinterval(bpnewpos2)+1 )];
                elseif (bpnew==bpnew2)
                  labelsadd1 = ['wait','stop'];
                  bptsadd1 = [bp1, bpnew];
                  piece1add = [waitfunc(1: maptothesmallinterval(bpnewpos)), zeros(1,
                   ptsintheinterval -maptothesmallinterval(bpnewpos)+1)];
                end
              end
    else
      labelsadd1 = ['wait', 'switch'];
      bptsadd1 = [bp1, bpnew, bp2];
      piece1add = [waitfunc(1:maptothesmallinterval(bpnewpos)),
      switchfunc(maptothesmallinterval(bpnewpos)+1:ptsintheinterval)];
    end
```

```
  elseif ( diff1  > 0)
        if ( last==1)
                if (waitfunc(bp2) <= 0)
                  labelsadd1= ['switch','wait',  'stop'];
                  bptsadd1 = [bp1,bpnew, bp2]
                  piece1add =  [switchfunc(1:maptothesmallinterval(bpnewpos))
                  waitfunc(maptothesmallinterval(bpnewpos)+1:ptsintheinterval+1)];
                  else
                    bpnew2 = binsearch(waitfunc,tolval)
                       if (bpnew2 > bpnew)
                         labelsadd1 = ['switch','wait',  'stop'];
                         bptsadd1 = [bp1, bpnew, bpnew2];
                         piece1add =  [switchfunc(1:maptothesmallinterval(bpnewpos)),
                         waitfunc(maptothesmallinterval(bpnewpos)
                         +1:maptothesmallinterval(bpnew2pos)), zeros(1,ptsintheinterval−
                         maptothesmallinterval(bpnewpos2)+1 )];
                       elseif (bpnew==bpnew2)
                         labelsadd1 = ['switch','stop'];
                         bptsadd1 = [bp1, bpnew];
                         piece1add = [switchfunc(1: maptothesmallinterval(bpnewpos)),
                         zeros(1,  ptsintheintervalmaptothesmallinterval(bpnewpos)+1)];
                      end
                  end
            else
             labelsadd1 = ['switch',  'wait']
             bptsadd1 = [bp1, bpnew, bp2];
             piece1add =  [switchfunc(1:maptothesmallinterval(bpnewpos)),
             waitfunc(maptothesmallinterval(bpnewpos)+1:ptsintheinterval)];
        end
  end
end
```

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