KARADENİZ TECHNICAL UNIVERSITY
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS

PROXIMAL HOMOTOPY OF PROXIMAL CONTINUOUS FUNCTIONS

MASTER'S THESIS

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JUNE 2023
TRABZON

KARADENİZ TECHNICAL UNIVERSITY
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This thesis is accepted to give the degree of
"MASTER OF SCIENCE"
By
The Graduate School of Natural and Applied Sciences at
Karadeniz Technical University

The Date of Submission : 29/05/2023
The Date of Examination : 19/06/2023

Supervisor : Assoc. Dr. Tane VERGILI

Trabzon 2023

ACKNOWLEDGMENTS

This thesis is a milestone on my way to obtaining a Master's degree in Mathematics from Karadeniz Technical University. At the end of the Master program, I thank God for giving me the strength to accomplish this thesis, and for His blessings, strength, and guidance.

First and foremost, I would like to express my sincere gratitude to my supervisor, Assoc. Dr. Tane VERGİLİ for her excellent guidance and continuous support.

I dedicate this work to my parents, sisters and brothers. I offer them my sincere love, gratitude, and appreciation for their love, support, patience, and understanding throughout the entire journey.

Maram Z M ALMAHARIQ

Trabzon 2023

DECLARATION PAGE

I have submitted this study titled "PROXIMAL HOMOTOPY OF PROXIMAL CONTINUOUS FUNCTIONS ", which I submitted as a Master's Thesis, from beginning to end, by my supervisor, Assoc. Dr. Tane VERGİLİ, that I have collected the data/samples myself, that I have fully indicated the information I received from other sources in the text and in the bibliography, that I have acted in accordance with scientific research and ethical rules during the working process, and that I have made all kinds of decisions in case the opposite arises. I declare that I accept the legal result. 19/06/2023

Maram Z M ALMAHARIQ

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SUMMARY

Master's Thesis

PROXIMAL HOMOTOPY OF PROXIMALLY CONTINUOUS FUNCTIONS

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Karadeniz Technical University
The Graduate School of Natural and Applied Sciences
Mathematics Graduate Program
Supervisor: Dr. Tane VERGİLİ
2023, Pages 108

In this thesis entitled "Proximal Homotopy of Proximal Continuous Functions", the concept of homotopy is a transformation that enables a continuous deformation between continuous functions defined in space. In addition to the fact that the main features of these already defined transformations are available in the literature, it is aimed to complete the deficiencies in this thesis. In addition, it is planned to obtain new theorems and results. Thus, in the first part of this thesis, we searched the literature and discussed the proximal and descriptive homotopy theory and the proximal and descriptive fixed point theorems. The second part of our thesis includes our original work. The second part was from the sub-heading. Under these headings, we have obtained new theorems and results about spatial (descriptive) homotopy, spatial (descriptive) path homotopy, spatial (descriptive) deformation retract, fixed point theorems in spatial and descriptive terms. In addition, we introduced new types of functions, which are proximal (descriptive) universal functions, and examined the link between the spatial (descriptive) approximate fixed set property and spatial (descriptive) universal functions.

Key Words: Proximal homotopic theory, Descriptive homotopic theory, Proximal (descriptive) approximate fixed set property, Proximal (descriptive) fixed set property with respect to a proximity (descriptive proximity)

ÖZET

Yüksek Lisans Tezi

YAKIN SÜREKLİ FONKSİYONLARIN YAKIN HOMOTOPİSİ

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2023, Sayfa 108

"Proximal Homotopy Of Proximal Continuous Functions" adlı tez çalışmasında, homotopi kavramı uzayda tanımlı sürekli fonksiyonlar arasında sürekli bir deformasyonu mümkün kılan dönüşümlerdir. Halı hazırda tanımlanmış bu dönüşümlerin belli başlı özellikleri literatürde mevcut olmasının yanı sıra, eksikliklerinin tamamlanması bu tez çalışmasında hedeflenmiştir. Ayrıca yeni teoremler ve sonuçlar elde edilmesi planlanmıştır. Böylece bu tez çalışmasında ilk bölümde literatür taraması yaptık ve yakınsal ve betimsel homotopi teorisi ile yakınsal ve betimsel sabit nokta teoremlerini ele aldık. Tezin ikinci bölümü orijinal çalışmamızı içermektedir. İkinci bölüm altı alt başlıktan oluşmuştur. Bu başlıklar altında yakın (betimleyici) homotopi, yakın (betimleyici) yol homotopi, yakın (betimleyici) deformasyon retrakt ve yakın (betimleyici) homotopi döngüleri, konumsal ve betimsel anlamda sabit nokta teoremleri ile ilgili yeni teorem ve sonuçlar elde ettik. Ek olarak, yakınsal (betimleyici) evrensel fonksiyonlar olan yeni fonksıyon türlerini tanıttık, (betimleyici) yaklaşık sabit küme özelliği ile yakınsal (betimleyici) evrensel fonksiyonlar arasındaki bağlantıyı inceledik.

Anahtar Kelimeler: Yakınsal homotopik teorem, Tanımlayıcı homotopik teorem, Yakınsal (tanımlayıcı) yaklaşık sabit küme özelliği, Yakınlığa göre yakınsal (tanımlayıcı) sabit küme özelliği (tanımlayıcı yakınlık)

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SYMBOLS INDEX

(X, δ)	: Proximity space
$A\ \delta\ B$	: A set A is near to set B
$A\ \underline{\delta}\ B$	: A set A is far from set B
$\{x\}\ \delta\ B$	: x is a closure point of B
clB	: The closure operator on a set B
$\grave{B}$	: The set of limit points of a set B
δ_L	: Lodato proximity
δ_o	: Wallman proximity
(X, δ_Φ)	: Descriptive proximity space
$A\ \delta_\Phi\ B$	: A set A is descriptively near to set B
$A\ \underline{\delta}_\Phi\ B$	: A set A is descriptively far from set B
$A \cap_\Phi B$	: Descriptive intersection in descriptive space
$cl_\Phi A$	: Descriptive closure of the set A
$A =_{des} B$	: A and B are descriptively proximal equal
$\alpha * \beta$	: The product path between α and β
$\bar{\alpha}$	: The inverse of the path α
$\alpha \sim_p \beta$	: Path homotopic paths
$[\alpha]$	: The equivalence class of path α
$f \sim_\delta g$	: Proximal homotopy maps
$X \approx_{\delta_1, \delta_2} Y$	: Proximal isomorphism spaces

$X \simeq_\delta Y$	: Proximally homotopy equivalent spaces
$f \sim_\delta g\ (relA)$	: f and g are proximally homotopic relative to A
$\alpha \sim_{\delta,p} \beta$	: α and β are proximal path homotopy
$f \sim_\Phi g$	: Descriptive proximally homotopic maps
$X \approx_{\Phi_1,\Phi_2} Y$	: Descriptive proximal isomorphism spaces
$X \simeq_\Phi Y$	: Descriptive proximally homotopy equivalent spaces
$f \sim_\Phi g\ (relA)$	: Descriptive proximally homotopic maps relative to A
$\alpha \sim_{\Phi,p} \beta$	: α and β are descriptive proximal path homotopic
(ADFSP)	: Approximate descriptive fixed set property
(APFSP)	: Approximate proximal fixed set property
(PDT)	: Proximal descriptive transitive
(PFSP)	: Proximal fixed set property
(PT)	: Proximal transitive

1. BASIC CONSTRUCTIONS

1.1. Introduction

The word proximity refers to the quality, state of being proximate or closeness, and thus the proximity of two points on the axis of real numbers can be measured using the scale obtained from the absolute map. Similarly, the proximity of two points in the plane $\mathbb{R}^2$ and $\mathbb{R}^3$ can be calculated. If the point is on the plane, then we can say that the point is close to the plane if we consider that the distance between them is zero. However, if the point is not in the plane, then it can be expressed mathematically in any sense that the point can be considered close to the plane. The scale we use in these measurements is the Euclidean scale which is known to those who work in fields close to mathematics. So, using the gap map definition we will define the relation of a point is near a set and one set is near another set in a metric space (X, d).

Computational proximity (CP) is an approach to find what we will then call by proximity spaces consisting of a non-empty space X and a proximity relation. In CP, we have two main types of proximity nearness between the sets proximal (spatial) nearness and descriptive nearness which will help us to study a lot of properties like proximal (descriptive) homotopy, proximal (descriptive) connectedness, proximal (descriptive) fixed set property, etc. But before we study the proximity of any non-empty set, we will derive it from the metric space because it will be easier to understand.

In this section we will review some of the topics that will be required by our thesis, we will introduce proximity as a relation and show how it proceeds from a metric space and a map gap to any non-empty set X. Moreover, we will study some properties on proximity spaces and descriptive proximity spaces such as continuity, connectedness and conjugacy. Furthermore, we will introduce the simplicial complex and CW spaces terminated with homotopy, the path homotopy and the deformation retraction.

Definition 1.1.1. (Searcoid, 2007): Given a non-empty set X and a function $d: X \times X \to \mathbb{R}$, if the following conditions hold, we say that the pair (X, d) is a metric space and d is a metric on X. For all $x, y, z \in X$,

M(1) $d(x, y) \geq 0$,
M(2) $d(x, y) = 0$ if and only if $x = y$,
M(3) $d(x, y) = d(y, x)$,
M(4) $d(x, z) \leq d(x, y) + d(y, z)$,

Definition 1.1.2. (Naimpally & Peters, 2013): Suppose that (X, d) is a metric space and $A, B \subseteq X$. Then the map $D: P(X) \to P(X)$ defined by

$$D(A,B) = \begin{cases} \inf\{d(a,b): a \in A,\ b \in B\}, & if\ A\ and\ B\ are\ not\ empty \\ \infty, & if\ A\ or\ B\ is\ empty \end{cases}$$

is called the gap map.

Now we will define the metric proximity space.

Definition 1.1.3. (Naimpally & Peters, 2013): Assume that (X, d) is a metric space and $A, B \subseteq X$. We say that A is near B, denoted by $A\ \delta\ B$, if $D(A, B) = 0$.

Remark 1.1.4. (Naimpally & Peters, 2013): If (X, d) is a metric space and $A, B \subseteq X$ with $D(A, B) \neq 0$, then we say that A is far from B and denote it by $A\ \underline{\delta}\ B$.

Example 1.1.5. Consider the set of real numbers $\mathbb{R}$ with the Euclidean metric $d(x, y) = |x - y|$, where $x, y \in \mathbb{R}$. The open interval $A = (0,1)$ is far from the open interval $B = (4,5)$, since

$$\begin{aligned} D(A,B) &= \inf\{d(a,b): a \in A, b \in B\} \\ &= \inf\{|b-a|: a \in A, b \in B\} \\ &\neq 0 \end{aligned}$$

In fact, take $a = 1 - \frac{1}{n} \in A$ and $b = 4 + \frac{1}{m} \in B$. In that case

$$|b - a| = |(4 + \frac{1}{m}) - (1 - \frac{1}{n})|$$

$$= |3 + \left(\frac{1}{m} + \frac{1}{n}\right)|,$$

and $inf\{|b - a| : a \in A, b \in B\} = inf\ |3 + \left(\frac{1}{m} + \frac{1}{n}\right)| = 3$. Hence, $D(A, B) \neq 0$.

So, A is far from B.

We will give the definition of the closure operator for a subset of a metric space and the closure points according to the proximity relation.

Definition 1.1.6. (Naimpally & Peters, 2013): Suppose that (X, d) is a metric space, $x \in X$ and $B \subseteq X$. Then x is a closure point of B provided $\{x\}\ \delta\ B$.

We can restate Definition 1.1.6. for a metric space (X, d), $x \in X$, and $B \subseteq X$, hence x is a closure point of B if and only if $D(\{x\}, B) = 0$. For simplicity we will replace $D(\{x\}, B) = 0$ with $D(x, B) = 0$.

Example 1.1.7. Consider $\mathbb{R}$ with the Euclidean metric $d(x, y) = |x - y|$, where $x, y \in \mathbb{R}$. Then the point $x = 0$ is a closure point of the open interval $A = (0,1)$.

This is because

$$\begin{aligned} D(0, A) \quad &= inf\{|0 - a| : a \in A\} \\ &= inf\{a : a \in A\} \\ &= inf(0,1) \\ &= 0\ . \end{aligned}$$

Theorem 1.1.8. (Naimpally & Peters, 2013): For every subset $A, B, C \subset X$ and points $x, y \in X$ in a metric space (X, d), the metric proximity δ satisfies the following.

$P(1)$ $A\ \delta\ B$ implies $B\ \delta\ A$.

$P(2)$ $A\ \delta\ B$ implies $A \neq \emptyset$ and $B \neq \emptyset$.

$P(3)$ $A \cap B\ \neq \emptyset$ implies $A\ \delta\ B$.

$P(4)$ $A\ \delta\ (B \cup C)$ iff $A\ \delta\ B$ or $A\ \delta\ C$.

$P(5)$ $A\ \underline{\delta}\ B$ implies that there exists $E \subset X$ such that $A\ \underline{\delta}\ E$ and $E^c\ \underline{\delta}\ B$.

$P(6)$ $\{x\}\ \delta\ \{y\}$ implies $x = y$.

If we take the set A in Theorem 1.1.8. to be the singleton set $\{x\}$, then we will obtain Theorem 1.1.9.

Theorem 1.1.9. (Naimpally & Peters, 2013): Let X be a non-empty set. For every point $x \in X$ and subsets $B, C \subset X$, we have the following.

K(1) $\{x\}\ \delta\ B$ implies $B \neq \emptyset$.

K(2) $\{x\} \cap B \neq \emptyset$ implies $\{x\}\ \delta\ B$.

K(3) $\{x\}\ \delta\ (B \cup C)$ *if and only if* $\{x\}\ \delta\ B$ or $\{x\}\ \delta\ C$.

K(4) $\{x\}\ \delta\ B$ and $\{b\}\ \delta\ C$ for each $b \in B$ implies $\{x\}\ \delta\ C$.

Definition 1.1.10. (Naimpally & Peters, 2013): Given a metric space (X, d), the closure of $B \subset X$, denoted by clB or $\bar{B}$, is the set of all points that are near B, $i.e.$,

$$clB = \{\, x \in X \ : \ \{x\}\ \delta\ B\}.$$

If $\{x\}\ \delta\ B$, x is a closure point of B.

By Definition 1.1.10, we can rewrite Theorem 1.1.9. in terms of the closure operator as the following.

Cl(1) $cl(\emptyset) = \emptyset$.

Cl(2) $B \subset clB$.

Cl(3) $cl(B \cup C) = clB \cup clC$.

Cl(4) $cl\ (clB) = clB$.

Definition 1.1.11. (Naimpally & Peters, 2013): Suppose that B is a subset of a metric space X. The point $x \in X$ is called a limit point of B if x is near $B\backslash\{x\}$.

So, the point x is a limit point of B provided $D(x, B\backslash\{x\}) = 0$.

Remark 1.1.12. (Naimpally & Peters, 2013): The set of limit points of B, denoted by $\grave{B}$, is called the derived set of B.

Now we will introduce two main types of proximity for a non-empty set X: the spatial proximity and the descriptive proximity. Non-empty sets with spatial proximity are close to each other either asymptotically or by their common points. However, the sets are descriptively close provided the sets contain one or more elements with matching descriptions. A common example of descriptive intimacy is a pair of images with matching facial features, matching eyes, hair, skin color, or matching parts such as matching nose, mouth, ear shape.

Definition 1.1.13. (Peters, 2016): Given a non-empty set X and $A, B \subset X$, then A is said to be near (close to) B if they have one or more points in common or they contain one or more points that are sufficiently close to each other.

Čech introduced axioms for the basic form of proximity which denoted by δ as follows.

Čech Proximity Axioms (Čech, 1966)

P(1) $\emptyset \ \underline{\delta} \ A$, for all $A \subset X$.
P(2) $A \ \delta \ B$ iff $B \ \delta \ A$.
P(3) $A \cap B \neq \emptyset$ implies $A \ \delta \ B$.
P(4) $A \ \delta \ (B \cup C)$ iff $A \ \delta \ B$ or $A \ \delta \ C$.

Efremovič introduced the fifth axiom of Čech proximity axioms which is called "The Separation or EF axiom". Hence, the Efremovič proximity axioms satisfy the Čech proximity axioms and the separation axiom, $i.e$,

Efremovič *Proximity Axiom* (Čech, 1966)

P(5) If $A \ \underline{\delta} \ B$, then there exists $C, D \subset X$, $C \cup D = X$ such that $A \ \underline{\delta} \ C$ and $B \ \underline{\delta} \ D$.

M.W. Lodato added to Čech proximity axioms a fifth axiom which is called "The Lodato axiom". Hence, the Lodato proximity satisfies the Čech proximity axioms and the Lodato axiom, the Lodato proximity is donate by δ_L .

Lodato Proximity Axiom (Čech, 1966)

If $A \ \delta_L \ B$ and $\{b\} \ \delta_L \ C$ for each $b \in B$, then $A \ \delta_L \ C$ where $A, B, C \subset X$ and $X \neq \emptyset$.

As we defined the closure operator for a subset of a metric space, now we will define the closure operator for a subset of a non-empty set X.

Definition 1.1.14. (Peters, 2016): Given a non-empty set X, $x \in X$, and $A \subset X$, we define the closure of a set A, denoted by clA, as the following

$$clA = \{x \in X : \{x\} \ \delta \ A\}.$$

So, we can rewrite the properties of the closure map according to Theorem 1.1.9. with the Definition 1.1.14.

Cl(1) $clA \cup clB = cl(A \cup B)$.

Cl(2) $A \subset clA$.

Cl(3) $cl\ (\emptyset) = \emptyset$.

Cl(4) $cl(clA) = clA$.

The Wallman proximity or the fine proximity, which is denoted by δ_{o}, satisfies the Čech proximity axioms with the Wallman axiom which states the following.

Wallman Proximity Axiom (Čech, 1966)

$A\ \delta_{\mathrm{o}}\ B$ if and only if $clA \cap clB \neq \emptyset$.

Example 1.1.15. Let X be a non-empty set illustrated in Figure 1, and endowed with a Čech proximity δ. Then

$A\ \delta\ E$, since $A \cap E \neq \emptyset$.

$B\ \delta\ C$, since $B \cap C \neq \emptyset$.

$A\ \underline{\delta}\ B$, $A\ \underline{\delta}\ C$, $E\ \underline{\delta}\ B$ and $E\ \underline{\delta}\ C$.

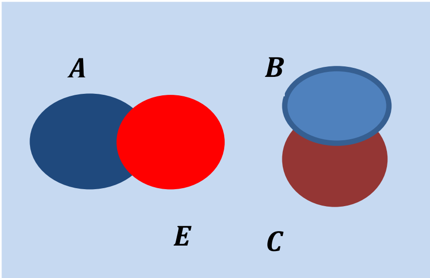

Figure 1. Remote sets are $A\underline{\delta}B$,$A\underline{\delta}C$,$E\underline{\delta}B$ and $E\underline{\delta}C$. Near sets are $A\delta E$ and $B\delta C$

In the following definition we will obtain the proximity product space and define the Čech proximity relation over it.

Definition 1.1.16. (Peters & Vergili, 2023): Given a family of proximity spaces $\{(X_i, \delta_i)\}_{i\in I}$ with an index set I, consider the product of X_i's: $X = \prod_{i\in I} X_i$ is the product of X_i's. Then define a relation $\delta = \prod_{i\in I} \delta_i$ on the subsets A and B of X, by declaring that $A\ \delta\ B$ if and only if $pr_i(A)\ \delta_i\ pr_i(B)$ for all $i \in I$, where pr_i is the i^{th} projection map of X onto X_i.

Here δ is a $\check{C}ech$ proximity on X. So, easily if we have the proximity product space $X \times Y$, then the subsets $A \times B$ and $C \times D$ of $X \times Y$ are proximally near if and only if $A\ \delta_1\ C$ and $B\ \delta_2\ D$.

We would like to study the nearness of two sets due to some descriptions like color, texture, shape and etc. In this work a description is a feature vector in an Euclidean space $\mathbb{R}^n$ which means that the description is a point in $\mathbb{R}^n$. So, to define the descriptive proximity between two subsets in a non-empty set, we need to present the definition of the probe map.

Definition 1.1.17. (Peters, 2016): Let $\phi: X \to \mathbb{R}$ be a map which represents the properties of a point of the set X such as density, color, or texture.

A probe map $\Phi: X \to \mathbb{R}^n$ is a collection of maps $(\phi_1, \phi_2, \ldots, \phi_n)$ with $\phi_i: X \to \mathbb{R}$ for $i \in \{1, 2, \ldots, n\}$.

For $p \in X$, the vector $\Phi(p) = (\phi_1(p), \phi_2(p), \ldots, \phi_n(p))$ is called a *feature vector* of p.

For $A \subset X$, $\Phi(A) = (\phi_1(A), \phi_2(A), \ldots, \phi_n(A))$ is called a *feature vector* of A.

Let $p, p' \in X$. We say that $\{p\}$ is descriptively near $\{p'\}$, denoted by $\{p\}\, \delta_\Phi\, \{p'\}$, if $\Phi(p) = \Phi(p')$.

Definition 1.1.18. (Peters, 2016): Let (X, δ_Φ) be a descriptive proximity space, and $\Phi: X \to \mathbb{R}^n$ a probe function. Given $A, B \subseteq X$, A is said to be descriptively near B, denoted by $A\, \delta_\Phi\, B$ if $\Phi(A) \cap \Phi(B) \neq \emptyset$. Otherwise, we say that A is descriptively far from B, and denote it by $A\, \underline{\delta}_\Phi\, B$.

In a descriptive proximity space, we have a descriptive intersection which does not concern about the common points between two sets, instead of this, it focuses about the common descriptive features as we will see in the following definition.

Definition 1.1.19. (Diconcillo et al., 2018): Let (X, δ_Φ) be a descriptive proximity space. The descriptive intersection of non-empty subsets A and B, denoted by $A \cap_\Phi B$, is defined by

$$A \cap_\Phi B = \{\, x \in A \cup B : \ \Phi(x) \in \Phi(A) \cap \Phi(B)\}.$$

Proposition 1.1.20. (Peters, 2019): Let (X, δ_Φ) be a descriptive proximity space. Then for non-empty subsets A and B, $A\, \delta_\Phi\, B$ if and only if $A \cap_\Phi B \neq \emptyset$.

Now we can replace the proximity δ in Čech, Efromovič, Lodato and Wallman proximity axioms with δ_Φ and get the descriptive Čech , Efromovič, Lodato and Wallman proximity axioms which are presented in the following.

Descriptive Čech Axioms (Diconcillo et al., 2018)

D(1) $\emptyset\, \underline{\delta}_\Phi\, A$, for all $A \subset X$.
D(2) $A\, \delta_\Phi\, B$ if and only if $B\, \delta_\Phi\, A$.
D(3) $A \cap_\Phi B \neq \emptyset$ if and only if $A\, \delta_\Phi\, B$.
D(4) $A\, \delta_\Phi\, (B \cup C)$ if and only if $A\, \delta_\Phi\, B$ or $A\, \delta_\Phi\, C$.

Descriptive Efromovič Axioms (Diconcillo et al., 2018)

D(1) $\emptyset\, \underline{\delta}_\Phi\, A$, for all $A \subset X$.
D(2) $A\, \delta_\Phi\, B$ if and only if $B\, \delta_\Phi\, A$.
D(3) $A \cap_\Phi B \neq \emptyset$ if and only if $A\, \delta_\Phi\, B$.

D(4) $A\ \delta_\Phi\ (B \cup C)$ if and only if $A\ \delta_\Phi\ B$ or $A\ \delta_\Phi\ C$.

D(5) If $A\ \underline{\delta}_\Phi\ B$, then there exists $C, D \subset X$, $C \cup D = X$ such that $A\ \underline{\delta}_\Phi\ C$ and $B\ \underline{\delta}_\Phi\ D$.

Descriptive Lodato Axiom (Diconcillo et al., 2018)

If $A\ \delta_\Phi\ B$ and $\{b\}\ \delta_\Phi C$ for each $b \in B$, then $A\ \delta_\Phi\ C$, where $A, B, C \subset X$ and $X \neq \emptyset$.

To define descriptive Wallman axiom, we need to define a descriptive form of the closure of a nonempty set.

Definition 1.1.21. (Peters, 2014): Given a non-empty set X, $x \in X$ and $A \subset X$, the descriptive closure of the set A, denoted by $cl_\Phi\ A$, is given by

$$cl_\Phi\ A = \{x \in X : \{x\}\ \delta_\Phi\ A\}.$$

Descriptive Wallman Axiom (Diconcillo et al., 2018)

$A\ \delta_\Phi\ B$ if and only if $cl_\Phi\ A \cap cl_\Phi\ B \neq \emptyset$.

Example 1.1.22. Consider the Rubik's Cube and its some subsets equipped with Wallman proximity δ and Wallman descriptive proximity δ_Φ respectively defined as follows

$A\ \delta_\Phi\ B :\Leftrightarrow A$ and B have the same surface color.

$A\ \delta\ B\ \ :\Leftrightarrow A$ and B have points in common.

By the illustration in Figure 2, we have

$A\ \delta_\Phi\ E$ and $A\ \delta\ E$, since $cl_\Phi\ A \cap cl_\Phi\ E \neq \emptyset$ and $clA \cap cl\ E \neq \emptyset$.

$A\ \underline{\delta}_\Phi C$ and $A\ \delta\ C$, since $cl_\Phi\ A \cap cl_\Phi\ C = \emptyset$ and $cl\ A \cap cl\ C \neq \emptyset$.

$B\ \delta_\Phi\ H$ and $B\ \underline{\delta}\ H$, since $cl_\Phi\ B \cap cl_\Phi\ H \neq \emptyset$ and $cl\ B \cap cl\ H = \emptyset$.

$A\ \underline{\delta}_\Phi\ H$ and $A\ \underline{\delta}\ H$, since $cl_\Phi\ A \cap cl_\Phi\ H = \emptyset$ and $cl\ A \cap cl\ H = \emptyset$.

From this example we can say that any two nonempty sets can be both spatially and descriptively near such as for A and E, spatially far and descriptively near such as B and H, spatially near and descriptively far such as A and C or neither spatially near nor descriptively near such as for A and H.

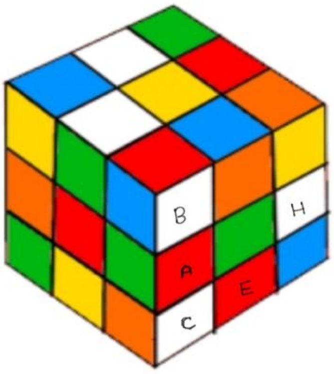

Figure 2. Descriptively close, spatially remote Rubik's Cube polygons where the picture is taken from (URL-1, 2009)

Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be two descriptive proximity spaces. We will discuss how one can define a descriptive proximal relation on $X \times Y$. In the following definition and proposition, we answer this question.

Definition 1.1.23. (Peters & Vergili, 2023): Given a family of descriptive proximity spaces $\{(X_i, \delta_{\Phi_i})\}_{i\in I}$ with an index set I, let $X = \prod_{i\in I} X_i$ be the product of X_i's. Define a probe map $\Phi = \prod_{i\in I} \Phi_i$ on the subsets A and B of X. Then $A \, \delta_\Phi \, B$ if and only if $pr_i(A) \, \delta_{\Phi_i} \, pr_i(B)$ for all $i \in I$ where pr_i is the i^{th} projection map of X onto X_i .

Proposition 1.1.24. (Peters & Vergili, 2023): The product of descriptive proximity spaces is a descriptive proximity space.

We will present the proximally continuous (pc) maps and descriptive proximally continuous (dpc) maps which come from two dynamical spaces with same or different proximities. Also, we will define the invariant and descriptive invariant property for a subset of proximity and descriptive proximity spaces respectively.

Define the proximally continuous maps between two proximity spaces as follows

Definition 1.1.25. (Peters & Vergili, 2020): Suppose (X, δ_1) and (Y, δ_2) are proximity spaces. Then a map $f: (X, \delta_1) \to (Y, \delta_2)$ is a proximally continuous (pc) map if and only if for $A, B \subseteq X$ if $A \, \delta_1 B$, then $f(A) \, \delta_2 \, f(B)$.

From Definition 1.1.25. we can say that the nearness of two subsets in a proximity space X is preserved by f.

Example 1.1.26. Let $(\mathbb{R}, \delta)$ be a proximity space, where the proximity relation δ is given by

$$A \; \delta \; B \text{ if and only if } D(A, B) = 0 \text{ for } A, B \subseteq \mathbb{R}.$$

If $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = a$ for a fixed $a \in \mathbb{R}$, then f is a pc map. In fact, Let $A \; \delta \; B$ in $\mathbb{R}$, then $f(A) = f(B) = \{a\}$. Thus, $f(A) \; \delta \; f(B)$, so f is pc.

Definition 1.1.27. (Peters & Vergili, 2020): Let (X, δ_1) and (Y, δ_2) be proximity spaces. A map $f: (X, \delta_1) \to (Y, \delta_2)$ is called a proximal isomorphism provided f has a proximally continuous inverse f^{-1}.

Example 1.1.28. Let $(\mathbb{R}, \delta)$ be a proximity space, where δ is defined by $A \; \delta \; B$ if and only if $D(A, B) = 0$ for $A, B \subseteq \mathbb{R}$. If $f: \mathbb{R} \to \mathbb{R}$ is the identity map, then f is a pc map, since for $A \; \delta \; B$ we have $A = f(A) \; \delta \; f(B) = B$. However, $f = f^{-1}$, so f is proximal isomorphism.

Example 1.1.29. Let $f: (X, \delta_1) \times (Y, \delta_2) \to (Y, \delta_2)$ be a map defined by $f(x, y) = y$. Then f is pc. In fact, let $A \times B$ and $C \times D$ be proximally near subsets of $X \times Y$ so that $A \; \delta_1 \; C$ and $B \; \delta_2 \; D$. Then $f(A \times B) = B$ and $f(C \times D) = D$, and $B \; \delta_2 \; D$. So, f is pc.

Example 1.1.30. For fixed element $x \in X$, let $g: (Y, \delta_2) \to (X, \delta_1) \times (Y, \delta_2)$ be a map defined by $g(y) = (x, y)$. Then g is pc.

Proof: Assume B and D are proximally near subsets of Y so that $B \; \delta_2 \; D$. Then $g(B) = \{x\} \times B$ and $g(D) = \{x\} \times D$. Hence, $g(B)$ and $g(D)$ are proximally near subsets of $X \times Y$ because $\{x\} \; \delta_1 \; \{x\}$ and $B \; \delta_2 \; D$.

Proposition 1.1.31. (Peters & Vergili, 2021): Let (X, δ_1), (Y, δ_2) and (Z, δ_3) be proximity spaces, and $f: (X, \delta_1) \to (Y, \delta_2)$ and $g: (Y, \delta_2) \to (Z, \delta_3)$ pc maps. Then their composition $g \circ f: (X, \delta_1) \to (Z, \delta_3)$ is also a pc map.

Definition 1.1.32. (Peters & Vergili, 2020): Given a proximity space (X, δ) and a pc map $f: (X, \delta) \to (X, \delta)$, a subset $B \subseteq X$ is said to be invariant with respect to f provided $f(B) \subseteq B$.

Theorem 1.1.33. (Peters & Vergili, 2020): If B is an invariant set with respect to f, then $f^n(B) \subseteq B$ for all positive integer n, where f^n means that f occurs n times.

Theorem 1.1.34. (Peters & Vergili, 2020): Suppose that (X, δ) is a proximity space and $f: (X, \delta) \to (X, \delta)$ is a pc map. If $\{B_i\}_{i \in I} \subseteq X$ is a collection of invariant sets with respect to f, then

i. $\cup_{i \in I} B_i$ is an invariant set with respect to f,

ii. $\cap_{i \in I} B_i$ is an invariant set with respect to f.

Theorem 1.1.35. (Peters & Vergili, 2020): Given a proximity space (X, δ) and a pc self-map $f: (X, \delta) \to (X, \delta)$, If $B \subseteq X$ is invariant with respect to f, then clB is also an invariant with respect to f.

Definition 1.1.36. (Peters & Vergili, 2020): Let (X, δ) be a proximity space, and $f: (X, \delta) \to (X, \delta)$ be a pc map. Then for $E \subseteq X$,

i. E is a fixed subset of f provided $f(E) = E$.

ii. E is an eventual fixed subset of f provided E is not a fixed subset while $f^n(E)$ is a fixed subset of f for some positive integer $n \neq 1$.

iii. E is an almost (approximate) fixed subset of f provided $f(E) = E$ or $f(E)\ \delta\ E$.

Remark 1.1.37. From Definition 1.1.32. we note that if E is a fixed subset of f, then E is an invariant with respect to f and E is an almost (approximate) fixed subset of f.

Example 1.1.38. Let X and its subsets be given as in Figure 3. We impose the nearness relation δ on the subsets of X as follows.

$$A\ \delta\ B \text{ if and only if } A \cap B \neq \emptyset\ for\ A, B \subseteq X.$$

Let $f: X \to X$ be a map defined on the subsets of X by

$f(A) = C$, $f(C) = A$, $f(E) = B$, $f(B) = E$ and $f(N) = E$.

Then f is a pc map because

$A\ \delta\ E$ implies $f(A) = C\ \delta\ B = f(E)$,

$C\ \delta\ B$ implies $f(C) = A\ \delta\ E = f(B)$,

$N\ \delta\ B$ implies $f(N) = E\ \delta\ E = f(B)$, and

$N\ \delta\ C$ implies $f(N) = E\ \delta\ A = f(C)$.

However, there is no fixed subset of f and no invariant subset too, but the set A is an eventual fixed subset, since $f^2(A) = f(C) = A$, and this is also true for the sets C, B, and E.

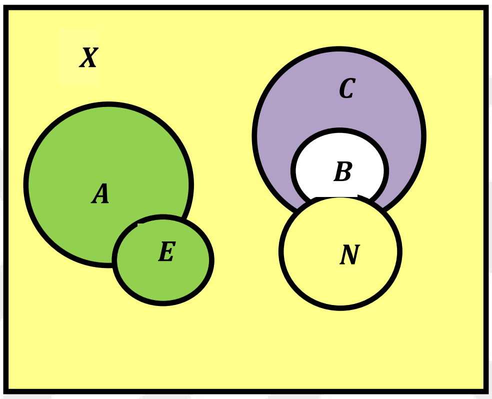

Figure 3. A proximally continuous map f in (X, δ)

Now we will introduce descriptive proximally continuous maps between descriptive proximity spaces.

Definition 1.1.39. (Peters & Vergili, 2020): Suppose (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) are descriptive proximity spaces with probe maps $\Phi_1: X \to \mathbb{R}^n$, $\Phi_2: Y \to \mathbb{R}^n$, respectively. Then a map $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is a descriptive proximally continuous (dpc) map if and only if for A,$B \subseteq X$, $A\ \delta_{\Phi_1} B$ implies $f(A)\ \delta_{\Phi_2}\ f(B)$.

Recall that $f(A)\ \delta_{\Phi_2}\ f(B)$ is equivalent to $f(A) \cap_{\Phi} f(B) \neq \emptyset$.

Definition 1.1.40. (Peters & Vergili, 2020): Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be descriptive proximity spaces. Then a map $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is a descriptive proximal isomorphism provided f is a dpc map with a descriptive proximally continuous inverse f^{-1}.

Proposition 1.1.41. (Peters & Vergili, 2021): Let (X, δ_{Φ_1}), (Y, δ_{Φ_2}) and (Z, δ_{Φ_3}) be descriptive proximity spaces, and $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ and $g: (Y, \delta_{\Phi_2}) \to (Z, \delta_{\Phi_3})$ be dpc maps. Then the composition $g \circ f: (X, \delta_{\Phi_1}) \to (Z, \delta_{\Phi_3})$ is a dpc map.

Definition 1.1.42. (Peters & Vergili, 2020): Given (X, δ_{Φ_1}) a descriptive proximity space and $f: (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ a dpc map, a subset $B \subseteq X$ is said to be descriptively invariant with respect to f provided $\Phi(f(B)) \subseteq \Phi(B)$.

Theorem 1.1.43. (Peters & Vergili, 2020): If B is a descriptively invariant set with respect to f, then $\Phi(f^n(B)) \subseteq \Phi(B)$ for all positive integer n.

Theorem 1.1.44. (Peters & Vergili, 2020): Suppose (X, δ_Φ) is a descriptive proximity space and $f: (X, \delta_\Phi) \to (X, \delta_\Phi)$ is a dpc map. If $\{B_i\}_{i \in I} \subseteq X$ is a collection of descriptively invariant sets with respect to f, then

i. $\cup_{i \in I} B_i$ is a descriptively invariant set with respect to f.

ii. $\cap_{i \in I} B_i$ is a descriptively invariant set with respect to f.

Theorem 1.1.45. (Peters & Vergili, 2020): Given a descriptive proximity space (X, δ_Φ) and a dpc map $f: (X, \delta_\Phi) \to (X, \delta_\Phi)$, if $B \subseteq X$ is descriptively invariant with respect to f, then $cl_\Phi\, B$ is also descriptively invariant with respect to f.

Now we will give the construction of invariant, fixed, eventual and almost fixed subsets of a descriptive proximity space.

Definition 1.1.46. (Peters & Vergili, 2020): Let (X, δ_Φ) be a descriptive proximity space, and $f: (X, \delta_\Phi) \to (X, \delta_\Phi)$ a dpc map. Then for $E \subseteq X$

i. E is a descriptive fixed subset of f provided $\Phi(f(E)) = \Phi(E)$,

ii. E is a descriptive eventual fixed subset of f provided E is not a descriptive fixed subset, but $f^n(E)$ is a descriptive fixed subset of f for some positive integer $n \neq 1$,

iii. E is an almost (approximate) descriptive fixed subset of f provided $\Phi(f(E)) = \Phi(E)$ or $\Phi\big(f(E)\big)\ \delta_\Phi\ \Phi(E)$, or equivalently $f(E) \cap_\Phi E \neq \emptyset$.

Let (X, δ_Φ) be a descriptive proximity space. Then one can define the descriptive equality between two subsets of X which is different from the usual equality.

Definition 1.1.47. (Peters & Vergili, 2020): Given a descriptive proximity space (X, δ_Φ), let $\Phi: X \to \mathbb{R}^n$ be its probe function, and $A, B \subseteq X$. A and B are said to be descriptively proximal equal provided $\Phi(A) = \Phi(B)$, denoted by $A =_{des} B$.

Remark 1.1.48. (Peters & Vergili, 2020): If (X, δ_Φ) is a descriptive proximity space with a probe map $\Phi: X \to \mathbb{R}^n$ and $A, B \subseteq X$, then $A = B$ implies $A =_{des} B$.

Remark 1.1.49. (Peters & Vergili, 2020): We can rewrite the Definition 1.1.46. using the Definition 1.1.47. as the following.

i. E is a descriptive fixed subset of f provided $f(E) =_{des} E$.

ii. E is a descriptive eventual fixed subset of f provided $f^n(E) =_{des} E$ for some $n \neq 1$.

iii. E is an almost (approximate) descriptive fixed subset of f provided $f(E) =_{des} E$ or $f(E) \cap_\Phi E \neq \emptyset$.

Example 1.1.50. Let (X, δ_Φ) be a descriptive proximity space as shown in Figure 4, where the descriptive nearness relation δ_Φ is defined as the following.

For $A, B \subseteq X$, $A \ \delta_\Phi \ B$ if and only if they have points that have the same color. Let $f: (X, \delta_\Phi) \to (X, \delta_\Phi)$ be defined as follows.

$f(A) = B, f(B) = B, f(C) = E$ and $f(E) = C$. Then f is a dpc map. Also notice that:

$A \ \delta_\Phi \ B$ implies $f(A) = B \ \delta_\Phi \ B = f(B)$.

$C \ \delta_\Phi \ E$ implies $f(C) = E \ \delta_\Phi \ C = f(E)$.

B is a descriptive fixed subset, since $f(B) =_{des} B$. Furthermore, C and E are descriptively eventual, since $f^2(C) =_{des} C$, $f^2(E) =_{des} E$, respectively.

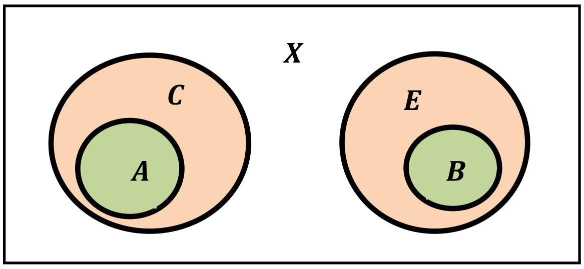

Figure 4. A descriptive proximally continuous map f in (X, δ_Φ)

Example 1.1.51. Let $f: (X, \delta_{\Phi_1}) \times (Y, \delta_{\Phi_2}) \rightarrow (Y, \delta_{\Phi_2})$ be a map defined by $f(x, y) = y$, then f is dpc.

To see this, let $A \times B$ and $C \times D$ be descriptive proximally near subsets of $X \times Y$ where

A, $C \subseteq X$ and B,$D \subseteq Y$. Then $f(A \times B) = B$ and $f(C \times D) = D$, but $B\ \delta_{\Phi_2} D$. So, f is a dpc map.

Example 1.1.52. For some fixed point element $x_0 \in X$, let

$$g: (Y, \delta_{\Phi_2}) \rightarrow (X, \delta_{\Phi_1}) \times (Y, \delta_{\Phi_2})$$

be a map defined by $g(y) = (x_0, y)$. Then g is a dpc map.

In fact, assume that B and D are descriptive proximally near subsets of Y: $B\ \delta_{\Phi_2} D$. Then $g(B) = \{x_0\} \times B$ and $g(D) = \{x_0\} \times D$. Hence $g(B)$ and $g(D)$ are descriptive proximally near subsets of $X \times Y$ because $\{x_0\}\ \ \delta_{\Phi_1} \{x_0\}$ and $B\ \delta_{\Phi_2} D$.

Theorem 1.1.53. (Peters & Vergili, 2020): Suppose rbE is a ribbon with intersecting cycles having n vertices in the intersection and with k bridge edges. Then

$$\beta_\alpha(rbE) = 2k + n.$$

Example 1.1.54. Let $X = \{rbE_1, rbE, rbE_2\}$ shown in Figure 5(a), and $Y = \{rbA, rbB\}$ shown in Figure 5(b). Assume that X and Y are equipped with the same descriptive proximity relation:

$$A\ \delta_\Phi\ B\ \ :\Leftrightarrow\ \ \Phi(A) = \beta_\alpha(A) =\ \beta_\alpha(B) = \Phi(B)$$

for a pair A, B subset of X or Y. Let $f: (X, \delta_\Phi) \to (Y, \delta_\Phi)$ be defined on the subsets rbE, rbE_1 and rbE_2 by

$$f(rbE) = rbB, f(rbE_1) = rbA \text{ and } f(rbE_2) = rbA.$$

Then f is a dpc map.

$rbE_1 \, \delta_\Phi \, rbE_2$ since $\beta_\alpha(rbE_1) = \beta_\alpha(rbE_2) = 1$ and $f(rbE_1) = rbA \, \delta_\Phi \, rbA = f(rbE_2)$. So f is a dpc map.

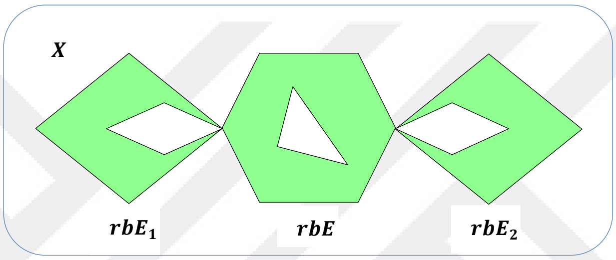

Figure 5(a). A cell complex X with three ribbons rbE_1, rbE and rbE_2

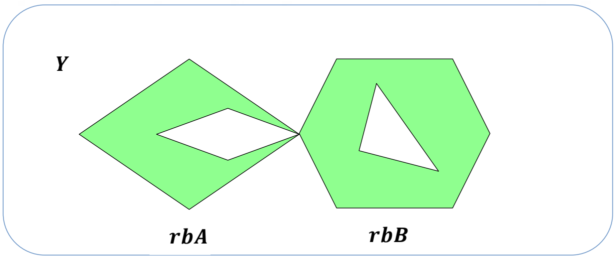

Figure 5(b). A cell complex Y with two ribbons rbA and rbB

Figure 5.

Remark 1.1.55. In Example 1.1.54. and 1.1.38, we examine only one case to show that f is a dpc / pc map. It is enough to consider the non-trivial cases, $i.e.$, we do not consider the cases where the set is related with itself, because it is clearly that the continuity condition will be directly satisfied. Also we do not consider the cases where the sets are not related, because the statement will remain true logically.

Topological conjugacy between maps is a very strong tool. Since, we can specify the properties of a space, and make a comment on the properties of other spaces, because we have the fact that most of the topological properties of the space are preserved under topological conjugacy map.

Definition 1.1.56. (Peters & Vergili, 2020): The pc maps $f:(X,\delta_1)\to(X,\delta_1)$ and $g:(Y,\delta_2)\to(Y,\delta_2)$ are said to be proximal conjugates if there is a proximal isomorphism $h:(X,\delta_1)\to(Y,\delta_2)$ such that $g\circ h=h\circ f$. Here, the map h is called a proximal conjugacy between f and g.

Theorem 1.1.57. (Peters & Vergili, 2020): Suppose h is a proximal conjugacy between $f:(X,\delta_1)\to(X,\delta_1)$ and $g:(Y,\delta_2)\to(Y,\delta_2)$. Then for each $A\subseteq X$ and $n\in\mathbb{Z}^+$, we have $g^n\circ h\,(A)=h\circ f^n(A)$.

Remark 1.1.58. For a proximal conjugacy h between f and g, we have $g\circ h\,(A)=h\circ f(A)$ for each $A\subseteq X$, since h is a proximal isomorphism. Then instead of h, we can rewrite the condition of the proximal conjugacy between f and g in terms of h^{-1} as the following. For each $C\subseteq Y$, $h^{-1}\circ g(C)=f\circ h^{-1}(C)$, which is obvious by composing both sides of $g\circ h\,(A)=h\circ f(A)$ with h^{-1}.

In the next theorem we will show that the proximal conjugacy map preserves some properties for a subspace of proximity space like the fixed set property, eventual fixed set property and almost fixed set property.

Theorem 1.1.59. (Peters & Vergili, 2020): For a proximal conjugacy h between $f:(X,\delta_1)\to(X,\delta_1)$ and $g:(Y,\delta_2)\to(Y,\delta_2)$, the following hold.

i. If E is a fixed subset of f, then $h(E)$ is a fixed subset of g.
ii. If E is an eventual fixed subset of f, then $h(E)$ is an eventual fixed subset of g.
iii. If E is an almost fixed subset of f, then $h(E)$ is an almost fixed subset of g.

Now we will present the weakly proximal conjugacy between two proximity spaces.

Definition 1.1.60. (Peters & Vergili, 2020): Two pc maps $f:(X,\delta_1)\to(X,\delta_1)$ and $g:(Y,\delta_2)\to(Y,\delta_2)$ are said to be weakly proximal conjugates if there is a proximal isomorphism $h:(X,\delta_1)\to(Y,\delta_2)$ such that for $A\subseteq X$, $g\circ h(A)\ \delta_2\ h\circ f(A)$. The map h is called a weakly proximal conjugacy between f and g.

Corollary 1.1.61. (Peters & Vergili, 2020): A proximal conjugacy h between $f:(X,\delta_1)\to(X,\delta_1)$ and $g:(Y,\delta_2)\to(Y,\delta_2)$, is also a weakly proximal conjugacy between f and g.

Theorem 1.1.62. (Peters & Vergili, 2020): Suppose h is a weakly proximal conjugacy between $f:(X,\delta_1)\to(X,\delta_1)$ and $g:(Y,\delta_2)\to(Y,\delta_2)$. Then for each $B\subseteq X$ and $n\in\mathbb{Z}^+$, we have $g^n\circ h\ (B)\ \delta_2\ h\circ f^n(B)$.

Remark 1.1.63. Since any proximal conjugacy map between two proximally continuous maps is weakly proximal conjugacy, we can conclude that any statement which is true for the proximal conjugacy will be also true for the weakly proximal conjugacy.

We will present the proximal descriptive conjugacy between two descriptive proximity spaces and show how some properties under descriptive conjugacy map are preserved.

Definition 1.1.64. (Peters & Vergili, 2020): Two dpc maps $f:\left(X,\delta_{\Phi_1}\right)\to\left(X,\delta_{\Phi_1}\right)$ and $g:\left(Y,\delta_{\Phi_2}\right)\to\left(Y,\delta_{\Phi_2}\right)$ are said to be descriptive proximal conjugates if there is a descriptive proximal isomorphism $h:\left(X,\delta_{\Phi_1}\right)\to\left(Y,\delta_{\Phi_2}\right)$ such that $g\circ h(A)=_{des} h\circ f(A)$ for all $A\subseteq X$. Here h is said to be descriptive proximal conjugacy between f and g.

Remark 1.1.65. From Definition 1.1.64. it is not necessary that $g\circ h(A)=h\circ f(A)$, but their descriptions with respect to the probe map Φ_2 must be equal, $i.e.$, $\Phi_2\big(g\circ h(A)\big)=\Phi_2\big(h\circ f(A)\big)$. Also, we can write this condition in terms of h^{-1} and get the following.

For any $C\subseteq Y$, $\Phi_1\big(h^{-1}\circ g(C)\big)=\Phi_1\big(f\circ h^{-1}(C)\big)$. Hence we have the following commutative diagrams (Peters & Vergili, 2021).

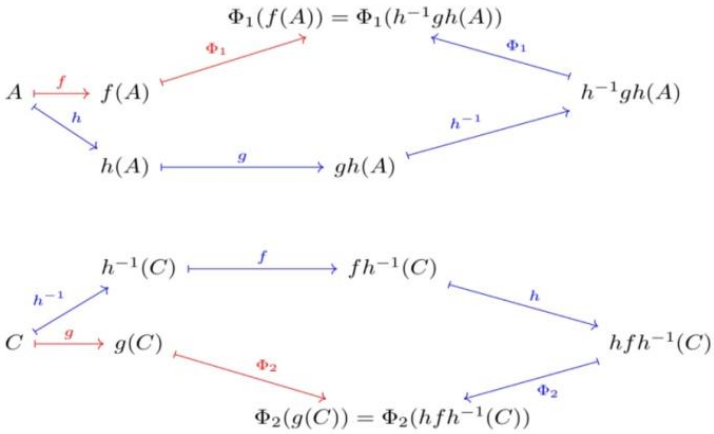

Figure 6. Descriptive proximal conjugacy commutative diagrams

A descriptive proximal conjugacy map preserves the descriptive equality of two subsets of descriptive proximity space as the following.

Lemma 1.1.66. (Peters & Vergili, 2020): For descriptive proximal conjugacy h between $f: (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g: (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$ and for $A, B \subseteq X$ if $A =_{des} B$, then $h(A) =_{des} h(B)$.

The descriptive proximal conjugacy between two descriptive proximally continuous maps remains true for iterations of maps for any positive integer.

Theorem 1.1.67. (Peters & Vergili, 2020): Suppose h is a descriptive proximal conjugacy between $f: (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g: (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$. Then for each $A \subseteq X$ and $n \in \mathbb{Z}^+$, we have $g^n \circ h\ (A) =_{des} h \circ f^n(A)$.

Theorem 1.1.68. (Peters & Vergili, 2020): For descriptive proximal conjugacy h between $f: (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g: (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$, we have the following.

i. If E is a descriptively fixed subset of f, then $h(E)$ is a descriptively fixed subset of g.

ii. If E is an eventual descriptively fixed subset of f, then $h(E)$ is an eventual descriptively fixed subset of g.

iii. If E is an almost descriptively fixed subset of f, then $h(E)$ is an almost descriptively fixed subset of g.

Since any descriptively equal subsets are descriptively near, then we can restrict the condition of the descriptive proximal conjugacy to the descriptive nearness as a necessary and sufficient condition, so in this case the conjugacy map is called a weakly descriptive proximal conjugacy map and defined as below.

Definition 1.1.69. (Peters & Vergili, 2020): Two dpc maps $f: (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g: (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$ are said to be weakly descriptive proximal conjugates if there is a descriptive proximal isomorphism $h: \left(X, \delta_{\Phi_1}\right) \to (Y, \delta_{\Phi_2})$ such that for $A \subseteq X$, $g \circ h(A)\, \delta_{\Phi_2} h \circ f(A)$. The map h is called weakly descriptive proximal conjugacy between f and g.

Corollary 1.1.70. A descriptive proximal conjugacy h between $f: (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g: (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$ is also weakly descriptive proximal conjugacy between f and g.

Proof: It is clear from Definition 1.1.64.

Theorem 1.1.71. (Peters & Vergili, 2020): Suppose h is a weakly descriptive proximal conjugacy between $f: (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g: (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$, then for all $A \subseteq X$ and $n \in \mathbb{Z}^+$, we have $g^n \circ h(A)\ \delta_{\Phi_2}\ h \circ f^n(A)$.

Remark 1.1.72. For weakly descriptive proximal conjugacy h between f and g we have $g \circ h(A)\ \delta_{\Phi_2}\ h \circ f(A)$ for all $A \subseteq X$, which means that $g \circ h(A) \cap_{\Phi_2} h \circ f(A) \neq \emptyset$. Since h is descriptive proximal isomorphism, then h^{-1} is also a descriptive proximally continuous map. Hence, $h^{-1} \circ g(C)\ \delta_{\Phi_1}\ f \circ h^{-1}(C)$ which means that

$$h^{-1} \circ g\ (C) \cap_{\Phi_1} f \circ h^{-1}(C) \neq \emptyset \text{ for all } C \subseteq Y.$$

1.2. CW Complex Spaces

Knowing the ways of solving problems facing a country, government, institution, home or individual is one of the eternal discussions that man has sought. René Descartes, 380 years ago, gave us some excellent ideas on this subject. In fact, Descartes introduced four main steps to solving any problem, and this is the second step which I will highlight as Descartes says "To divide each of the difficulties under examination into as many parts as possible, and as might be necessary for its adequate solution". What Descartes meant was to simplify the vision of the problem as much as possible, divide it into its components. The largest possible part, the smallest possible part, dealing with them individually will make the problem clearer and more understandable, and then solve the problem faster and more efficiently. This process is also called "decomposition". It is noteworthy that Descartes was not the only one who demonstrated the importance of the idea of decomposition, as George Pólya stated it, saying "If you cannot solve a problem, there is an easier problem that you can solve, find it". For example, to find the total area of the shapes, we decompose it into triangles and find the area of triangles separately, then find the total sum.

Now we will present simplicial complexes and CW-complexes and these are two examples of the idea of "decomposition". From the simplicial complex new point, a space is decomposed into triangles of different dimensions, aka simplices, and from the simplicial complex new point, a space is decomposed into balls, aka cells. But the question here is why we divide the topological spaces into triangles or balls. For the idea of homology, the singular homology, and the simplicial homology, we refer to (Hatcher, 2002 ; Rotman, 2009 ; Vick, 1994).

Definition 1.2.1. (Boyd & Vandenberghe, 2004): Let $\mathbb{R}^n$ be an Euclidean space, and $A \subseteq \mathbb{R}^n$. Then we say that A is an affine space provided for every distinct points x and y in A, the line joining x and y is contained in A.

Definition 1.2.2. (Boyd & Vandenberghe, 2004): A subset E in a Euclidean space $\mathbb{R}^n$ is said to be convex provided for every pair of distinct points x and y in E, the line segment $tx + (1-t)y$ should lie in E.

Definition 1.2.3. (Boyd & Vandenberghe, 2004): Let $p_0, p_1, \dots, p_k$ be points in $\mathbb{R}^n$. Then

i. The affine combination of $p_0, p_1, \dots, p_k$ is a point x in $\mathbb{R}^n$ with $x = t_0p_0 + t_1p_1 + \cdots + t_kp_k$ such that $\sum_{i=0}^{k} t_i = 1$.

ii. The convex combination of $p_0, p_1, ..., p_k$ is a point x in $\mathbb{R}^n$ with $x = t_0p_0 + t_1p_1 + \cdots + t_kp_k$ such that $\sum_{i=0}^{k} t_i = 1$ and for all i, $t_i \geq 0$.

iii. The convex hull of a set of points $\{p_0, p_1, ..., p_k\}$ in $\mathbb{R}^n$ is the set of all points $x \in \mathbb{R}^n$ that are convex combination of the given points.

Definition 1.2.4. (Yan, 2016): Let $\{p_0, p_1, ..., p_k\} \subseteq \mathbb{R}^n$ be an ordered set of points. Then the set is affine independent, if the set $\{p_1 - p_0, p_2 - p_0, ..., p_k - p_0\}$ is a linearly independent subset of $\mathbb{R}^n$.

Definition 1.2.5. (Hatcher, 2002): Let $\{p_0, p_1, ..., p_k\}$ be an affine independent subset of $\mathbb{R}^n$.

i. Let A be the affine set spanned by this set, that is if $x \in A$ such that $x = \sum_{i=0}^{k} t_i p_i$ and $\sum_{i=0}^{k} t_i = 1$. In that case $(t_0, t_1, \dots, t_k)$ is called the barycentric coordinates of x.

ii. A k-simplex $[p_0, p_1, \dots, p_k]$ is the smallest convex set spanned by these points, $i.e.$ if $x \in [p_0, p_1, \dots, p_k]$, then $x = \sum_{i=0}^{k} t_i p_i$, $\sum_{i=0}^{k} t_i = 1$ and for all i, $t_i \geq 0$.

As shown in the Figure 7, a 0-dimensional simplex, called vertex, is a point, a 1-dimensional simplex, called edge, is a line segment, a 2-dimensional simplex is a triangle, and a 3-dimensional simplex is a tetrahedron. So, in general, an n-dimensional simplex is an n-dimensional triangle.

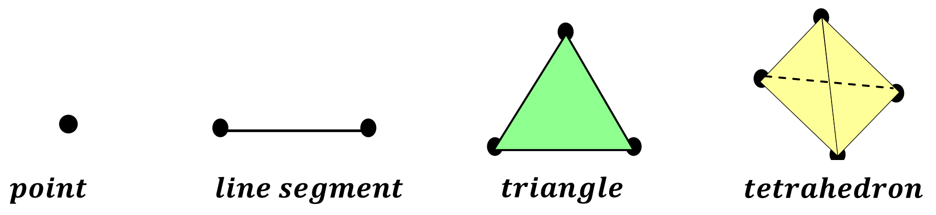

Figure 7. Simplices up to dimension 3

Definition 1.2.6. (Hatcher, 2002): Let σ be a k-simplex. Then for $0 \leq m \leq k$ an m-face of σ is an m-simplex that is the convex hull of a nonempty set of the vertices.

The Figure 8, shows 0-face, 1-face and 2-face, respectively of a 3-simplex. Notice that the 3-face of a 3-simplex is itself.

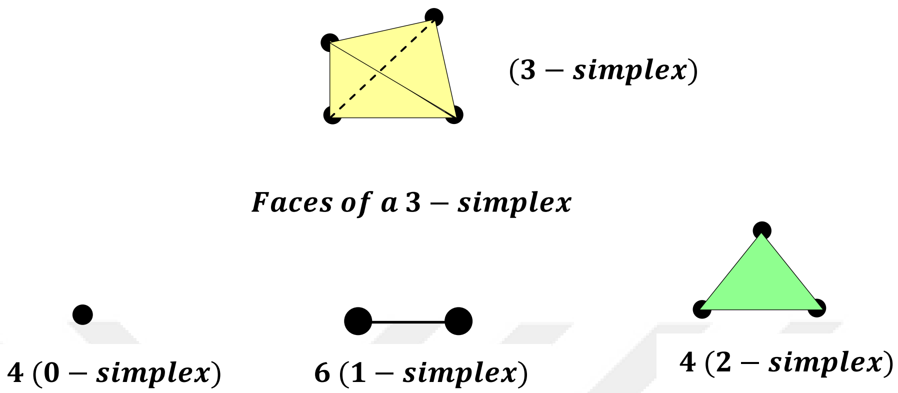

Figure 8. All faces of a 3-simplex

Definition 1.2.7. (Yan, 2016): A simplicial complex is a collection K of simplices in a Euclidean space $\mathbb{R}^n$ such that the following conditions hold.

i. If $\sigma \in K$, then the faces of σ are also in K.

ii. If $\sigma, \tau \in K$, then $\sigma \cap \tau$ is either empty or one common face of σ and τ.

In this thesis, we assume that K is locally finite. That is, every point of $\mathbb{R}^n$ has a neighborhood that intersects with only finitely many simplices in K.

Definition 1.2.8. (Gray, 1975): The geometric realization $|K|$ of a simplicial complex K is the union of all simplices in K.

Proposition 1.2.9. (Gray, 1975): If the ordered set of points $\{p_0, p_1, ..., p_k\}$ in $\mathbb{R}^n$ is affine independent, then the affine set spanned by $\{p_0, p_1, ..., p_k\}$ has a unique expression as an affine combination $x = \sum_{i=0}^{k} t_i\, p_i$ and $\sum_{i=0}^{k} t_i = 1$.

To construct a simplicial complex, we will follow these steps iteratively:

1) Start by adding 0-dimensional vertices (0-simplices).

2) Next add 1-dimensional edges (1-simplices).
3) Add 2-dimensional triangles (2-simplices).
4) Add 3-dimensional tetrahedrons (3-simplices).
5) Add n-dimensional n-simplices.

Note that for adding 1-simplices there must be one edge between two vertices which means that there is no loops and the edge is unique according to Proposition 1.2.9. In Figure 9, we can see the steps for constructing a simplicial complex of dimension 3.

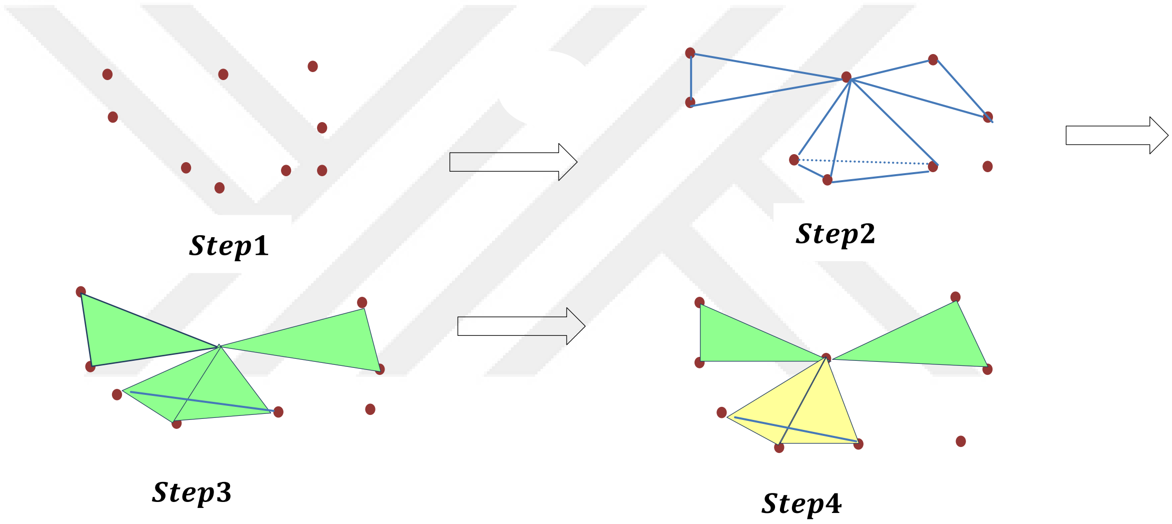

Figure 9. Construction of a simplicial complex of dimension 3

Definition 1.2.10. (Gray, 1975): A topological space homeomorphic to $|K|$ for some K is called a polyhedron. By a triangulation of a space X, we mean a complex K such that $|K|$ and X are homeomorphic $|K| \equiv X$.

Since the simplices are straight objects and for spaces such as circle, torus and sphere are curved spaces, so to study their topological properties by simplicial method, we need to find a homeomorphism from simplices to the curved objects.

Example 1.2.11. We find the triangulation of the following objects.

i. Unit circle $S^1 = \{ x \in \mathbb{R}^2 : ||x|| = 1 \}$.

ii. Unit disk $D^2 = \{ x \in \mathbb{R}^2 : ||x|| \leq 1 \}$.

iii. Unit sphere $S^2 = \{\, x \in \mathbb{R}^3 : \|x\| = 1 \,\}$.

For S^1, we need three 0 −simplices and three 1 −simplices, so it is homeomorphic to non-filled triangle. The disk is homeomorphic to the geometric realization of a 2 −simplex and the sphere is homeomorphic to the geometric realization of a 3 −simplex. Why is CW complex important if we already have simplicial complex which solves the problem of decomposing the space into smaller pieces? Min Yan answered this question in his book, and said "simplicity comes at a cost", which means that if we break down the spaces into such smaller pieces, the process will be very complex and take time at length. He also added that "the concept of CW-complexes strikes a good balance between simplicity and usability " (Yan, 2016).

Definition 1.2.12. (Yan, 2016): A CW −structure on a topological space X is a sequence of subspaces

$$X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^n \subset \cdots \subset X = \cup X^n.$$

such that X^n is obtained from X^{n-1} by attaching some n−dimensional closed disks D^n, and X carries a suitable topology.

Definition 1.2.13. (Munkres, 1984):

i. A zero-dimensional cellular space X^0: is just a discrete space coming from attaching the empty set with the boundary of non-infinite points by a continuous map.

ii. A one-dimensional cellular space X^1: is a space that can be obtained as follows. Take any zero-dimensional cellular space X^0 and glue it with the boundary of D^1 by a continuous map.

iii. A two-dimensional cellular space X^2: is a space that can be obtained by taking any one-dimensional cellular space X^1 and attach it to the boundary of D^2 by a continuous map.

iv. A cellular space of dimension n, X^n: is defined in a similar way by taking the $(n-1)$ −dimensional cellular space X^{n-1} and glue it with the boundary of the n −disk D^n by a continuous map.

From Definition 1.2.13, we can see how we can construct a finite CW −complex and we obtained that the building blocks for CW −complex are the closed discs.

$$D^n = \{ x \in R^n : |x| \leq 1\}$$

Called as cells. So, 0-cells are points, 1-cells are closed intervals, 2-cells are closed disks and 3-cells are closed filled spheres. One of the difference between the simplicial complexes and CW−complexes can be obtained from the Definition 1.2.13. Since we can attach $(n-1)$ −dimensional cellular space X^{n-1} and glue it with the boundary of the disk D^n, $i.e.$ we can attach the points in X^0 space with the boundary of D^1 as in Figure 10. For example we attach the point on the left twice with different boundary of D^1, which is the opposite of what happened in simplicial complexes, since the attaching just uniquely. We also note that any simplicial complex has a CW−structure.

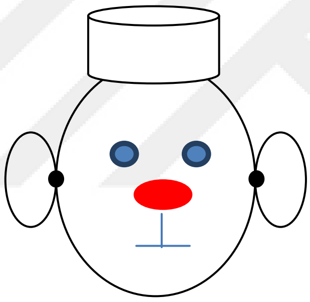

Figure 10. CW−structure

Example 1.2.14. We show that a torus T^2 is a 2 −dimensional CW−complex. Let X be presented by figure-8 which is constructed by attaching a point with the boundaries of two closed intervals, so X is a 1 −dimensional space. Then, take a square filled inside which is homeomorphic to a closed disk D^2 and attach the boundary of the square to X as presents in Figure 11. Then we get the filled square by joining the opposite sides, the upper and lower sides together and the left and right sides together, respectively, to get a torus T^2, so the dimension of T^2 is 2.

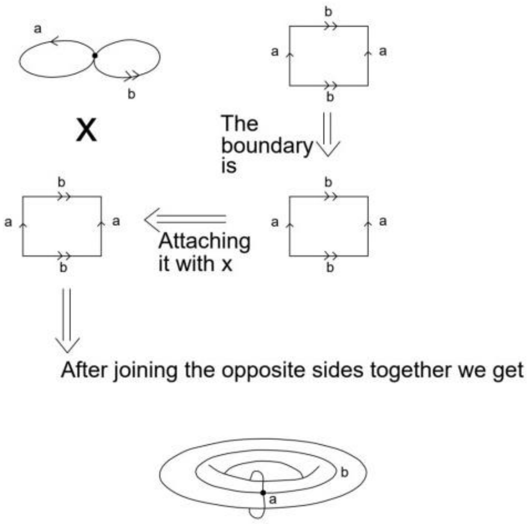

Figure 11**.** Triangulation of the torus

Definition 1.2.15. (Yan, 2016): Given a topological space X, a continuous map $f: S^{n-1} \to X$ is called the attaching map that joins the space X with the boundary of D^n denoted by S^{n-1}.

After the attaching process we get the space $X \cup_f D^n = X \coprod \dot{D}^n$ where $\dot{D}^n$ is the interior of D^n as shown in Figure 12. Now we can define an onto map $F: X \coprod D^n \to X \cup_f D^n$ such that

$$F(a) = \begin{cases} a, & if\ a \in X \\ f(a), & a \in \partial D^k \\ a, & a \in \dot{D}^k \end{cases}$$

Then F is a quotient map and induces the quotient topology on $X \cup_f D^n$.

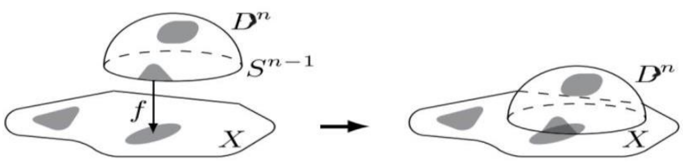

Figure 12. Open subsets after gluing a disk (Yan, 2016)

Now we can generalize the idea by defining a continuous map $f_i: S^{n-1} \to X$ for $i = 1, \dots, k$ and we can get a quotient space $X \cup_{f_i} D^n$ which is obtained by attaching k −cells to X along the attaching maps f_i for $i = 1, \dots, k$.

Remark 1.2.16. To construct an infinite CW−complex, we can follow the same steps as in the finite case but with non-stop steps. However, the obtained topology will be the weak topology.

2. ANALYSIS OF MADE WORK

We will present some approval propositions that will be used through the study. In Proposition 2.1. and Proposition 2.3, we will show that the sum and scalar product operations preserve the proximal continuity as the following.

Proposition 2.1. Suppose (X,δ) is a proximity space and t is a real number. If $f: X \to \mathbb{R}$ is a pc map, then $t.f$ is also a pc map.

Proof: Let A, B be near subsets of (X,δ), $i.e$, $A\ \delta\ B$. To show that $t.f$ is a pc map we need to prove that $D\big(t.f(A), t.f(B)\big) = 0$, where

$$D\big(t.f(A), t.f(B)\big) = \ inf\{|t.f(a) - t.f(b)|: a \in A, b \in B\}$$

$$= |t| inf\{\ |f(a) - f(b)|: a \in A, b \in B\}$$

$$= |t| D\big(f(A), f(B)\big) = 0.$$

Since f is a pc map, then $D(f(A), f(B)) = 0$. Thus, $t.f$ is a pc map.

Corollary 2.2. Given a convex proximity space X, if $f: X \to X$ is a pc map, then $t.f$ is also a pc map for all $t \in [0,1]$.

Proof: By substituting the gap map $D(x,y)$ in Proposition 2.1. as usual where the metric map $d(x,y) = ||x - y||$, we conclude that $t.f$ is a pc map.

Proposition 2.3. Given (X,δ) a proximity space, if $f, g: (X,\delta) \to \mathbb{R}$ are pc maps, then so is the map $f + g$.

Proof: Let A, B be near subsets of (X,δ). If we show that $D\big(f + g(A), f + g(B)\big) = 0$, then $f + g$ is a pc map.

$D\big(f + g(A), f + g(B)\big) = inf\{|f(a) - f(b) + g(a) - g(b)| a \in A$ and $b \in B\}$

$\leq inf\{|f(\text{a}) - f(b)|\ a \in A$ and $b \in B\} + inf\ \{|g(a) - g(b)|\ |\ a \in A$ and $b \in B\}$

$= D(f(A), f(B)) + D(g(A), g(B)) = 0.$

Since f and g are pc maps, then $D\big(f(A), f(B)\big) = D\big(g(A), g(B)\big) = 0$. Thus $D\big(f + g(A), f + g(B)\big) = 0$, so $f + g$ is a pc map.

Corollary 2.4. Suppose that (X, δ) is a convex set in an appropriate Euclidean space. If $f, g: (X, \delta) \to (X, \delta)$ are pc maps, then so is $f + g$.

Proof: Define the gap map $D(x, y)$ as usual where the metric map $d(x, y) = ||x - y||$, then substitute $D(x, y)$ in the Proposition 2.3. Hence, $f + g$ is a pc map.

In the next Propositions 2.5. and 2.7. we will show that the sum of dpc maps is also dpc. Furthermore, the scalar product operation preserves the descriptive proximal continuity.

Proposition 2.5. Suppose (X, δ_Φ) is a descriptive proximity space. If $f: X \to \mathbb{R}$ is a dpc map, then $t.f$ is also a dpc map for $\forall t \in \mathbb{R}$.

Proof: Let A, B be near subsets of (X, δ_Φ), and $A\ \delta_\Phi\ B$. To show that $t.f$ is a dpc map, we need to prove that $D\big(t.f(A), t.f(B)\big) = 0$, where

$$\begin{aligned} D\big(t.f(A), t.f(B)\big) &= inf\{|t.f(a) - t.f(b)|: a \in A, b \in B\} \\ &= |t|.inf\{\,|f(a) - f(b)|: a \in A, b \in B\} \\ &= |t|.D\big(f(A), f(B)\big) \\ &= 0. \end{aligned}$$

Since f is a dpc map, we have $D(f(A), f(B)) = 0$. Thus, $t.f$ is a dpc map.

Corollary 2.6. Given (X, δ_Φ) is a convex descriptive proximity space. If $f: X \to X$ is a dpc map, then $t.f$ is also a dpc map for all $t \in [0,1]$.

Proof: By substituting the gap map $D(x, y)$ in Proposition 2.5. as usual where the metric map $d(x, y) = ||x - y||$, then we conclude that $t.f$ is a dpc map.

Proposition 2.7. Given (X, δ_Φ) is a descriptive proximity space, if $f, g: (X, \delta_\Phi) \to (\mathbb{R}, |.|)$ are dpc maps, then so is $f + g$.

Proof: Let A, B be near subsets of (X, δ_Φ). If we show that $D((f+g)(A), (f+g(B))) = 0$, then $f+g$ is a dpc map.

$$D\big(f+g(A), f+g(B)\big) = inf\{|f(a) - f(b) + g(a) - g(b)| a \in A \text{ and } b \in B\}$$

$$\leq inf\ \{|f(\text{a}) - f(b)| a \in A \text{ and } b \in B\}$$

$$+\ inf\ \{|g(a) - g(b)|\ |\ a \in A \text{ and } b \in B\}$$

$$=\ D\big(f(A), f(B)\big) + D\big(g(A), g(B)\big)\ =\ 0$$

Since f and g are dpc maps, we have $D\big(f(A), f(B)\big) = D\big(g(A), g(B)\big) = 0$. Thus, $D\big((f+g)(A), (f+g)(B)\big) = 0$, so $f+g$ is a dpc map.

Corollary 2.8. Suppose that (X, δ_Φ) is a convex set in an appropriate Euclidean space. If $f, g: (X, \delta_\Phi) \to (X, \delta_\Phi)$ are dpc maps, then so is $f+g$.

Proof: Define the gap map $D(x, y)$ as usual where the metric map $d(x, y) = ||x - y||$, then substitute $D(x, y)$ in Proposition 2.7. Hence, $f+g$ is a dpc map.

We will discuss another type of proximal conjugacy called proximal semi conjugate which has been studied for topological spaces in (Alkfari & Alftllawy, 2022). In this section, we will explain this concept on proximity and descriptive proximity spaces.

Definition 2.9. Two pc maps $f: (X, \delta_1) \to (X, \delta_1)$ and $g: (Y, \delta_2) \to (Y, \delta_2)$ are said to be proximal semi conjugates if there is a surjective pc map $h: (X, \delta_1) \to (Y, \delta_2)$ such that $g \circ h = h \circ f$. The map h is called proximal semi conjugacy between f and g.

Definition 2.10. Let $f: (X, \delta) \to (X, \delta)$ be a pc map. Then f has the proximal transitive property (PT) provided for any non-empty open sets $A, B \subseteq X$ there is $n \in \mathbb{Z}^+$such that $f^n(A)\ \delta\ B$.

Definition 2.11. Let $f: (X, \delta) \to (X, \delta)$ be a pc map. Then f is called a proximal mixing map provided for any nonempty open sets $A, B \subseteq X$ there is $N > 0$ such that $f^n(A)\ \delta\ B$ for $n \geq N$.

Proposition 2.12. Given two proximal conjugate maps $f:(X,\delta_1)\to(X,\delta_1)$ and $g:(Y,\delta_2)\to(Y,\delta_2)$, if f has the PT property, then so does g.

Proof: To show that g has the PT property, assume that U and V are two nonempty open subsets of Y. Then we want to find $n\in\mathbb{Z}^+$such that $g^n(U)\ \delta_2\ V$. Since f and g are proximal conjugate, there is proximal isomorphism $h:(X,\delta_1)\to(Y,\delta_2)$ such that $h^{-1}(U)$ and $h^{-1}(V)$ are open subsets of X. We know that f has the PT property, so there is $n\in\mathbb{Z}^+$ such that $f^n\left(h^{-1}(U)\right)\delta_1\ h^{-1}(V)$. Since h is proximal conjugacy between f and g, then $f^n\left(h^{-1}(U)\right)=h^{-1}(g^n(U))$, so $h^{-1}\left(g^n(U)\right)\delta_1\ h^{-1}(V)$. Being h a proximal isomorphism, we have $h(h^{-1}(g^n(U)))\ \delta_2\ h(h^{-1}(V))$. Hence, $g^n(U)\ \delta_2\ V$, and this completes the proof.

Proposition 2.13. Suppose $f:(X,\delta_1)\to(X,\delta_1)$ and $g:(Y,\delta_2)\to(Y,\delta_2)$ are two proximal conjugate maps. If f is proximal mixing, then g is also proximal mixing.

Proof: To show that g is proximal mixing, assume that U and V are two non-empty open subsets of Y. Then we want to find $N>0$ such that $g^n(U)\ \delta_2\ V$ for $n\geq N$. Since f and g are proximal conjugate, then there is proximal isomorphism $h:(X,\delta_1)\to(Y,\delta_2)$, so $h^{-1}(U)$ and $h^{-1}(V)$ are open subsets of X. We know that f is proximal mixing, so there is $N>0$ such that $f^n\left(h^{-1}(U)\right)\delta_1\ h^{-1}(V)$ for $n\geq N$. Since h is proximal conjugacy between f and g, then $f^n\left(h^{-1}(U)\right)=h^{-1}(g^n(U))$, so $h^{-1}\left(g^n(U)\right)\delta_1\ h^{-1}(V)$. Since h is a proximal isomorphism, we have $h(h^{-1}(g^n(U)))\ \delta_2\ h(h^{-1}(V))$. Hence, $g^n(U)\ \delta_2\ V$ for $n\geq N$, then this shows that g is proximal mixing.

Proximal descriptive semi conjugate is a type of a conjugacy between two descriptive proximally continuous maps and defined in the following way.

Definition 2.14. Two dpc maps $f:(X,\delta_{\Phi_1})\to(X,\delta_{\Phi_1})$ and $g:(Y,\delta_{\Phi_2})\to(Y,\delta_{\Phi_2})$ are said to be descriptive proximal semi conjugates if there is a descriptive proximally surjective map $h:(X,\delta_{\Phi_1})\to(Y,\delta_{\Phi_2})$ such that $g\circ h=_{des}h\circ f$. The map h is called descriptive proximal semi conjugacy between f and g.

Definition 2.15. Let $f: (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ be a dpc map, then f is proximal descriptive transitive (PDT) provided for any non-empty open subsets $A, B \subseteq X$ there is $n \in \mathbb{Z}^+$ such that $f^n(A)\ \delta_{\Phi_1} B$.

Definition 2.16. A dpc map $f: (X, \delta_{\Phi_1}) \to \left(X, \delta_{\Phi_1}\right)$ is called a proximal descriptive mixing provided for any nonempty open subsets $A, B \subseteq X$ there is $N > 0$ such that $f^n(A)\ \delta_{\Phi_1} B$ for $n \geq N$.

The proximal descriptive transitive and proximal descriptive mixing properties are preserved under a descriptive proximal conjugacy.

Proposition 2.17. Given $f: (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g: (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$ two descriptive proximal conjugate maps, if f is PDT, then g is also PDT.

Proof: To show that g is PDT, assume that U and V are two nonempty open subsets of Y. Then we want to find $n \in \mathbb{Z}^+$ such that $g^n(U)\ \delta_{\Phi_2} V$. Since f and g are descriptive proximal conjugate, then there is a descriptive proximal isomorphism $h: \left(X, \delta_{\Phi_1}\right) \to (Y, \delta_{\Phi_2})$, so that $h^{-1}(U)$ and $h^{-1}(V)$ are open subsets of X. We know that f is PDT, so there is $n \in \mathbb{Z}^+$ such that $f^n\left(h^{-1}(U)\right)\ \delta_{\Phi_1}\ h^{-1}(V)$. Since h is a descriptive proximal conjugacy between f and g, we have $f^n\left(h^{-1}(U)\right) =_{des} h^{-1}(g^n(U))$, so $h^{-1}\left(g^n(U)\right)\ \delta_{\Phi_1} h^{-1}(V)$. However, h is descriptive proximal isomorphism, then $h(h^{-1}(g^n(U)))\ \ \delta_{\Phi_2}\ h(h^{-1}(V))$. Hence, $g^n(U)\ \delta_{\Phi_2} V$, so g is PDT.

Proposition 2.18. Suppose $f: (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ and $g: (Y, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_2})$ are two descriptive proximal conjugate maps. If f is proximal descriptive mixing, then g is also proximal descriptive mixing.

Proof: To show that g is proximal descriptive mixing, assume that U and V are two nonempty open subsets of Y. We want to find $N > 0$ such that $g^n(U)\ \delta_{\Phi_2}\ V$ for $n \geq N$. f and g are proximal descriptive conjugate so that there is a descriptive proximal isomorphism $h: \left(X, \delta_{\Phi_1}\right) \to (Y, \delta_{\Phi_2})$, so $h^{-1}(U)$ and $h^{-1}(V)$ are open subsets of X. Since f is proximal descriptive mixing, there is $N > 0$ such that $f^n\left(h^{-1}(U)\right)\ \delta_{\Phi_1} h^{-1}(V)$ for $n \geq N$. Being h descriptive proximal conjugacy between f and g, we obtain $f^n\left(h^{-1}(U)\right) =_{des} h^{-1}(g^n(U))$, so we have

$$h^{-1}\big(g^n(U)\big)\ \delta_{\Phi_1} h^{-1}(V).$$

However, h is a descriptive proximal isomorphism, then

$$h(h^{-1}(g^n(U)))\ \ \delta_{\Phi_2}\ h(h^{-1}(V)).$$

Hence, $g^n(U)\ \delta_{\Phi_2} V$ for $n \geq N$, and this completes the proof.

A topological space is connected if we cannot split it into two disjoint open subsets, so this is not easy to make accurate. Therefore, we define it's negative analogue disconnectedness. Let X be a topological space. Then X is called disconnected provided there are disjoint nonempty open (closed) sets $A, B \subseteq X$ such that $A \cup B = X$. In that case X is connected if and only if it is not disconnected (Munkres, 2000).

We will present the definition of proximal connectedness and descriptive connectedness.

Definition 2.19. (Peters & Vergili, 2023): Let (X, δ) be a proximity space. Then X is a $\delta -$ connected space provided for every pair of different subsets $A, B \subseteq X$, there is a collection $\{E_0, E_1, \ldots, E_r\}$ of subsets of X such that $A = E_0$, $B = E_r$ and $E_i\ \delta\ E_{i+1}$ for all $i \in \{0,1, \ldots, r-1\}$.

Example 2.20. Let (X, δ) be a proximity space shown in Figure 13. Then X is $\delta -$ connected, since for $A, B \subseteq X$ there is a set $\{E_0, E_1\}$ where $A = E_0$, $B = E_1$ and $A\ \delta\ B$, since $A \cap B \neq \emptyset$.

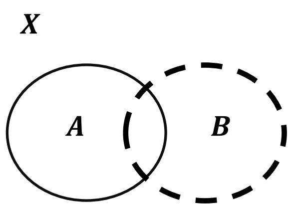

Figure 13. A $\delta -$connected space X

The image of $\delta -$connected set under a proximally continuous map is $\delta -$connected. Thus, the $\delta -$connectedness is preserved under proximal continuity.

Theorem 2.21. If $f: (X, \delta_1) \to (Y, \delta_2)$ is a pc map and X is δ_1 −connected, then $f(X)$ is also δ_2 −connected.

Proof: Let C, D be two subsets of $f(X)$. Then there are two subsets of X, say A and B, such that $f(A) = C$ and $f(B) = D$. Since X is δ_1 −connected, there is a collection $\{E_0, E_1, \dots, E_r\}$ of subsets of X such that $A = E_0$, $B = E_r$ and $E_i\ \delta_1\ E_{i+1}$ for all $i \in \{0,1, \dots, r-1\}$. However, f is a pc map, then $\{f(E_0), f(E_1), \dots, f(E_r)\}$ of subsets of $f(X)$ such that $C = f(A) = f(E_0)$, $D = f(B) = f(E_r)$ and $f(E_i)\ \delta_2\ f(E_{i+1})$ for all $i \in \{0,1, \dots, r-1\}$ .Thus $f(X)$ is δ_2 −connected.

Corollary 2.22. If $f: (X, \delta_1) \to (Y, \delta_2)$ is a pc map and E is a δ_1 −connected subset of X, then $f(E)$ is δ_2 − connected subset of Y.

Definition 2.23. (Peters & Vergili, 2021): Suppose (X, δ) is a proximity space and $x_0, x_1 \in X$. A proximal path from x_0 to x_1 is a pc map $\alpha: [0,1] \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$.

Definition 2.24. A proximity space (X, δ) is called proximally path connected, provided for all pairs x_0, $x_1 \in X$, there is a proximal path $\alpha: [0,1] \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$.

The following theorem shows that proximal path connectedness implies δ –connectedness.

Theorem 2.25. Suppose (X, δ) is proximally path connected. Then X is δ −connected.

Proof: Let A, B two subsets of X and $x_0 \in A$, $x_1 \in B$. Since X is proximal path connected, there is a proximal path α between x_0 and x_1 such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Define $E_0 = A$, $E_1 = \alpha((0,1))$ and $E_2 = B$. Then $E_i\ \delta\ E_{i+1}$ for all $i = 0,1$. Hence, X is δ −connected.

The proximally path connectedness is preserved under proximally continuous maps as in the following theorem.

Theorem 2.26. Given a pc map $g: (X, \delta_1) \to (Y, \delta_2)$, If (X, δ_1) is proximally path connected, then $g(X)$ is also proximally path connected.

Proof: Let $y_0, y_1 \in g(X)$ so that there is $x_0, x_1 \in X$ such that $g(x_0) = y_0$ and $g(x_1) = y_1$. We know that (X, δ_1) is proximally path connected, then there is a proximal path $\alpha: [0,1] \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Define $\beta \coloneqq g \circ \alpha$, then β is a pc map with $\beta(0) = g\big(\alpha(0)\big) = g(x_0) = y_0$ and $\beta(1) = g\big(\alpha(1)\big) = g(x_1) = y_1$. Hence g is proximally path connected.

Definition 2.27. (Peters & Vergili, 2023): A descriptive proximity space (X, δ_Φ) is said to be a δ_Φ −connected space, provided for every pair of different subsets $A, B \subseteq X$, there is a collection $\{E_0, E_1, \dots, E_r\}$ of subsets of X such that $A = E_0$, $B = E_r$ and $E_i\ \delta_\Phi\ E_{i+1}$ for all $i \in \{0,1, \dots, r-1\}$.

Example 2.28. For a descriptive proximity space (X, δ_Φ) given in Example 1.1.54, X is not descriptive connected, since for rbE_1 and rbE there is only one set $\{rbE_1, rbE\}$ but $rbE_1\ \underline{\delta}_\Phi\ rbE$.

In the following theorem, it is given that the descriptive proximal continuity preserves the descriptively connectedness, $i.e.$, the image of a descriptively connected set under a descriptive proximally continuous map is also descriptively connected.

Theorem 2.29. If $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is a dpc map and X is δ_{Φ_1} −connected, then $f(X)$ is δ_{Φ_2} −connected.

Proof: Let C, D be two subsets of $f(X)$. Then there are two subsets of X, say A and B, such that $f(A) = C$ and $f(B) = D$ since X is δ_{Φ_1} −connected, so there is a collection $\{E_0, E_1, \dots, E_r\}$ of subsets of X such that $A = E_0$, $B = E_r$ and $E_i\ \delta_{\Phi_1} E_{i+1}$ for all $i \in \{0,1, \dots, r-1\}$. However, f is a dpc map, then we have the collection $\{f(E_0), f(E_1), \dots, f(E_r)\}$ of subsets of $f(X)$ such that $C = f(A) = f(E_0)$, $D = f(B) = f(E_r)$ and $f(E_i)\ \delta_{\Phi_2}\ f(E_{i+1})$ for all $i \in \{0,1, \dots, r-1\}$. Thus $f(X)$ is δ_Φ −connected.

Corollary 2.30. If $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is a dpc map and E is a δ_Φ −connected subset of X, then $f(E)$ is a δ_Φ −connected subset of Y.

Definition 2.31. (Peters & Vergili, 2021): Suppose (X, δ_Φ) is a descriptive proximity space and $x_0, x_1 \in X$. A descriptive proximal path from x_0 to x_1 is a dpc map $\alpha: [0,1] \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$.

Definition 2.32. A descriptive proximity space (X, δ_Φ) is called descriptively path connected provided for all pairs x_0, $x_1 \in X$ there is a descriptive proximal path $\alpha: [0,1] \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$.

Theorem 2.33. Suppose (X, δ_Φ) is descriptively path connected. Then X is $\delta_\Phi -$ connected.

Proof: Let A, B two subsets of X and $x_0 \in A$, $x_1 \in B$. Since X is descriptively path connected, there is a descriptive proximal path α between x_0 and x_1 such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Define $E_0 = A$, $E_1 = \alpha((0,1))$ and $E_2 = B$. Then $E_i \; \delta_\Phi \; E_{i+1}$ for all $i = 0,1$. Hence, X is δ_Φ-connected.

In the following theorem we will show that the image of a descriptively path connected set under a descriptive proximally continuous map is descriptively path connected.

Theorem 2.34. Given a dpc map $g: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$. If X is descriptively path connected, then $g(X)$ is also descriptively path connected.

Proof: Let $y_0, y_1 \in g(X)$. Then there are $x_0, x_1 \in X$ such that $g(x_0) = y_0$ and $g(x_1) = y_1$. Since (X, δ_{Φ_1}) is descriptively path connected, there is a descriptive proximal path $\alpha: [0,1] \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Define $\beta \coloneqq g \circ \alpha$, then β is a dpc map with $\beta(0) = g\big(\alpha(0)\big) = g(x_0) = y_0$ and $\beta(1) = g\big(\alpha(1)\big) = g(x_1) = y_1$. Hence, $g(X)$ is descriptively path connected.

2.1. Proximal Homotopic Theory

Homotopy theory is one of the most important topics in algebraic topology. Therefore, it is important to adapt it to proximity spaces which is of great importance in analyzing spaces and distinguishing between different spaces.

In this section, we give the definition of proximal homotopy, path proximal homotopy and proximal retraction and deformation. Furthermore, obtain some related definitions and prove some propositions.

Definition 2.1.1. (Peters & Vergili, 2021): Let (X, δ_1) and (Y, δ_2) be proximity spaces, and $f, g: (X, \delta_1) \to (Y, \delta_2)$ be two pc maps. Then we say that f and g are proximally homotopic provided there exists a map

$$F: X \times [0,1] \to Y,$$

such that

i. F is a pc map, and

ii. $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

The map F is called a proximal homotopy between f and g. We write $f \sim_\delta g$ provided there is a proximal homotopy between them.

Figure 14. shows two proximally homotopic maps, and the continuous deformation between them.

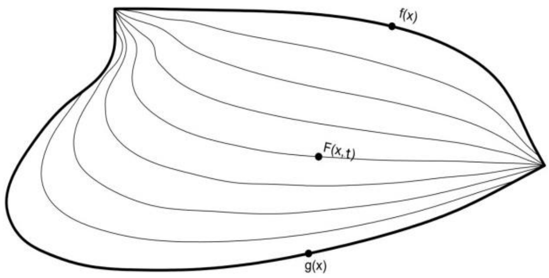

Figure 14. Proximally homotopic maps f and g maps

Proposition 2.1.2. and Proposition 2.1.3. we will prove that the proximal homotopy is preserved under from the left and right composition of proximally continuous maps.

Proposition 2.1.2. (Peters & Vergili, 2021): Given $f, g: (X, \delta_1) \to (Y, \delta_2)$ two pc maps, and $h: (Y, \delta_2) \to (Z, \delta_3)$ a pc map, If $f \sim_\delta g$, then $h \circ f \sim_\delta h \circ g$.

Proof: Since $f \sim_\delta g$ there is a pc map $F: X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We want to show $h \circ f \sim_\delta h \circ g$. Notice that $h \circ f$, $h \circ g: (X, \delta_1) \to (Z, \delta_3)$ are pc maps. Now we should find a pc map $G: X \times [0,1] \to Z$ such that $G(x,0) = h \circ f(x)$ and $G(x,1) = h \circ g(x)$. Define $G := h \circ F$, then $G: X \times [0,1] \to Z$ is a pc map according to Proposition 1.1.31, and

$$G(x,0) = h \circ F(x,0) = h\big(F(x,0)\big) = h\big(f(x)\big) = h \circ f(x)$$

and

$$G(x,1) = h \circ F(x,1) = h\big(F(x,1)\big) = h\big(g(x)\big) = h \circ g(x),$$

so $h \circ f \sim_\delta h \circ g$.

Proposition 2.1.3. (Peters & Vergili, 2021): Given $f, g: (X, \delta_1) \to (Y, \delta_2)$ two pc maps, and $k: (W, \delta_0) \to (X, \delta_1)$ a pc map. If $f \sim_\delta g$, then $f \circ k \sim_\delta g \circ k$.

Proof: Let $H: X \times [0,1] \to Y$ be the proximal homotopy map between f and g, then H is a pc map such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. To show that the two pc maps $f \circ k$ and $g \circ k$ are proximally homotopic, define

$$F: W \times [0,1] \to Y,$$

by $F(w,t) = H(k(w),t)$. Then F is a pc map and $F(w,0) = H(k(w),0) = f\big(k(w)\big) = f \circ k(w)$ and $F(w,1) = H(k(w),1) = g\big(k(w)\big) = g \circ k(w)$. Hence, $f \circ k \sim_\delta g \circ k$.

The Gluing Lemma for the proximity spaces and descriptive proximity spaces is a transition of the Gluing Lemma (aka Pasting Lemma) for topological spaces, which is an important result that we can construct a continuous map by gluing two continuous maps. In other words, we can get a piecewise defined map which is helpful to analyze the fundamental group in topology. Also, when we talk about the concept of the gluing lemma in topological spaces, we can say that it is a tool to help us construct new continuous paths from others. Thus, the Gluing Lemma for (descriptive) proximity spaces has the same purpose in the (descriptive) proximal homotopic theory.

Lemma 2.1.4. (Peters & Vergili, 2021): Let (X, δ_1) and (Y, δ_2) be proximity spaces, and A, B closed subsets of X such that $X = A \cup B$. Suppose $f: (A, \delta_1) \to (Y, \delta_2)$ and

$g:(B,\delta_1) \to (Y,\delta_2)$ are pc maps such that $f(x) = g(x)$ for all $x \in A \cap B$. Then the map $h:(X,\delta_1) \to (Y,\delta_2)$ defined by

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

is also a pc map.

Proposition 2.1.5. (Peters & Vergili, 2021): The proximal homotopy relation $\sim_\delta$ is an equivalence relation.

Proof: To show that $\sim_\delta$ is an equivalence relation, we need to prove that it is reflexive, symmetric and transitive. Let $f, g, h:(X,\delta_1) \to (Y,\delta_2)$ be pc maps.

i. $\sim_\delta$ *is reflexive:* We will show that $f \sim_\delta f$. Define $F: X \times [0,1] \to Y$ such that $F(x,t) = f(x)$. Then clearly F is a pc map and $F(x,0) = F(x,1) = f(x)$, then $f \sim_\delta f$.

ii. $\sim_\delta$ *is symmetric:* Assume $f \sim_\delta g$, want to show that $g \sim_\delta f$. Since $f \sim_\delta g$ there is $H: X \times [0,1] \to Y$ such that H is a pc map, $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Define $G := H(x, 1-t)$, then $G: X \times [0,1] \to Y$ is a pc map, $G(x,0) = H(x,1) = g(x)$ and $G(x,1) = H(x,0) = f(x)$, so we have $g \sim_\delta f$.

iii. $\sim_\delta$ *is transitive:* Suppose $f \sim_\delta g$ and $g \sim_\delta h$. Since $f \sim_\delta g$ there is $F: X \times [0,1] \to Y$ such that F is a pc map, $F(x,0) = f(x)$ and $F(x,1) = g(x)$ and since $g \sim_\delta h$ there is $G: X \times [0,1] \to Y$ such that G is a pc map, $G(x,0) = g(x)$ and $G(x,1) = h(x)$. Now to show that $f \sim_\delta h$. Define

$$H: X \times [0,1] \to Y \text{ such that } H(x,t) = \begin{cases} F(x,2t), & 0 \le t \le \frac{1}{2} \\ G(x,2t-1), & \frac{1}{2} \le t \le 1 \end{cases}.$$

Then H is a pc map by Lemma 2.1.4 and $H(x,0) = F(x,0) = f(x)$ and $H(x,1) = G(x,1) = h(x)$, so $f \sim_\delta h$. Thus, the proximal homotopy relation $\sim_\delta$ is an equivalence relation.

Proposition 2.1.6. Let $f, \tilde{f}:(X,\delta_1) \to (Y,\delta_2)$, and $g, \tilde{g}:(Y,\delta_2) \to (Z,\delta_3)$ be pc maps. If $f \sim_\delta \tilde{f}$ and $g \sim_\delta \tilde{g}$, then $g \circ f \sim_\delta \tilde{g} o \tilde{f}$.

Proof: We have $g \circ f \sim_\delta g \circ \tilde{f}$ From Proposition 2.1.2, and $g \circ \tilde{f} \sim_\delta \tilde{g} \circ \tilde{f}$ from Proposition 2.1.3. Furthermore, since $\sim_\delta$ is transitive, then $g \circ f \sim_\delta \tilde{g} \circ \tilde{f}$, and this completes the proof.

Lemma 2.1.7. Let (X, δ) be a proximity space. In that case any two pc maps $f, g: (X, \delta) \to [0,1]$ are proximally homotopic.

Proof: Define $H: X \times [0,1] \to [0,1]$ by $H(x, t) = tf(x) + (1 - t)g(x)$, then H is a pc map, $H(x, 0) = g(x)$ and $H(x, 1) = f(x)$. Thus $g \sim_\delta f$.

Definition 2.1.8. (Peters & Vergili, 2021): Let (X, δ) be a proximity space, then X is called a proximally contractible space provided the identity map on it is proximally homotopic to a constant map.

Proposition 2.1.9. Convex Čech proximity spaces are proximally contractible.

Proof: Let X be a convex proximity space to show that X is proximally contractible, define the identity map $id_X: (X, \delta) \to (X, \delta)$ and a constant map $c: (X, \delta) \to (X, \delta)$ at $x_0 \in X$, so that $c(x) = x_0$. Then we claim that $id_X \sim_\delta c$. Since X is a convex set , we can define

$$F: X \times [0,1] \to X \text{ by } F(x, t) = tx_0 + (1 - t)x.$$

F is clearly a pc map, $F(x, 0) = x = id_X(x)$ and $F(x, 1) = x_0 = c(x)$. Thus $id_X \sim_\delta c$, and X is proximally contractible.

Proposition 2.1.10. If (X, δ) is a proximally contractible space, then (X, δ) is a proximally path connected.

Proof: Since X is proximally contractible, then we have $id_X \sim_\delta c$, $i.e.$, there is a pc map

$$H: X \times [0,1] \to X,$$

such that $H(x, 0) = id_X(x) = x$ and $H(x, 1) = c(x) = x_0$. To show that X is proximally path connected, it is suffices to take a point $x_1 \in X$ and prove that there is a proximal path from x_1 to x_0. Define a proximal path $\beta: [0,1] \to X$ by $\beta(t) = H \circ \alpha$ where $\alpha: [0,1] \to X \times [0,1]$ is defined by $\alpha(t) = (x_1, t)$. Since α and H are pc maps, then so is β, and $\beta(0) = H\big(\alpha(0)\big) = H(x_1, 0) = id_X(x_1) = x_1$ and $\beta(1) = H\big(\alpha(1)\big) = H(x_1, 1) = c(x_1) = x_0$, so β is a proximal path between x_0 and x_1. Thus (X, δ) is proximally path connected.

The following theorem shows that proximal contractibility is preserved under a proximal isomorphism.

Proposition 2.1.11. Let (X, δ_1) be a proximally contractible space. If $X \approx_{\delta_1,\delta_2} Y$, then (Y, δ_2) is also a proximally contractible space.

Proof: To show that (Y, δ_2) is a proximally contractible space, we need to find a proximal constant map that is proximally homotopic to id_Y on Y. Since $X \approx_{\delta_1,\delta_2} Y$, then there is a bijective pc map $f: (X, \delta_1) \to (Y, \delta_2)$ and its inverse $f^{-1}: (Y, \delta_2) \to (X, \delta_1)$ is also a pc map. If (X, δ_1) is proximally contractible, then there is a proximal constant map $c: (X, \delta_1) \to (X, \delta_1)$ defined by $c(x) = x_0$ such that $id_X \sim_\delta c$ which means that there is a proximal homotopy $H: X \times [0,1] \to X$ such that $H(x, 0) = x$ and $H(x, 1) = x_0$. Define $G: Y \times [0,1] \to Y$ by $G(y, t) = f \circ H\ (f^{-1}(y), t)$. Then G is a pc map, and

$G(y, 0) = f\big(H(f^{-1}(y), 0)\big) = f\big(f^{-1}(y)\big) = y = id_Y$ and
$G(y, 1) = f\big(H(f^{-1}(y), 1)\big) = f(x_0)$ which is a constant map.

So we conclude that there is a constant map on Y, which is proximally homotopic to id_Y. Hence, (Y, δ_2) is a proximally contractible space.

Definition 2.1.12. Let (X, δ_1) be a proximity space, and P is a proximal property of (X, δ_1). If $f: (X, \delta_1) \to (Y, \delta_2)$ is proximal isomorphism and (Y, δ_2) has the proximal property P, then such a proximal property P is called a proximal invariant property.

Corollary 2.1.13. Proximally contractibility is a proximal invariant property.

Proof: The proof follows from Proposition 2.1.11 and Definition 2.1.12.

Proposition 2.1.14. The product of two proximally contractible spaces is proximally contractible.

Proof: Let (X, δ_1) and (Y, δ_2) be two proximally contractible spaces. To show that $X \times Y$ is proximally contractible, we have (X, δ_1) and (Y, δ_2) are proximally contractible spaces, then $id_X \sim_\delta c_{x_0}$ where c_{x_0} is a proximal constant map at $x_0 \in X$ and $id_Y \sim_\delta c_{y_0}$ where c_{y_0} is a proximal constant map at $y_0 \in Y$. It is natural to consider

$$id_X \times id_Y \coloneqq id_{X \times Y} \text{ and } c_{x_0} \times c_{y_0} \coloneqq c_{(x_0, y_0)}$$

We claim that $id_{X\times Y} \sim_\delta c_{(x_0,y_0)}$ where $id_{X\times Y}: X\times Y \to X\times Y$ is the identity map on $X\times Y$ and $c_{(x_0,y_0)}: X\times Y \to X\times Y$ is the proximal constant map defined by $c_{(x_0,y_0)}(x,y) = (x_0, y_0)$. Since $id_X \sim_\delta c_{x_0}$ and $id_Y \sim_\delta c_{y_0}$, there exist H_1 and H_2 proximal homotopies between id_X and c_{x_0}, id_Y and c_{y_0} respectively. Define

$$pr_X \times id_{[0,1]}: X\times Y\times[0,1] \to X\times[0,1] \text{ such that } pr_X(x,y,t) = (x,t) \text{ and}$$

$$pr_Y \times id_{[0,1]}: X\times Y\times[0,1] \to Y\times[0,1] \text{ such that } pr_Y(x,y,t) = (y,t)$$

Then $pr_X \times id_{[0,1]}$ and $pr_Y \times id_{[0,1]}$ are pc maps. Let $H: X\times Y\times[0,1] \to X\times Y$ be defined by $H(x,y,t) = (H_1 \circ pr_X \times id_{[0,1]},\ H_2 \circ pr_Y \times id_{[0,1]})$, then H is a pc map, $H(x,y,0) = \big(H_1(x,0), H_2(y,0)\big) = (x,y) = id_{X\times Y}(x,y)$ and $H(x,y,1) = \big(H_1(x,1), H_2(y,1)\big) = (x_0, y_0) = c_{(x_0,y_0)}(x,y)$, hence $id_{X\times Y} \sim_\delta c_{(x_0,y_0)}$. Thus, $X\times Y$ is proximally contractible.

Corollary 2.1.15. Let $\{(X_i, \delta_i)\}_{i\in I}$ be a family of proximally contractible spaces where I is a finite index set, then $X = \prod_{i\in I} X_i$ is also a proximally contractible space.

Proof: The proof follows from mathematical induction method.

i. For $n = 1$: It is trivial.

ii. For $n = 2$: Let X_1 and X_2 be proximally contractible spaces, then $X_1 \times X_2$ is proximally contractible by Proposition 2.1.14.

iii. For $n = k$: Assume $X_1, \dots, X_k$ are proximally contractible spaces, then $X_1 \times X_2 \dots \times X_k$ is proximally contractible. We want to show that If $X_1, \dots, X_k, X_{k+1}$ are proximally contractible spaces, then $X = (\prod_{i=1}^{i=k} X_i) \times X_{k+1}$ is proximally contractible. Since X_{k+1} is proximally contractible and $\prod_{i=1}^{i=k} X_i$ is proximally contractible from $n = k$, so from (ii) we have $X = \prod_{i=1}^{i=k+1} X_i$ is proximally contractible.

Definition 2.1.16. (Peters & Vergili, 2021): A pc map $f: X \to Y$ between two proximity topological spaces X, Y is said to be proximally null homotopic if it is proximally homotopic to a proximal constant map.

Definition 2.1.17. (Peters & Vergili, 2021): Let (X, δ_1) and (Y, δ_2) be two proximity spaces, then we say X and Y are proximally homotopically equivalent if there are two pc maps $f:(X,\delta_1)\rightarrow(Y,\delta_2)$ and $g:(Y,\delta_2)\rightarrow(X,\ \delta_1)$ such that $f\circ g\sim_\delta id_Y$ and $g\circ f\sim_\delta id_X$. If X and Y are proximally homotopically equivalent, we write $X\simeq_\delta Y$.

Proposition 2.1.18. Let (X,δ_1) and (Y,δ_2) be two proximity spaces. If $X\approx_{\delta_1,\delta_2} Y$, then X and Y are proximally homotopically equivalent.

Proof: Since $X\approx_{\delta_1,\delta_2} Y$ there is a bijective pc map $f:(X,\delta_1)\rightarrow(Y,\delta_2)$, and its inverse $f^{-1}:(Y,\delta_2)\rightarrow(X,\delta_1)$ is also a pc map. Then $f\circ f^{-1}=id_Y$ and $f^{-1}\circ f=id_X$, hence $f^{-1}\circ f\sim_\delta id_X$ and $f\circ f^{-1}\sim_\delta id_Y$. Thus, X and Y are proximally homotopically equivalent.

Proposition 2.1.19. If (X,δ) is proximally contractible, then for any proximity space Y is proximally homotopically equivalent to $X\times Y$.

Proof: Let $f:X\times Y\rightarrow Y$ be a map defined by $f(x,y)=y$. Then f is a pc map and $g:Y\rightarrow X\times Y$ defined as $g(y)=(x_0,y)$, $x_0\in X$ is also a pc map. In that case $f\circ g:Y\rightarrow Y$ is equal to id_Y since $f\circ g(y)=f(x_0,y)=y$. Thus, $f\circ g\sim_\delta id_Y$. Moreover, $g\circ f:X\times Y\rightarrow X\times Y$ is equal to $id_{\{x_0\}\times Y}$ because $g\circ f(x,y)=g(y)=(x_0,y)$ which means that $g\circ f$ is a constant map. X is proximally contractible, then $g\circ f$ is proximally homotopic to $id_{X\times Y}$. Thus Y is proximally homotopically equivalent to $X\times Y$.

Proposition 2.1.20. Let (X,δ) be a proximity space and $x_0\in X$. Then X is proximally contractible if and only if X is proximally homotopically equivalent to $\{x_0\}$.

Proof: Define pc maps $f:X\rightarrow\{x_0\}$ by $f(x)=x_0$ and $g:\{x_0\}\rightarrow X$ by $g(x_0)=x_0$. Then $f\circ g:\{x_0\}\rightarrow\{x_0\}$ is proximally homotopic to id_{x_0}, since $f\circ g(x_0)=f(x_0)=x_0$. Moreover, $g\circ f:X\rightarrow X$ is proximally homotopic to id_X because $g\circ f(x)=g(x_0)=x_0$ which means that $g\circ f$ is a pc constant map. X is proximally contractible, then $g\circ f$ is proximally homotopic to id_X. Therefore, X is proximally homotopic equivalent to $\{x_0\}$.

Conversely, assume that X is proximally homotopy equivalent to $\{x_0\}$. We will show that X is proximally contractible. Since X is proximally homotopically equivalent to $\{x_0\}$, there are pc maps $f:X\rightarrow\{x_0\}$ and $g:\{x_0\}\rightarrow X$ with $f\circ g\sim_\delta id_{x_0}$ and $g\circ f\sim_\delta id_X$. Notice that $g\circ f$ is a proximal constant map proximally homotopic to id_X. Therefore, X is proximally contractible.

Proposition 2.1.21. The proximally homotopically equivalent relation $\simeq_\delta$ is an equivalence relation.

Proof: We will show that $\simeq_\delta$ is reflexive, symmetric and transitive.

i. $\simeq_\delta$ *is reflexive:* Since (X,δ_1) is proximal isomorphism to itself, $i.e.$, $X \approx_{\delta_1,\delta_1} X$. Then by Proposition 2.1.18. $(X,\delta_1) \simeq_\delta (X,\delta_1)$. Therefore, $\simeq_\delta$ is reflexive.

ii. $\simeq_\delta$ *is symmetric:* Assume that $(X,\delta_1) \simeq_\delta (Y,\delta_2)$. We need to show that $(Y,\delta_2) \simeq_\delta (X,\delta_1)$. Since $(X,\delta_1) \simeq_\delta (Y,\delta_2)$, then there are two pc maps

$$f:(X,\delta_1) \to (Y,\delta_2) \text{ and } g:(Y,\delta_2) \to (X,\delta_1)$$

such that $f \circ g \sim_\delta id_Y$ and $g \circ f \sim_\delta id_X$. To show that $(Y,\delta_2) \simeq_\delta (X,\delta_1)$ we will find two pc maps $h:(Y,\delta_2) \to (X,\delta_1)$ and $k:(X,\delta_1) \to (Y,\delta_2)$ such that $h \circ k \sim_\delta id_X$ and $k \circ h \sim_\delta id_Y$, so take $h \coloneqq g$ and $k \coloneqq f$. So, the proof is complete.

iii. $\simeq_\delta$ *is transitive*: Given $(X,\delta_1) \simeq_\delta (Y,\delta_2)$, and $(Y,\delta_2) \simeq_\delta (Z,\delta_3)$, we will show that $(X,\delta_1) \simeq_\delta (Z,\delta_3)$. Since $(X,\delta_1) \simeq_\delta (Y,\delta_2)$, then there exist pc maps $f:X \to Y$ and $g:Y \to X$ such that $g \circ f \sim_\delta id_X$ and $f \circ g \sim_\delta id_Y$. Since we have $(Y,\delta_2) \simeq_\delta (Z,\delta_3)$, there exist pc maps $h:Y \to Z$ and $k:Z \to Y$ such that $h \circ k \sim_\delta id_z$ and $k \circ h \sim_\delta id_Y$. We claim that the pc map $(h \circ f) \circ (g \circ k)$ is proximally homotopic to id_Z since $f \circ g \sim_\delta id_Y$ and this implies $h \circ (f \circ g) \sim_\delta h \circ id_Y$ by Proposition 2.1.2. and $(h \circ f \circ g) \circ k \sim_\delta (h \circ id_Y) \circ k$ by Proposition 2.1.3. Hence,

$$(h \circ f) \circ (g \circ k) \sim_\delta h \circ k \sim_\delta id_z.$$

Since $\sim_\delta$ is transitive, $(h \circ f) \circ (g \circ k) \sim_\delta id_z$. Moreover, for the pc map $(g \circ k) \circ (h \circ f)$, we have $(g \circ k) \circ (h \circ f) \sim_\delta id_X$ because $k \circ h \sim_\delta id_Y$ implies $g \circ (k \circ h) \sim_\delta g \circ id_Y$ and $(g \circ k \circ h) \circ f \sim_\delta (g \circ id_Y) \circ f$, but $g \circ f \sim_\delta id_X$. However, $\sim_\delta$ is transitive and

$$(g \circ k) \circ (h \circ f) \sim_\delta id_X$$

Thus $(X,\delta_1) \simeq_\delta (Z,\delta_3)$.

Proposition 2.1.22. Suppose that the proximal isomorphism maps $f, g: (X, \delta_1) \to (Y, \delta_2)$ are proximally homotopic maps. Then $f^{-1}, g^{-1}: (Y, \delta_2) \to (X, \delta_1)$ are also proximally homotopic.

Proof: Since $f \sim_\delta g$ there is a pc homotopy $H: X \times [0,1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. However, f and g are proximal isomorphism, so are f^{-1} and g^{-1}. Define

$$G: Y \times [0,1] \to X \text{ by } G(y, t) = f^{-1}(H(g^{-1}(y), 1 - t))$$

Then

G is a pc map.

Moreover, $G(y, 0) = f^{-1}(H(g^{-1}(y), 1)) = f^{-1}(g(g^{-1}(y)) = f^{-1}(y)$ and $G(y, 1) = f^{-1}(H(g^{-1}(y), 0)) = f^{-1}(f(g^{-1}(y)) = g^{-1}(y)$. Hence, $f^{-1} \sim_\delta g^{-1}$.

As we have seen in the definition of the proximal homotopy, we studied a homotopy relation over a proximity space X, however in the following definition we will define the proximal homotopy relative to any subset of the proximity space X, and this definition leads to an important application, Proximal Fundamental Group.

Definition 2.1.23. (Peters & Vergili, 2021): Suppose (X, δ_1) and (Y, δ_2) are proximity spaces and A is a subset of X. Then two pc maps $f: (X, \delta_1) \to (Y, \delta_2)$ and $g: (X, \delta_1) \to (Y, \delta_2)$ are said to be proximally homotopic relative to A, denoted by $f \sim_\delta g \ (rel\, A)$ provided there exists a proximal homotopy H between f and g such that $H(a, t) = f(a) = g(a)$ for all $a \in A$ and $t \in [0,1]$.

Proposition 2.1.24. Given $A \subseteq X$ and $B \subseteq Y$. Let $f_0, f_1: (X, \delta_1) \to (Y, \delta_2)$ be pc maps with $f_0|_A = f_1|_A$ and $f_i(A) \subseteq B$ for $i = 0,1$. Also assume that $g_0, g_1: (Y, \delta_2) \to (Z, \delta_3)$ are pc maps with $g_0|_B = g_1|_B$. If $f_0 \sim_\delta f_1 \ (rel\, A)$ and $g_0 \sim_\delta g_1 \ (rel\, B)$, then $g_0 \circ f_0 \sim_\delta g_1 \circ f_1 \ (rel\, A)$.

Proof: Since $f_0 \sim_\delta f_1 \ (rel\, A)$, there is a pc map $F: X \times [0,1] \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. Furthermore, $F(a, t) = f_0(a) = f_1(a)$ for all $a \in A$ and $t \in [0,1]$. Since $g_0 \sim_\delta g_1 \ (rel\, B)$, then there is a pc map $G: Y \times [0,1] \to Z$ such that

$G(x,0) = g_0(x)$ and $G(x,1) = g_1(x)$. Moreover, $G(b,t) = g_0(b) = g_1(b)$ for all $b \in B$ and $t \in [0,1]$. Define

$$H: X \times [0,1] \to Z \text{ by } H(x,t) = G(F(x,t),t)$$

Then H is a pc map, and

$H(x,0) = G(F(x,0),0) = G(f_0(x),0) = g_0(f_0(x))$,

$H(x,1) = G(F(x,1),1) = G(f_1(x),1) = g_1(f_1(x))$.

Also, for all $a \in A$ and $t \in [0,1]$ we have

$$H(a,t) = G(F(a,t),t) = G(f_0(a),t) = G(f_1(a),t)$$

and since $f_i(A) \subseteq B$ for $i = 0,1$,

$$G(f_0(a),t) = g_0(f_0(a))$$

and

$$G(f_1(a),t) = g_1(f_1(a))$$

Hence, $H(a,t) = g_0\big(f_0(a)\big) = g_1(f_1(a))$ for all $a \in A$ and $t \in [0,1]$. Thus, $g_0 \circ f_0 \sim_\delta g_1 \circ f_1$ $(rel\ A)$.

Now we can introduce the proximal path homotopy and prove some important related results.

Definition 2.1.25. Let α and β be two proximal paths in (X,δ) such that $\alpha: [0,1] \to X$ is a pc map with $\alpha(0) = x_0$, $\alpha(1) = x_1$ and $\beta: [0,1] \to X$ is a pc map, $\beta(0) = x_0$, $\beta(1) = x_1$. Then we say that α and β are proximally path homotopic, provided there is a pc map

$$H: [0,1] \times [0,1] \to X$$

such that

$H(s,0) = \alpha(s)$, $H(s,1) = \beta(s)$,

$H(0,t) = x_0$ and $H(1,t) = x_1$.

Such a map H is called a proximal path homotopy between α and β, we write $\alpha \sim_{\delta,p} \beta$ which means that there is a proximal path homotopy between them.

In Figure 15, we can see that the proximal path α is deformed to the proximal path β continuously through the time $t \in [0,1]$. So $\alpha \sim_{\delta,p} \beta$.

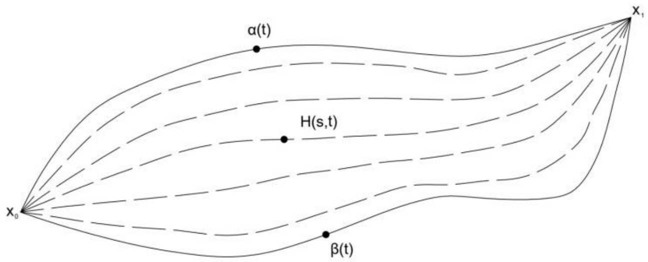

Figure 15. α and β proximally path homotopic paths

Proposition 2.1.26. A proximal path homotopy relation $\sim_{\delta,p}$ is an equivalence relation.

Proof : The proof will follow the strategy given in Proposition 2.1.5.

Definition 2.1.27. Let α be a proximal path between x_0 and x_1 in the proximity space (X,δ). Then the reverse of α is the proximal path $\bar{\alpha}:[0,1] \to X$ defined by $\bar{\alpha}(s) = \alpha(1-s)$ between x_1 and x_0.

Proposition 2.1.28. Let (X,δ) be a proximity space, and α, β are proximal paths from x_0 to x_1 where $x_0, x_1 \in X$. If $\alpha \sim_{\delta,p} \beta$, then $\bar{\alpha} \sim_{\delta,p} \bar{\beta}$.

Proof: Since $\alpha \sim_{\delta,p} \beta$ there is a pc map $H:[0,1] \times [0,1] \to X$ such that $H(s,0) = \alpha(s)$, $H(s,1) = \beta(s)$, $H(0,t) = x_0$ and $H(1,t) = x_1$. Let $F:[0,1] \times [0,1] \to X$ be defined by $F(s,t) = H(1-s,t)$. Then F is a pc map with $F(s,0) = H(1-s,0) = \alpha(1-s) = \bar{\alpha}(s)$, and $F(s,1) = H(1-s,1) = \beta(1-s) = \bar{\beta}(s)$, $F(0,t) = H(1,t) = x_1$ and $F(1,t) = H(0,t) = x_0$. Thus $\bar{\alpha} \sim_{\delta,p} \bar{\beta}$.

Proposition 2.1.29. Suppose (X,δ_1) and (Y,δ_2) are proximity spaces, and $f: X \to Y$ is a pc map. If proximally paths α and β between x_0 and x_1 are homotopic, then $f\circ\alpha \sim_{\delta,p} f\circ\beta$.

Proof: Since $\alpha \sim_{\delta,p} \beta$ we have a proximal path homotopy H between α and β. Define

$$F := f\circ H \ \text{ then } \ F: [0,1]\times[0,1] \to Y$$

is a pc map such that $F(s,0) = f\circ H(s,0) = f\big(H(s,0)\big) = f(\alpha(s))$, $F(s,1) = f\circ H(s,1) = f\big(H(s,1)\big) = f(\beta(s))$, $F(0,t) = f(x_0)$ and $F(1,t) = f(x_1)$. Hence, $f\circ\alpha \sim_{\delta,p} f\circ\beta$.

Definition 2.1.30. Suppose (X,δ) is a proximity space, α is a proximal path between x_0 and x_1 and β is a proximal path between x_1 and x_2, then $\alpha*\beta$ defined by

$$\alpha*\beta\,(s) = \begin{cases} \alpha(2s), & 0\le s\le \dfrac{1}{2} \\ \beta(2s-1), & \dfrac{1}{2}\le s\le 1 \end{cases}$$

is a proximal path between x_0 and x_2.

Remark 2.1.31. The product of two proximal paths, α and β, $\alpha*\beta$ in Definition 2.1.30. is a proximal path provided it is a pc map according to Lemma 2.1.4. In addition $\alpha*\beta\,(0) = x_0$ and $\alpha*\beta\,(1) = x_2$.

Definition 2.1.32. (Peters & Vergili, 2021): Let (X,δ) be a proximity space. The proximal constant path $c: [0,1]\to X$ at $x_0\in X$ is the proximal path such that $c(t) = x_0$ for every $t\in[0,1]$.

Example 2.1.33. Let $(\mathbb{R},|.|)$ be a proximity space. Then any proximal loop $\alpha: [0,1]\to\mathbb{R}$ is proximally path homotopic to a proximal constant map.

Proof : Let $c: [0,1]\to\mathbb{R}$ defined by $c(s) = x_0$ be a proximal constant map at $x_0\in X$ and α is a proximal loop at x_0. Then $\alpha \sim_{\delta,p} c$ since $\mathbb{R}$ is a convex set: there is a pc map $H: [0,1]\times[0,1]\to\mathbb{R}$ given by $H(s,t) = (1-t)\,\alpha(s) + tc(s)$ such that $H(s,0) = \alpha(s)$, $H(s,1) = c(s)$, $H(0,t) = (1-t)x_0 + tx_0 = x_0$ and $H(1,t) = (1-t)x_0 + tx_0 = x_0$ so that α is proximally null-homotopic.

Proposition 2.1.34. Suppose that (X,δ) is a proximity space, α and ω are two proximal paths between x_0 and x_1, and β is a proximal path between x_1 and x_2. If $\alpha \sim_{\delta,p} \omega$, then $\alpha * \beta \sim_{\delta,p} \omega * \beta$.

Proof: Since $\alpha \sim_{\delta,p} \omega$, there is a pc map $H: [0,1] \times [0,1] \to X$ such that $H(s,0) = \alpha(s)$, $H(s,1) = \omega(s)$, $H(0,t) = x_0$ and $H(1,t) = x_1$. Define

$$G: [0,1] \times [0,1] \to X$$

by

$$G(s,t) = \begin{cases} H(2s,t), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

In that case, G is a pc map by Lemma 2.1.4, and

$$G(s,0) = \begin{cases} H(2s,0), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

$$= \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

$$= \alpha * \beta(s)$$

$$G(s,1) = \begin{cases} H(2s,1), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

$$= \begin{cases} \omega(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

$$= \omega * \beta(s).$$

Also we have $G(0,t) = H(0,t) = x_0$ and $G(1,t) = \beta(1) = x_2$. Thus, $\alpha * \beta \sim_{\delta,p} \omega * \beta$.

We can illustrate Proposition 2.1.34. by the Figure 16. As we can see in Figure 16(a), we have $\alpha \sim_{\delta,p} \omega$ and a proximal path β from x_1 to x_2, and in Figure 16(b) we have $\alpha * \beta \sim_{\delta,p} \omega * \beta$.

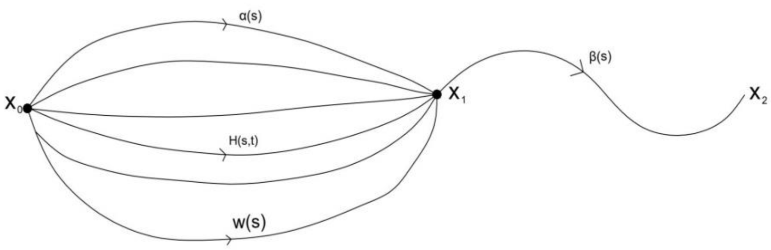

Figure 16(a). $\alpha \sim_{\delta,p} \omega$ and a proximal path β

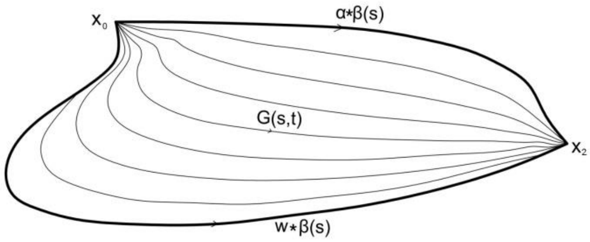

Figure 16(b). $\alpha * \beta \sim_{\delta,p} \omega * \beta$

Figure 16.

Proposition 2.1.35. Assume that (X,δ) is a proximity space, α and γ are two proximal paths between x_1 and x_2, and β is a proximal path between x_0 and x_1. If $\alpha \sim_{\delta,p} \gamma$, then $\beta * \alpha \sim_{\delta,p} \beta * \gamma$.

Proof: Since $\alpha \sim_{\delta,p} \gamma$, there is a pc map $F: [0,1] \times [0,1] \to X$ such that $F(s,0) = \alpha(s)$, $F(s,1) = \gamma(s)$, $F(0,t) = x_1$ and $F(1,t) = x_2$. Define

$$H: [0,1] \times [0,1] \to X$$

by

$$H(s,t) = \begin{cases} \beta(2s), & 0 \le s \le \frac{1}{2} \\ F(2s-1\,,t), & \frac{1}{2} \le s \le 1 \end{cases}$$

Then H is a pc map by Lemma 2.1.4., and

$$H(s,0) \;=\begin{cases} \beta(2s), & 0 \le s \le \frac{1}{2} \\ F(2s-1\,,0), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$=\begin{cases} \beta(2s), & 0 \le s \le \frac{1}{2} \\ \alpha(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$= \quad \beta * \alpha(s)$$

$$H(s,1) \quad =\begin{cases} \beta(2s), & 0 \le s \le \frac{1}{2} \\ F(2s-1\,,1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$=\begin{cases} \beta(2s), & 0 \le s \le \frac{1}{2} \\ \gamma(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$= \beta * \gamma(s).$$

Also, $H(0,t) = \beta(0) = x_0$ and $H(1,t) = F(1,t) = x_2$. Thus, $\beta * \alpha \sim_{\delta,p} \beta * \gamma$.

As we state in Proposition 2.1.26, the proximal path homotopy relation $\sim_{\delta,p}$ is an equivalence relation, so we can define the proximal equivalence classes as the following.

Definition 2.1.36. Let (X,δ) be a proximity space, and $\alpha: [0,1] \to X$ a proximal path between x_0 and x_1. Then $[\alpha] = \{\, \beta: [0,1] \to X \,:\, \alpha \sim_{\delta,p} \beta \,\}$ is the proximal equivalence classes of α.

Remark 2.1.37. Let (X, δ) be a proximity space, γ and ρ proximal paths between x_1 and x_2. Then $\gamma \sim_{\delta,p} \rho$ if and only if $[\gamma] = [\rho]$.

Proposition 2.1.38. Let (X, δ_1) and (Y, δ_2) be proximity spaces, $\alpha: [0,1] \to X$ a proximal path between x_0 and x_1, $\beta: [0,1] \to X$ a proximal path between x_1 and x_2. If $k: (X, \delta_1) \to (Y, \delta_2)$ is proximally continuous, then $[k \circ (\alpha * \beta)] = [(k \circ \alpha) * (k \circ \beta)]$.

Proof: To show that $[k \circ (\alpha * \beta)] = [(k \circ \alpha) * (k \circ \beta)]$, we need to prove that $k \circ (\alpha * \beta) \sim_{\delta,p} (k \circ \alpha) * (k \circ \beta)$. In that case

$$\alpha * \beta\,(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

$$k \circ (\alpha * \beta) = \begin{cases} k \circ \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ k \circ \beta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

$$= (k \circ \alpha) * (k \circ \beta)(s)\,.$$

So we conclude that $k \circ (\alpha * \beta) \sim_{\delta,p} (k \circ \alpha) * (k \circ \beta)$. Hence, $[k \circ (\alpha * \beta)] = [(k \circ \alpha) * (k \circ \beta)]$.

Now we will present the proximal retract and proximal deformation for a subset of a proximity space.

Definition 2.1.39. Assume $\emptyset \neq A \subset X$ and δ is a proximity for A and X. Then we say that A is a proximal retract of X provided there is a pc map $r: X \to A$ such that $r(a) = a$ for all $a \in A$. Such a map r is called a proximal retraction of X onto A.

Example 2.1.40. Let (X, δ) be a proximity space, and $X = A \cup B$ where A and B are closed subspaces of X with $A \cap B = \{x_0\}$. Then A is a proximal retract of X.

Proof : Let $r: X \to A$ be defined by

$$r(x) = \begin{cases} x, & x \in A \\ x_0, & x \in B \end{cases}$$

Then r is a pc map by Lemma 2.1.4. Furthermore, $r(a) = a$, for all $a \in A$. Thus, r is a proximal retraction and A is a proximal retract of X.

Definition 2.1.41. Let (X, δ) be a proximity space, and $A \subset X$. Then A is called a proximal deformation retract of X provided there is a pc map

$$H: X \times [0,1] \to X$$

by

i. $H(x, 1) \in A$ and $H(x, 0) = x$ for every $x \in X$;
ii. $H(a, t) = a$ for every $a \in A$, $t \in [0,1]$.

Such a map H is called a proximal deformation of X into A.

Remark 2.1.42. Note that Definition 2.1.39. can be rewritten as the following:

Let A be subspace of a proximity space X, and $i: A \hookrightarrow X$ is a proximal inclusion. If there is a pc map $r: X \to A$ such that $r \circ i = id_A$, then we say that A is a proximal retract of X and r is a proximal retraction. However, the two definitions are equivalent since $r \circ i = id_A$, for all $a \in A$, then we have $r \circ i\ (a) = a$, so $r(a) = a$.

Proposition 2.1.43. A proximal deformation retract $A \subset X$ is a proximal retract of X, and for the proximal inclusion $i: A \hookrightarrow X$, we have $i \circ r \sim_\delta id_X$. Moreover, $i \circ r$ and $r \circ i$ are proximal homotopy equivalence.

Proof: Since A is a proximal deformation retract of X, there is a proximal deformation map $H: X \times [0,1] \to X$, so we have $H(x, 1) \in A$. Then we can write $H(x, 1) = i(r(x))$ where $r: X \to A$ is a pc map. Hence, for $a \in A$ we have $H(a, 1) = i\big(r(a)\big) = a$ so that $r(a) = a$. Thus, r is a proximal retraction. Now we notice that H is a proximal homotopy between $i \circ r$ and id_X which follows directly from the definition of H and $r \circ i = id_A$. Since r is a proximal retraction, then we can conclude that $i \circ r$ and $r \circ i$ are proximal homotopy equivalence.

Proposition 2.1.44. (Peters & Vergili, 2023): Let (X, δ) be a proximally contractible space, and A a proximal retract of X, then (A, δ) is a proximally contractible space.

Proof: Since X is proximally contractible, then $id_x \sim_\delta c$ where $c: X \to X$ such that $c(x) = x_0$. Then there is a pc map $H: X \times [0,1] \to X$ such that $H(x, 0) = id_x(x) = x$ and $H(x, 1) = c(x) = x_0$. We will find a pc map $G: A \times [0,1] \to A$ such that $G(a, 0) = id_A$ and $G(a, 1) = k(a) = a_0$, where $k: A \to A$ is a proximal constant map at $a_0 \in A$. Define

$$G \coloneqq r \circ H \circ (i \times id_{[0,1]}).$$

In that case $G: A \times [0,1] \to A$ is a pc where r and i are proximal retraction and proximal inclusion, respectively. Since A is a proximal retract of X, then $G(a, 0) = r\big(H(a, 0)\big) = r(a) = a = id_A$ and $G(a, 1) = r\big(H(a, 1)\big) = r(x_0) = r(c(a))$ where $r(c)$ is a constant map. Letting $r \circ c = k$, then (A, δ) is a proximally contractible space.

Proposition 2.1.45. Let (X, δ) be a proximally path connected space, and A a proximal retract of X. Then (A, δ) is a proximally path connected space.

Proof: Let $a, b \in A$. We will find a proximal path $\beta: [0,1] \to A$ from a to b so that $\beta(0) = a$ and $\beta(1) = b$. However, X is proximally path connected, there is a pc map $\alpha: [0,1] \to X$ such that $\alpha(0) = a$ and $\alpha(1) = b$. Define $\beta \coloneqq r \circ \alpha$, where r is a proximal retraction. Then $\beta: [0,1] \to A$ is a pc map with $\beta(0) = r\big(\alpha(0)\big) = r(a) = a$ and $\beta(1) = r\big(\alpha(1)\big) = r(b) = b$. Thus, (A, δ) is a proximally path connected space.

Proposition 2.1.46. Suppose (X, δ) is a proximity space, $A, B \subset X$. If A and B are proximal deformation retracts of X, then $A \ \simeq_\delta B$.

Proof: To show that $A \simeq_\delta B$, we need to find pc maps $f: A \to B$ and $g: B \to A$ such that $f \circ g \sim_\delta id_B$ and $g \circ f \sim_\delta id_A$. Since A is a proximal deformation retract of X, there a proximal inclusion $i: A \hookrightarrow X$ and a proximal retraction $r: X \to A$ such that $r \circ i = id_A$ and $i \circ r \sim_\delta id_X$. Moreover since B is a proximal deformation retraction of X, then there are a proximal inclusion $j: B \hookrightarrow X$ and a proximal retract $s: X \to B$ such that $s \circ j = id_B$ and $j \circ s \sim_\delta id_X$. Define $f: A \to B$ by $f \coloneqq s \circ i$ and $g: B \to A$ by $g \coloneqq r \circ j$. In that case f and g are pc maps. Furthermore,

$f \circ g = s \circ (i \circ r) \circ j \sim_\delta s \circ id_X \circ j = s \circ j = id_B$. Thus,

$$f \circ g \sim_\delta id_B.$$

$g \circ f = r \circ (j \ \circ s) \circ i \sim_\delta r \circ id_X \circ i = r \circ i = id_A$. So,

$$g \circ f \sim_\delta id_A.$$

Hence, $A \simeq_\delta B$.

2.2. Descriptive Homotopic Theory

We proved in Section 1.1. that the product space of descriptive proximity spaces is also a descriptive proximity space. In this section, we will present the descriptive proximal homotopy between two descriptive proximally maps and some of its properties.

Definition 2.2.1. (Peters & Vergili, 2021): Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be descriptive proximity spaces and $f, g: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ two dpc maps. Then we say that f and g are descriptive proximally homotopic if there exists a map

$$F: X \times [0,1] \to Y,$$

such that

i. F is a dpc map, and

ii. $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

The map F is called a descriptive proximal homotopy between f and g. If f and g are descriptive proximally homotopic, we write $f \sim_\Phi g$.

In Figure 17, we have two descriptive proximally continuous maps f and g, also they are descriptive proximally homotopic since we can continuously deform f to g during the time $t \in [0,1]$.

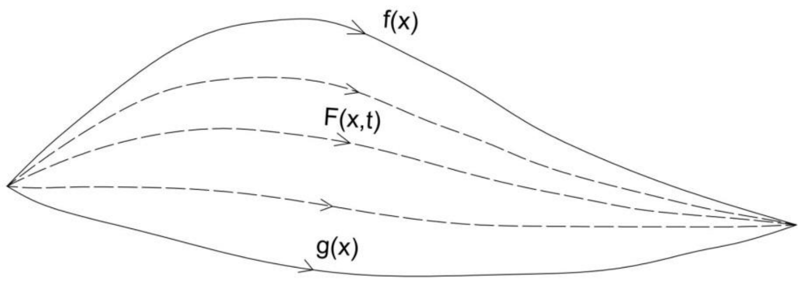

Figure 17. Descriptive proximally homotopic maps f and g

Remark 2.2.2. The descriptive nearness relation on [0,1] is defined in the following manner: Let A and B be two subsets of $[0,1]$, then A is descriptively near to B provided $D(A,B) = 0$.

In Proposition 2.2.3. and Proposition 2.2.4. we will prove that the descriptive homotopy is preserved under from the left and right composition of descriptive proximally continuous maps.

Proposition 2.2.3. Given two dpc maps $f,g:(X,\delta_{\Phi_1}) \to \left(Y,\delta_{\Phi_2}\right)$ and $h:(Y,\delta_{\Phi_2}) \to (Z,\delta_{\Phi_3})$, if $f \sim_\Phi g$, then $h \circ f \sim_\Phi h \circ g$.

Proof: Since $f \sim_\Phi g$, there is $F: X \times [0,1] \to Y$ such that F is a dpc map with $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We want to show that $h \circ f \sim_\Phi h \circ g$. Notice that $h \circ f, h \circ g$: $(X,\delta_{\Phi_1}) \to \left(Z,\delta_{\Phi_3}\right)$ are dpc maps. Now we should find a map $G: X \times [0,1] \to Z$ such that G is a dpc map with $G(x,0) = h \circ f(x)$ and $G(x,1) = h \circ g(x)$. Define $G := h \circ F$, G is a dpc map according to Proposition 1.1.41, and $G(x,0) = h \circ F(x,0) = h\big(F(x,0)\big) = h\big(f(x)\big) = h \circ f(x)$ and $G(x,1) = h \circ F(x,1) = h\big(F(x,1)\big) = h\big(g(x)\big) = h \circ g(x)$. Hence, $h \circ f \sim_\Phi h \circ g$.

Proposition 2.2.4. Given two dpc maps f,g: $(X,\delta_{\Phi_1}) \to (Y,\delta_{\Phi_2})$ and $k:(W,\delta_{\Phi_0}) \to (X,\delta_{\Phi_1})$ a dpc map, if $f \sim_\Phi g$, then $f \circ k \sim_\Phi g \circ k$.

Proof: Let $H: X \times [0,1] \to Y$ be a descriptive proximal homotopy map between f and g. Then H is a dpc map with $H(x,0) = f(x)$ and $H(x,1) = g(x)$. To show that the two dpc maps $f \circ k$ and $g \circ k$ are descriptive proximally homotopic, define

$$F: W \times [0,1] \to Y \text{ by } F(w,t) = H(k(w),t).$$

Then F is a dpc map and $F(w,0) = H(k(w),0) = f\big(k(w)\big) = f \circ k(w)$, and $F(w,1) = H(k(w),1) = g\big(k(w)\big) = g \circ k(w)$. Thus, $f \circ k \sim_\Phi g \circ k$.

Lemma 2.2.5. (Peters & Vergili, 2021): Let (X,δ_{Φ_1}) and (Y,δ_{Φ_2}) be descriptive proximity spaces, A and B be two descriptively closed subsets of X such that $X = A \cup B$. Suppose $f:(A,\delta_{\Phi_1}) \to (Y,\delta_{\Phi_2})$ and $g:(B,\delta_{\Phi_1}) \to (Y,\delta_{\Phi_2})$ are dpc maps such that $f(x) = g(x)$ for all $x \in A \cap B$, then the map $h: \left(X,\delta_{\Phi_1}\right) \to \left(Y,\delta_{\Phi_2}\right)$ defined by

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

is also a dpc map.

Proposition 2.2.6. (Peters & Vergili, 2021): The descriptive proximal homotopy relation $\sim_\Phi$ is an equivalence relation.

Proof: To show that $\sim_\Phi$ is an equivalence relation, then the reflexive, transitivity and symmetry properties will be proved. Let $f, g, h: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ be dpc maps.

i. $\sim_\Phi$ *is reflexive*: We want to show that $f \sim_\Phi f$. Define $F: X \times [0,1] \to Y$ such that $F(x,t) = f(x)$. Then it is obvious that F is a dpc map and $F(x,0) = F(x,1) = f(x)$. Hence $f \sim_\Phi f$.

ii. $\sim_\Phi$ *is symmetric*: Assume $f \sim_\Phi g$, we want to show that $g \sim_\Phi f$. Since $f \sim_\Phi g$, there is a dpc map $H: X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Define $G := H(x, 1-t)$, then $G: X \times [0,1] \to Y$ is a dpc map, $G(x,0) = H(x,1) = g(x)$, and $G(x,1) = H(x,0) = f(x)$. So that $g \sim_\Phi f$.

iii. $\sim_\Phi$ *is transitive:* Suppose $f \sim_\Phi g$ and $g \sim_\Phi h$. We want to show that $f \sim_\Phi h$. Since $f \sim_\Phi g$, there is a dpc map $F: X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$, and since $g \sim_\Phi h$, there is a dpc map $G: X \times [0,1] \to Y$ such that $G(x,0) = g(x)$ and $G(x,1) = h(x)$. Now to show that $f \sim_\Phi h$, define $H : X \times [0,1] \to Y$ by

$$H(x,t) = \begin{cases} F(x,2t), & 0 \le t \le \dfrac{1}{2} \\ G(x,2t-1), & \dfrac{1}{2} \le t \le 1 \end{cases}$$

Then H is a dpc map by Lemma 2.2.5. and $H(x,0) = F(x,0) = f(x)$, $H(x,1) = G(x,1) = h(x)$. Hence $f \sim_\Phi h$.

As a result, the descriptive proximal homotopy relation $\sim_\Phi$ is an equivalence relation.

Proposition 2.2.7. Let $f,\tilde{f}: (X,\delta_{\Phi_1}) \to (Y,\delta_{\Phi_2})$ and $g,\tilde{g}: (Y,\delta_{\Phi_2}) \to (Z,\delta_{\Phi_3})$ be dpc maps. If $f \sim_\Phi \tilde{f}$ and $g \sim_\Phi \tilde{g}$, then $g \circ f \sim_\Phi \tilde{g} \circ \tilde{f}$.

Proof : From Proposition 2.2.3, we have $g \circ f \sim_\Phi g \circ \tilde{f}$ and from Proposition 2.2.4, we conclude that $g \circ \tilde{f} \sim_\Phi \tilde{g} \circ \tilde{f}$. Furthermore, since $\sim_\Phi$ is a transitive relation, $g \circ f \sim_\Phi \tilde{g} \circ \tilde{f}$.

Definition 2.2.8. (Peters & Vergili, 2021): Suppose (X,δ_{Φ_1}) and (Y,δ_{Φ_2}) are descriptive proximity spaces. A map $c: X \to Y$ is said to be descriptive constant if $c(x) = y$ for all $x \in X$ and for some $y \in Y$.

Definition 2.2.9. (Peters & Vergili, 2021): Let (X,δ_Φ) be a descriptive proximity space. Then X is descriptively contractible space provided the identity map on it is descriptive proximally homotopic to a descriptive constant map.

Proposition 2.2.10. Convex spaces are descriptively contractible.

Proof: Let X be a convex space. To show that X is descriptively contractible, consider the identity map $id_X: (X,\delta_\Phi) \to (X,\delta_\Phi)$ and a descriptive constant map $c: (X,\delta_\Phi) \to (X,\delta_\Phi)$, where $c(x) = x_0$. We claim that $id_X \sim_\Phi c$. Since X is convex, one can define a map $F: X \times [0,1] \to X$ by

$$F(x,t) = tx_0 + (1-t)x$$

Which is clearly a dpc map by Proposition 2.5. and Proposition 2.7. Moreover F satisfies $F(x,0) = x = id_X(x)$ and $F(x,1) = x_0 = c(x)$. Thus, $id_X \sim_\Phi c$. Hence, X is descriptively contractible.

Proposition 2.2.11. If (X,δ_Φ) is a descriptively contractible space, then (X,δ_Φ) is descriptively path connected.

Proof: Since X is descriptively contractible, then $id_X \sim_\Phi c$, $i.e.$, there is a dpc map $H: X \times [0,1] \to X$ such that $H(x,0) = id_X(x) = x$ and $H(x,1) = c(x) = x_0$. To show that X is descriptively path connected, it is enough to take a point $x_1 \in X$ and prove that there is a descriptive path from x_1 to x_0. Define a descriptive path

$$\beta: [0,1] \to X \text{ by } \beta(t) = H \circ \alpha.$$

where $\alpha: [0,1] \to X \times [0,1]$ is a map defined by $\alpha(t) = (x_1, t)$. Since α and H are dpc maps, then so is β. Moreover, $\beta(0) = H(\alpha(0)) = H(x_1, 0) = id_X(x_1) = x_1$ and $\beta(1) = H(\alpha(1)) = H(x_1, 1) = c(x_1) = x_0$. So β is a descriptive path from x_1 to x_0. Thus, (X, δ_Φ) is descriptively path connected.

Proposition 2.2.12. Let (X, δ_{Φ_1}) be descriptively contractible. If $X \approx_{\Phi_1,\Phi_2} Y$, then (Y, δ_{Φ_2}) is also a descriptively contractible space.

Proof: To show that (Y, δ_{Φ_2}) is a descriptively contractible space, we will find a descriptive constant map that is descriptive proximally homotopic to id_Y. Since $X \approx_{\Phi_1,\Phi_2} Y$, then there is a bijective dpc map $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$, and its inverse $f^{-1}: (Y, \delta_{\Phi_2}) \to (X, \delta_{\Phi_1})$ is also a dpc map. Since (X, δ_{Φ_1}) is descriptively contractible, there is a descriptive constant map $c: (X, \delta_{\Phi_1}) \to (X, \delta_{\Phi_1})$ defined by $c(x) = x_0$ so that $id_X \sim_\Phi c$, which means that there is a dpc map $H: X \times [0,1] \to X$ such that $H(x, 0) = x$ and $H(x, 1) = x_0$. Define

$$G: Y \times [0,1] \to Y \text{ by } G(y, t) = f \circ H\ (f^{-1}(y), t).$$

Then G is a dpc map satisfying, $G(y, 0) = f(H(f^{-1}(y), 0)) = f(f^{-1}(y)) = y = id_Y(y)$ and $G(y, 1) = f(H(f^{-1}(y), 1)) = f(x_0)$ which is constant. So, we can conclude that there is a descriptive constant map which is descriptively homotopic to id_Y. This implies that (Y, δ_{Φ_2}) is a descriptively contractible space.

Definition 2.2.13. Let (X, δ_{Φ_1}) be a descriptive proximity space, P a descriptive proximal property in (X, δ_{Φ_1}), and $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ a descriptive proximal isomorphism. If (Y, δ_{Φ_2}) also has the descriptive proximal property P, then we call P a descriptive invariant property.

From Definition 2.2.13, we can conclude that the descriptively contractible space is a descriptive invariant property as shown in Proposition 2.2.12.

Corollary 2.2.14. Descriptively contractible property is a descriptive invariant property.

Proposition 2.2.15. The product of two descriptively contractible spaces is descriptively contractible.

Proof: Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be two descriptive proximally contractible spaces. We need to show that $X \times Y$ is also descriptive proximally contractible. Since (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) are descriptively contractible spaces, $id_X \sim_\Phi c_{x_0}$ where c_{x_0} is a descriptive constant map at $x_0 \in X$ and $id_Y \sim_\Phi c_{y_0}$where c_{y_0} is a descriptive constant map at $y_0 \in Y$, respectively. It is natural to consider

$$id_X \times id_Y := id_{X \times Y} \text{ and } c_{x_0} \times c_{y_0} \coloneqq c_{(x_0,y_0)}$$

We claim that $id_{X \times Y} \sim_\Phi c_{(x_0,y_0)}$, where $id_{X \times Y}: X \times Y \to X \times Y$ is the identity map on $X \times Y$ and $c_{(x_0,y_0)}: X \times Y \to X \times Y$ is a dpc constant map defined by $c_{(x_0,y_0)}(x, y) = (x_0\,, y_0)$. However $id_X \sim_\Phi c_{x_0}$ and $id_Y \sim_\Phi c_{y_0}$, then there exist descriptive proximal homotopies H_1 and H_2 between id_X and c_{x_0}, id_Y and c_{y_0}, respectively. Define

$$pr_X \times id_{[0,1]}: X \times Y \times [0,1] \to X \times [0,1] \text{ by } pr_X(x, y, t) = (x, t)$$

and

$$pr_Y \times id_{[0,1]}: X \times Y \times [0,1] \to Y \times [0,1] \text{ by } pr_Y(x, y, t) = (y, t)$$

Then $pr_X \times id_{[0,1]}$ and $pr_Y \times id_{[0,1]}$ are dpc maps. Let

$$H: X \times Y \times [0,1] \to X \times Y$$

Be defined by

$$H(x, y, t) = (H_1 \circ\, pr_X \times id_{[0,1]}, H_2 \,\circ\, pr_Y \times id_{[0,1]}).$$

Then H is a dpc map satisfying $H(x, y, 0) = \big(H_1(x, 0), H_2(y, 0)\big) = (x, y) = id_{X \times Y}(x, y)$, and $H(x, y, 1) = \big(H_1(x, 1), H_2(y, 1)\big) = (x_0, y_0) = c_{(x_0,y_0)}(x, y)$. Hence $id_{X \times Y} \sim_\Phi c_{(x_0,y_0)}$ and $X \times Y$ is descriptively contractible.

Corollary 2.2.16. Let $\{(X_i\,, \delta_{\Phi_i})\}_{i \in I}$ be a collection of descriptively contractible spaces where I is a finite index set. Then the product $X = \prod_{i \in I} X_i$ is also a descriptively contractible space.

Proof: The proof follows from the mathematical induction method.

i. For $n = 1$, it is straight forward.

ii. For $n = 2$: Let X_1 and X_2 be descriptively contractible spaces. Then $X_1 \times X_2$ is descriptively contractible by Proposition 2.2.15.

iii. For $n = k$: Assume that if $X_1, \ldots, X_k$ are descriptively contractible spaces, then $X_1 \times X_2 \times \ldots \times X_k$ is descriptively contractible. Now we will prove it for the case $n = k + 1$: If $X_1, \ldots, X_k, X_{k+1}$ are descriptively contractible spaces, then $X = (\prod_{i=1}^{i=k} X_i) \times X_{k+1}$ is descriptively contractible, since X_{k+1} is descriptively contractible and $\prod_{i=1}^{i=k} X_i$ is descriptively contractible for $n = k$.

Definition 2.2.17. (Peters & Vergili, 2021): A dpc map $f: X \longrightarrow Y$ between two descriptive proximity spaces X, Y is said to be descriptive proximally null-homotopic if it is descriptive proximally homotopic to a descriptive constant map.

In the article (Peters & Vergili, 2021), spatial homotopy equivalent was given, so in the following definition we will define homotopy equivalent in a descriptive proximity space.

Definition 2.2.18. Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be two descriptive proximity spaces. We say X and Y are descriptive proximally homotopy equivalent, denoted by $X \simeq_\Phi Y$ provided there are two dpc maps $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ and $g: (Y, \delta_{\Phi_2}) \to (X, \delta_{\Phi_1})$ such that $f \circ g \sim_\Phi id_Y$ and $g \circ f \sim_\Phi id_X$.

Proposition 2.2.19. Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be two descriptive proximity spaces. If $X \approx_{\Phi_1, \Phi_2} Y$, then $X \simeq_\Phi Y$.

Proof: Since $X \approx_{\Phi_1, \Phi_2} Y$, there is a bijective dpc map $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ such that its inverse $f^{-1}: (Y, \delta_{\Phi_2}) \to (X, \delta_{\Phi_1})$ is also a dpc map. Then $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$. Hence, $f^{-1} \circ f \sim_\Phi id_X$ and $f \circ f^{-1} \sim_\Phi id_Y$. Thus, X and Y are descriptive proximally homotopy equivalent.

Proposition 2.2.20. If (X, δ_Φ) is descriptively contractible, then any descriptive proximity space Y is descriptive proximally homotopy equivalent to $X \times Y$.

Proof: Let $f: X \times Y \to Y$ be a map defined by $f(x, y) = y$. Then f is a dpc map. Define $g: Y \to X \times Y$ by $g(y) = (x_0, y)$, for a fixed $x_0 \in X$. Then g is also a dpc map. Now $f \circ g: Y \to Y$ is equal to id_Y, since $f \circ g(y) = f(x, y) = y$. Thus, $f \circ g \sim_\Phi id_Y$. Moreover,

$g \circ f: X \times Y \to X \times Y$ is a descriptive constant map because $g \circ f(x,y) = g(y) = (x_0, y)$. However, X is descriptively contractible, then $g \circ f \sim_{\Phi} id_{X\times Y}$. Thus Y is descriptive proximally homotopy equivalent to $X \times Y$.

Proposition 2.2.21. Let (X, δ_Φ) be a descriptive proximity space, and $x_0 \in X$. Then X is descriptively contractible if and only if X is descriptive proximally homotopy equivalent to $\{x_0\}$.

Proof: Let $f: X \to \{x_0\}$ defined by $f(x) = x_0$ and $g: \{x_0\} \to X$ defined by $g(x_0) = x_0$. Then $f \circ g: \{x_0\} \to \{x_0\}$ is descriptive proximally homotopic to $id_{\{x_0\}}$. Also $g \circ f: X \to X$ is descriptive constant map because $g\big(f(x)\big) = g(x_0) = x_0$. Since X is descriptively contractible, then $g \circ f$ is descriptive proximally homotopic to id_X. Thus, X is descriptive proximally homotopy equivalent to $\{x_0\}$.

Conversely, assume that X is descriptive proximally homotopy equivalent to $\{x_0\}$. We will show that X is descriptively contractible. Since X is descriptive proximally homotopy equivalent to $\{x_0\}$, then there are dpc maps $f: X \to \{x_0\}$ and $g: \{x_0\} \to X$ with $f \circ g \sim_\Phi id_{x_0}$ and $g \circ f \sim_\Phi id_X$. Notice that the descriptive constant map $g \circ f$ is descriptive proximally homotopic to id_X. So, X is descriptively contractible.

Proposition 2.2.22. The descriptive proximally homotopy equivalent relation $\simeq_\Phi$ is an equivalence relation.

Proof: We will show that $\simeq_\Phi$ is reflexive, symmetric, and transitive.

i. $\simeq_\Phi$ *is reflexive:* Since $(X, \delta_{\Phi_1}) \approx_{\Phi_1, \Phi_2} (X, \delta_{\Phi_1})$, by Proposition 2.2.19. $(X, \delta_{\Phi_1}) \simeq_\Phi (X, \delta_{\Phi_1})$

ii. $\simeq_\Phi$ *is symmetric*: Assume that $(X, \delta_{\Phi_1}) \simeq_\Phi (Y, \delta_{\Phi_2})$. We need to show that $(Y, \delta_{\Phi_2}) \simeq_\Phi (X, \delta_{\Phi_1})$. Since $(X, \delta_{\Phi_1}) \simeq_\Phi (Y, \delta_{\Phi_2})$, then there are two dpc maps

$$f: \big(X, \delta_{\Phi_1}\big) \to (Y, \delta_{\Phi_2}) \text{ and } g: \big(Y, \delta_{\Phi_2}\big) \to (X, \delta_{\Phi_1})$$

such that $f \circ g \sim_\Phi id_Y$ and $g \circ f \sim_\Phi id_X$. To show that $(Y, \delta_{\Phi_2}) \simeq_\Phi (X, \delta_{\Phi_1})$ we need to find two dpc maps $h: \big(Y, \delta_{\Phi_2}\big) \to (X, \delta_{\Phi_1})$ and $k: \big(X, \delta_{\Phi_1}\big) \to (Y, \delta_{\Phi_2})$ such that $h \circ k \sim_\Phi id_X$ and $k \circ h \sim_\Phi id_Y$. Take $h \coloneqq g$ and $k \coloneqq f$. So the proof is complete.

iii. $\simeq_\Phi$ *is transitive*: Let $(X,\delta_{\Phi_1}) \simeq_\Phi (Y,\delta_{\Phi_2})$ and $(Y,\delta_{\Phi_2}) \simeq_\Phi (Z,\delta_{\Phi_3})$. We will show that $(X,\delta_{\Phi_1}) \simeq_\Phi (Z,\delta_{\Phi_3})$. Since $(X,\delta_{\Phi_1}) \simeq_\Phi (Y,\delta_{\Phi_2})$, then there exist dpc maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f \sim_\Phi id_X$ and $f \circ g \sim_\Phi id_Y$. We also have $(Y,\delta_{\Phi_2}) \simeq_\Phi (Z,\delta_{\Phi_3})$, so there exist dpc maps $h: Y \to Z$ and $k: Z \to Y$ such that $h \circ k \sim_\Phi id_z$ and $k \circ h \sim_\Phi id_Y$. We claim that the dpc map $(h \circ f) \circ (g \circ k)$ is descriptive proximally homotopic to id_Z. Since $f \circ g \sim_\Phi id_Y$ implies $h \circ (f \circ g) \sim_\Phi h \circ id_Y$ by Proposition 2.2.3 and $(h \circ f \circ g) \circ k \sim_\Phi (h \circ id_Y) \circ k$ by Proposition 2.2.4. We have,

$$(h \circ f) \circ (g \circ k) \sim_\Phi h \circ k$$

Since $\sim_\Phi$ is transitive then $h \circ k \sim_\Phi id_z$, we have $(h \circ f) \circ (g \circ k) \sim_\Phi id_z$. Moreover, we have $(g \circ k) \circ (h \circ f) \sim_\Phi id_X$, because $k \circ h \sim_\Phi id_Y$ implies $g \circ (k \circ h) \sim_\Phi g \circ id_Y$ and $(g \circ k \circ h) \circ f \sim_\Phi (g \circ id_Y) \circ f$. Since $\sim_\Phi$ is transitive then $g \circ f \sim_\Phi id_X$, we have $(g \circ k) \circ (h \circ f) \sim_\Phi id_X$. Thus, $(X,\delta_{\Phi_1}) \simeq_\Phi (Z,\delta_{\Phi_3})$.

Proposition 2.2.23. Suppose descriptive proximal isomorphism maps $f, g: (X,\delta_{\Phi_1}) \to (Y,\delta_{\Phi_2})$ are descriptively homotopic maps. Then $f^{-1}, g^{-1}: (Y,\delta_{\Phi_2}) \to (X,\delta_{\Phi_1})$ are descriptive proximally homotopic.

Proof: Since $f \sim_\Phi g$, there is a dpc map $H: X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Because f and g are descriptive proximally isomorphisms, f^{-1} and g^{-1} are dpc maps. Define

$$G: Y \times [0,1] \to X \text{ by } G(y,t) = f^{-1}(H(g^{-1}(y), 1-t)).$$

Then G is a dpc map. Moreover, $G(y,0) = f^{-1}(H(g^{-1}(y),1)) = f^{-1}(g(g^{-1}(y))) = f^{-1}(y)$ and $G(y,1) = f^{-1}(H(g^{-1}(y),0)) = f^{-1}(f(g^{-1}(y))) = g^{-1}(y)$. Hence, $f^{-1} \sim_\Phi g^{-1}$.

Definition 2.2.24. (Peters & Vergili, 2021): Suppose (X,δ_{Φ_1}) and (Y,δ_{Φ_2}) are descriptive proximity spaces. Let A be a subset of X. Then two dpc maps $f: (X,\delta_{\Phi_1}) \to (Y,\delta_{\Phi_2})$ and $g: (X,\delta_{\Phi_1}) \to (Y,\delta_{\Phi_2})$ are said to be descriptive proximally homotopic relative to A, denoted by $f \sim_\Phi g\ (rel\ A)$ provided there exists a descriptive proximal homotopy H between f and g such that $H(a,t) = f(a) = g(a)$ for all $a \in A$ and $t \in [0,1]$.

Proposition 2.1.25. Given $A \subseteq X$ and $B \subseteq Y$, let $f_0, f_1: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ be dpc maps with $f_0|_A = f_1|_A$ and $f_i(A) \subseteq$ B for $i = 0,1$. Also assume that $g_0, g_1: (Y, \delta_{\Phi_2}) \to (Z, \delta_{\Phi_3})$ are dpc maps with $g_0|_B = g_1|_B$. If $f_0 \sim_\Phi f_1$ $(rel\, A)$ and $g_0 \sim_\Phi g_1$ $(rel\, B)$, then $g_0 \circ f_0 \sim_\Phi g_1 \circ f_1$ $(rel\, A)$.

Proof: Since $f_0 \sim_\Phi f_1$ $(rel\, A)$, there is a dpc map $F: X \times [0,1] \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. Furthermore, $F(a, t) = f_0(a) = f_1(a)$ for all $a \in A$ and $t \in [0,1]$. Since $g_0 \sim_\Phi g_1$ $(rel\, B)$, there is a dpc map $G: Y \times [0,1] \to Z$ such that $G(x, 0) = g_0(x)$ and $G(x, 1) = g_1(x)$. Moreover, $G(b, t) = g_0(b) = g_1(b)$ for all $b \in B$ and $t \in [0,1]$. Define

$$H: X \times [0,1] \to Z \text{ by } H(x, t) = G(F(x, t), t)$$

Then H is a dpc map, and

$H(x, 0) = G(F(x, 0), 0) = G(f_0(x), 0) = g_0(f_0(x)),$

$H(x, 1) = G(F(x, 1), 1) = G(f_1(x), 1) = g_1(f_1(x)).$

Also for all $a \in A$ and $t \in [0,1]$, we have

$$H(a, t) = G(F(a, t), t) = G(f_0(a), t) = G(f_1(a), t)$$

and since $f_i(A) \subseteq B$ for $i = 0,1$, then

$$G(f_0(a), t) = g_0(f_0(a))$$

and

$$G(f_1(a), t) = g_1(f_1(a)).$$

Hence $H(a, t) = g_0(f_0(a)) = g_1(f_1(a))$ for all $a \in A$ and $t \in [0,1]$. Thus, $g_0 \circ f_0 \sim_\Phi g_1 \circ f_1$ $(rel\, A)$.

Next, we will introduce the definition of the descriptive proximal path homotopy up to the definition of the descriptive proximal equivalent classes.

Definition 2.2.26. Let α and β be two descriptive proximal paths from x_0 to x_1 in (X, δ_Φ). Then we say α and β are descriptive proximally path homotopic, provided there is a dpc map

$$H: [0,1] \times [0,1] \to X,$$

such that

i. $H(s, 0) = \alpha(s), H(s, 1) = \beta(s),$

ii. $H(0, t) = x_0$ and $H(1, t) = x_1$.

Such a map H is called a descriptive proximal path homotopy between α and β and we write $\alpha \sim_{\Phi,p} \beta$ which means that there is a descriptive proximal path homotopy between them.

As shown in Figure 18, for α and β are descriptive proximal paths between x_0 and x_1. We can transform α continuously to get β through the time $t \in [0,1]$.

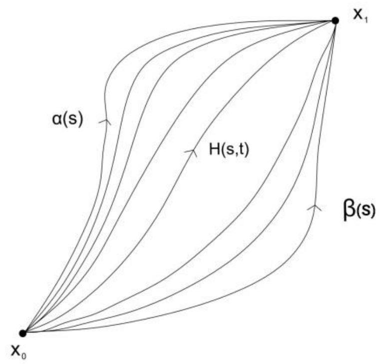

Figure 18. Descriptive proximally path homotopic α and β

Proposition 2.2.27. Proximal path homotopy relation $\sim_{\Phi,p}$ is an equivalence relation.

Proof: The proof follows similarly to that of Proposition 2.2.6.

Definition 2.2.28. Let α be a descriptive proximal path between x_0 and x_1 in a proximity space (X, δ_Φ). Then the reverse of α is the descriptive proximal path $\overline{\alpha}: [0,1] \to X$ defined by $\overline{\alpha}\,(s) = \alpha(1-s)$ from x_1 to x_0.

Proposition 2.2.29. Let (X, δ_Φ) be a descriptive proximity space, and α, β are descriptive proximal paths from x_0 to x_1 in X. If $\alpha \sim_{\Phi,p} \beta$, then $\overline{\alpha} \sim_{\Phi,p} \bar{\beta}$.

Proof: Since $\alpha \sim_{\Phi,p} \beta$, there is a dpc map $H: [0,1] \times [0,1] \to X$ such that $H(s,0) = \alpha(s)$, $H(s,1) = \beta(s)$, $H(0,t) = x_0$ and $H(1,t) = x_1$. Let $F: [0,1] \times [0,1] \to X$ be defined by $F(s,t) = H(1-s,t)$. Then F is a dpc map with $F(s,0) = H(1-s,0) = \alpha(1-s) = \bar{\alpha}(s)$, $F(s,1) = H(1-s,1) = \beta(1-s) = \bar{\beta}(s)$, $F(0,t) = H(1\,,t) = x_1$ and $F(1,t) = H(0,t) = x_0$. Thus, $\bar{\alpha} \sim_{\Phi,p} \bar{\beta}$.

Proposition 2.2.30. Suppose (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) are descriptive proximity spaces, and $f: X \to Y$ is a dpc map. If α and β are descriptive proximally paths between x_0 and x_1 such that $\alpha \sim_{\Phi,p} \beta$, then $f \circ \alpha \sim_{\Phi,p} f \circ \beta$.

Proof: Since $\alpha \sim_{\Phi,p} \beta$ we have a descriptive path homotopy $H: [0,1] \times [0,1] \to X$ between α and β. Define $F: = f \circ H : [0,1] \times [0,1] \to Y$ which is

a dpc map satisfying $F(s,0) = f \circ H(s,0) = f\big(H(s,0)\big) = f(\alpha(s))$, $F(s,1) = f \circ H(s,1) = f\big(H(s,1)\big) = f(\beta(s))$, $F(0,t) = f(x_0)$ and $F(1,t) = f(x_1)$. Hence, $f \circ \alpha \sim_{\Phi,p} f \circ \beta$.

Definition 2.2.31. Suppose (X, δ_Φ) is a descriptive proximity space, α is a descriptive proximal path between x_0 and x_1, β is a descriptive proximal path between x_1 and x_2. Then the product of α and β, $\alpha * \beta$, is defined by

$$\alpha * \beta\ (s) = \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \le s \le 1\,. \end{cases}$$

is a descriptive proximal path between x_0 and x_2.

Remark 2.2.32. The product of two descriptive proximal paths $\alpha * \beta$ is a descriptive proximal path provided it is a dpc map according to Lemma 2.2.5. with $\alpha * \beta(0) = x_0$ and $\alpha * \beta(1) = x_1$.

Definition 2.2.33. Let (X,δ_{Φ_1}) be a descriptive proximity space. The descriptive constant path $c\colon [0,1] \to X$ at $x_0 \in X$ is a descriptive path such that $c(t) = x_0$ for every $t \in [0,1]$.

Proposition 2.2.34. Suppose that (X,δ_Φ) is a descriptive proximity space, α and ω are a descriptive paths from x_0 to x_1, and β is a descriptive path from x_1 to x_2. If $\alpha \sim_{\Phi,p} \omega$, then $\alpha * \beta \sim_{\Phi,p} \omega * \beta$.

Proof: Since $\alpha \sim_{\Phi,p} \omega$, there is a dpc map $H\colon [0,1] \times [0,1] \to X$ such that $H(s,0) = \alpha(s)$, $H(s,1) = \omega(s)$, $H(0,t) = x_0$ and $H(1,t) = x_1$. Define

$G\colon [0,1] \times [0,1] \to X$ by

$$G(s,t) = \begin{cases} H(2s,t), & 0 \le s \le \dfrac{1}{2} \\ \beta(2s-1), & \dfrac{1}{2} \le s \le 1 \end{cases}.$$

In that case, G is a dpc map by Lemma 2.2.5. Moreover,

$$G(s,0) = \begin{cases} H(2s,0), & 0 \le s \le \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$= \begin{cases} \alpha(2s), & 0 \le s \le \dfrac{1}{2} \\ \beta(2s-1), & \dfrac{1}{2} \le s \le 1 \end{cases}$$

$$= \alpha * \beta(s)$$

$$G(s,1) = \begin{cases} H(2s,1), & 0 \le s \le \dfrac{1}{2} \\ \beta(2s-1), & \dfrac{1}{2} \le s \le 1 \end{cases}$$

$$= \begin{cases} \omega(2s), & 0 \le s \le \dfrac{1}{2} \\ \beta(2s-1), & \dfrac{1}{2} \le s \le 1 \end{cases}$$

$$= \omega * \beta(s)\,.$$

We also have, $G(0,t) = H(0,t) = x_0$ and $G(1,t) = \beta(1) = x_2$. Thus, $\alpha * \beta \sim_{\Phi,p} \omega * \beta$.

Proposition 2.2.35. Assume that (X, δ_Φ) is a descriptive proximity space, α and γ are descriptive paths from x_1 to x_2, and β is a descriptive path from x_0 to x_1. If $\alpha \sim_{\Phi,p} \gamma$, then $\beta * \alpha \sim_{\Phi,p} \beta * \gamma$.

Proof: Since $\alpha \sim_{\Phi,p} \gamma$ there is a dpc map $F: [0,1] \times [0,1] \to X$ such that $F(s,0) = \alpha(s)$, $F(s,1) = \gamma(s)$, $F(0,t) = x_1$ and $F(1,t) = x_2$. Define

$H: [0,1] \times [0,1] \to X$ by

$$H(s,t) = \begin{cases} \beta(2s), & 0 \le s \le \frac{1}{2} \\ F(2s-1\,,t), & \frac{1}{2} \le s \le 1 \end{cases} .$$

Then H is a dpc map by Lemma 2.2.5. Moreover,

$$H(s,0) = \begin{cases} \beta(2s), & 0 \le s \le \frac{1}{2} \\ F(2s-1\,,0), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$= \begin{cases} \beta(2s), & 0 \le s \le \frac{1}{2} \\ \alpha(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$= \quad \beta * \alpha(s) .$$

$$H(s,1) = \begin{cases} \beta(2s), & 0 \le s \le \frac{1}{2} \\ F(2s-1\,,1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$= \begin{cases} \beta(2s), & 0 \le s \le \frac{1}{2} \\ \gamma(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$= \beta * \gamma(s) .$$

We also have, $H(0,t) = \beta(0) = x_0$ and $H(1,t) = F(1,t) = x_2$. Thus $\beta * \alpha \sim_{\Phi,p} \beta * \gamma$.

As shown in Proposition 2.2.27, the descriptive proximal path homotopy relation $\sim_{\Phi,p}$ is an equivalence relation, so we can define the descriptive proximal equivalence classes as follows.

Definition 2.2.36. Let (X, δ_Φ) be a descriptive proximity space, and $\alpha: [0,1] \to X$ be a descriptive proximal path from x_0 to x_1. Then $[\alpha] = \{\, \beta: [0,1] \to X \,:\, \alpha \sim_{\Phi,p} \beta \,\}$ is the descriptive proximal equivalence class of α.

Remark 2.2.37. Let (X, δ_Φ) be a descriptive proximity space, γ and ρ be descriptive proximal paths from x_1 to x_2. Then $\gamma \sim_{\Phi,p} \rho$ if and only if $[\gamma] = [\rho]$.

Proposition 2.2.38. Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be descriptive proximity spaces, $\alpha: [0,1] \to X$ be a descriptive path from x_0 to x_1, $\beta: [0,1] \to X$ be a descriptive path from x_1 to x_2. If $k: \left(X, \delta_{\Phi_1}\right) \to (Y, \delta_{\Phi_2})$ is a dpc map, then $[k \circ (\alpha * \beta)] = [(k \circ \alpha\,) * (k \circ \beta)]$.

Proof: To show that $[k \circ (\alpha * \beta)] = [(k \circ \alpha\,) * (k \circ \beta)]$, we need to prove that $k \circ (\alpha * \beta)$ $\sim_{\Phi,p} (k \circ \alpha) * (k \circ \beta)$. Start with

$$\alpha * \beta\,(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \dfrac{1}{2} \\ \beta(2s-1), & \dfrac{1}{2} \leq s \leq 1 \end{cases}$$

so

$$k \circ (\alpha * \beta) = \begin{cases} k \circ \alpha(2s), & 0 \leq s \leq \dfrac{1}{2} \\ k \circ \beta(2s-1), & \dfrac{1}{2} \leq s \leq 1 \end{cases}$$

$$= (k \circ \alpha) * (k \circ \beta)(s).$$

Thus we conclude that $k \circ (\alpha * \beta) \sim_{\Phi,p} (k \circ \alpha\,) * (k \circ \beta)$. Hence, $[k \circ (\alpha * \beta)] = [(k \circ \alpha) * (k \circ \beta)]$.

Definition 2.2.39. (Peters & Vergili, 2021): Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be descriptive proximity spaces. Then for a map $d: X \to Y$ is said to be a degenerate descriptive constant map if $\Phi_2\big(d(x_0)\big) = \Phi_2(d(x_1))$ for all $x_0, x_1 \in X$.

We improve the proof of the following theorem stated in (Peters & Vergili, 2021).

Theorem 2.2.40. (Peters & Vergili, 2021): Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be descriptive proximity spaces, and $d: X \to Y$ is a degenerate descriptive constant map, then d is a dpc map.

Proof: Let A and B be subsets of X such that $A\, \delta_{\Phi_1} B$. Then we will show that $d(A)\, \delta_{\Phi_2} d(B)$. Since d is degenerate descriptive constant map, then $\Phi_2\big(d(A)\big) = \Phi_2\big(d(B)\big)$ because for any $x_0, x_1 \in X$ we have $\Phi_2\big(d(x_0)\big) = \Phi_2(d(x_1))$, hence $d(A) =_{des} d(B)$. Thus $d(A)\, \delta_{\Phi_2} d(B)$, so we can say that d is a dpc map.

Note that the converse of Theorem 2.2.40. is not true in general as in the following example.

Example 2.2.41. Consider the dpc map f in Example 1.1.54. We claim that this map is not degenerate descriptive constant map since for rbE and rbE_1 we have $f(rbE) = rbA$ and $f(rbE_1) = rbB$. However $\Phi_2(rbA) = 1$ and $\Phi_2(rbB) = 2$. Hence, f is not degenerate descriptive constant map.

Definition 2.2.42. (Peters & Vergili, 2021): Suppose (X, δ_Φ) is a descriptive proximity space. Then X is said to be degenerate descriptively contractible space if the identity map on it is descriptive proximally homotopic to a degenerate descriptive constant map.

Proposition 2.2.43. A convex space is a degenerate descriptively contractible space.

Proof: Let X be a convex space. To show that X is descriptively contractible, define the identity map $id_X: (X, \delta_\Phi) \to (X, \delta_\Phi)$ which is a dpc map and define $c: (X, \delta_\Phi) \to (X, \delta_\Phi)$ where $c(x) = x_0$ is a degenerate descriptive constant map. We claim that $id_X \sim_\Phi c$. Since X is a convex space, one can define a map $F: X \times [0,1] \to X$ by $F(x,t) = tx_0 + (1-t)x$. F is clearly a dpc map satisfying $F(x, 0) = x = id_X(x)$ and $F(x, 1) = x_0 = c(x)$. Thus $id_X \sim_\Phi c$. Hence, X is degenerate descriptively contractible.

The proof of the following theorem is missing in details (Peters & Vergili, 2021), so we give an explanation of the proof below.

Theorem 2.2.44. (Peters & Vergili, 2021): If (X, δ_Φ) is a degenerate descriptively contractible space, then it is also a descriptively contractible.

Proof: To show that (X, δ_Φ) is descriptively contractible, we need to prove that the identity map id_X is descriptive proximally homotopic to descriptive constant map. We claim that for a degenerate constant map c_d it is descriptive proximally homotopic to the descriptive constant map c_{x_0} for $x_0 \in Im(c_d)$. Define

$$H: X \times [0,1] \to X \text{ by } H(x,t) = \begin{cases} c_d(x), & 0 \leq t < 1 \\ c_{x_0}(x) = x_0, & t = 1 \end{cases}$$

Then H is a dpc map since if $(x_1, t_1) \sim_\Phi (x_2, t_2)$, then if $0 \leq t_0,\ t_1 < 1$ we have $H(x_1, t_1) = c_d(x_1)$ and $H(x_2, t_2) = c_d(x_2)$, so $c_d(x_1) \sim_\Phi c_d(x_2)$ by Theorem 2.2.40. In addition, W.L.O.G if $0 \leq t_0 < 1$ and $t_1 = 1$, then $H(x_1, t_1) = c_d(x_1)$ and $H(x_2, t_2) = x_0$, but $x_0 \in Im(c_d)$, then $H(x_1, t_1) \sim_\Phi H(x_2, t_2)$. Hence, H is a dpc map with $H(x, 0) = c_d(x)$ and $H(x, 1) = c_{x_0}(x) = x_0$, so $c_{x_0} \sim_\Phi c_d$. Since (X, δ_Φ) is a degenerate descriptive contractible space, then $c_d \sim_\Phi id_X$. Moreover, $\sim_\Phi$ is an equivalence relation, then we have $c_{x_0} \sim_\Phi id_X$. Thus, (X, δ_Φ) is descriptively contractible.

Corollary 2.2.45. Let (X, δ_Φ) be a degenerate descriptively contractible space, then (X, δ_Φ) is descriptively path connected.

Proof: Since (X, δ_Φ) is a degenerate descriptively contractible space, then by Theorem 2.2.44. it is descriptively contractible and from Proposition 2.2.11. (X, δ_Φ) is descriptively path connected.

Remark 2.2.46. Since any degenerate descriptive contractible space is descriptively contractible space, then the properties, such as given in Propositions 2.2.12, 2.2.20, 2.2.21, which are valid for the descriptively contractible spaces are also valid for the degenerate descriptive contractible spaces.

For a descriptive proximity space (X, δ_Φ) we will define descriptive retraction and descriptive deformation.

Definition 2.2.47. Assume (X, δ_Φ) is a descriptive proximity space with a probe map $\Phi: X \to \mathbb{R}^n$, $\emptyset \neq A \subseteq X$. Then we say that A is a descriptive retract of X provided there is a dpc map $r: (X, \delta_\Phi) \to (A, \delta_\Phi)$ such that $r(a) = a$ for all $a \in A$. Such a map r is called a descriptive retraction of X onto A.

In Figure 19, we can see that for a descriptive proximity space (X, δ_Φ), a subspace A is a descriptive retract since every element in X can be descriptively transformed successfully onto A, so A is a descriptive retract.

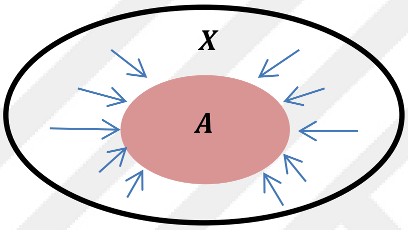

Figure 19. A descriptive retract A of (X, δ_Φ)

Example 2.2.48. Let (X, δ_Φ) be a descriptive proximity space illustrated in Figure 4. Where the probe map defined as the color of the subset of X. In that case E is a descriptive retract of X.

In fact, define $r: X \to E$ such that $r(E) = E, r(C) = E, r(A) = B$ and $r(B) = B$. Then r is a dpc map and as $r(E) = E$ and $r(B) = B$, we can conclude that E is a descriptive retract of X.

Definition 2.2.49. Let (X, δ_Φ) be a descriptive proximity space, and $A \subseteq X$. Then A is called a descriptive deformation retract of X provided there is a dpc map $H: X \times [0,1] \to X$ such that

1. $H(x, 1) \in A$ and $H(x, 0) = x$ for every $x \in X$

2. $H(a, t) = a$ for every $a \in A$ and $t \in [0,1]$

Such a map H is called a descriptive deformation of X into A.

Remark 2.2.50. We can rewrite the definition of the descriptive retraction as the follows. For a subspace A of a descriptive proximity space X and a descriptive inclusion $i: A \hookrightarrow X$ if there is a dpc map $r: X \to A$ such that $r \circ i = id_A$, then we say that A is a descriptive retract of X and r is a descriptive retraction. The two definitions are equivalent since $r \circ i = id_A$ for all $a \in A$, and we have $r \circ i(a) = a$, so $r(a) = a$.

Proposition 2.2.51. A descriptive deformation retract $A \subseteq X$ is a descriptive retract of X, and for the descriptive inclusion $i: A \hookrightarrow X$, we have $i \circ r \sim_{\Phi} id_X$. Moreover, $i \circ r$ and $r \circ i$ are descriptive homotopy equivalent.

Proof: Since A is a descriptive deformation retract of X, there is a descriptive deformation map $H: X \times [0,1] \to X$, such that $H(x,1) \in A$. Then we can write $H(x,1) = i(r(x))$ where $r: X \to A$ is a dpc map. Hence, for $a \in A$ we have $H(a,1) = i\big(r(a)\big) = a$, so that $r(a) = a$. Thus, r is a descriptive retraction. Now notice that H is a descriptive proximal homotopy between $i \circ r$ and id_X which comes directly from the definition of H and $r \circ i = id_A$. Since r is a descriptive retraction, we can conclude that $i \circ r$ and $r \circ i$ are descriptive proximal homotopy equivalent.

Proposition 2.2.52. Let (X, δ_Φ) be a descriptive proximally contractible space, and A is a descriptive retract of X. Then (A, δ_Φ) is a descriptive proximally contractible space.

Proof: Since X is descriptive proximally contractible, then $id_x \sim_\Phi c$, where $c: X \to X$ is a descriptive constant map at x_0, $i.e.$, $c(x) = x_0$. Then there is a dpc map $H: X \times [0,1] \to [0,1]$ such that $H(x,0) = id_x(x) = x$ and $H(x,1) = c(x) = x_0$. We will find a dpc map $G: A \times [0,1] \to A$ such that $G(a,0) = id_A(a)$ and $G(a,1) = k(a) = a_0$ where $k: A \to A$ is a descriptive constant map at a_0. Define

$$G: A \times [0,1] \to A \text{ by } G \coloneqq r \circ H \circ (i \times id_{[0,1]}).$$

Then G is a dpc map satisfying $G(a,0) = r\big(H(a,0)\big) = r(a) = a = id_A(a)$ and $G(a,1) = r\big(H(a,1)\big) = r(x_0) = r(c(a))$ where $r \circ c$ is a descriptive constant map. Now take $r(c) = k$, then (A, δ_Φ) is a descriptive proximally contractible space.

Proposition 2.2.53. Let (X, δ_Φ) be a descriptively path connected space, and A a descriptive retract of X. Then (A, δ_Φ) is a descriptively path connected space.

Proof: Let $a, b \in A$. We need to find a descriptive proximal path $\beta: [0,1] \to A$ from a to b so that $\beta(0) = a$ and $\beta(1) = b$. However, X is descriptively proximal path connected, then there is a dpc map $\alpha: [0,1] \to X$ such that $\alpha(0) = a$ and $\alpha(1) = b$. Define

$$\beta \coloneqq r \circ \alpha$$

where r is a descriptive retraction. Then $\beta: [0,1] \to A$ is a dpc map with $\beta(0) = r(\alpha(0)) = r(x_0) = a$ and $\beta(1) = r(\alpha(1)) = r(x_1) = b$. Thus (A, δ_Φ) is a descriptive proximally path connected space.

Proposition 2.2.54. Suppose (X, δ_Φ) is descriptive proximity space and A, $B \subseteq X$. If A and B are descriptive deformation retracts of X, then $A \simeq_\Phi B$.

Proof: To show that $A \simeq_\Phi B$, we need to find dpc maps $f: A \to B$ and $g: B \to A$ such that $f \circ g \sim_\Phi id_B$ and $g \circ f \sim_\Phi id_A$. Since A is descriptive deformation retraction of X, then there are a descriptive inclusion $i: A \hookrightarrow X$ and a descriptive retraction $r: X \to A$ such that $r \circ i = id_A$ and $i \circ r \sim_\Phi id_X$. Also since B is descriptive deformation retraction, then there are a descriptive inclusion $j: B \hookrightarrow X$ and a descriptive retraction $s: X \to B$ such that $s \circ j = id_B$ and $j \circ s \sim_\Phi id_X$. Define $f: A \to B$ by $f \coloneqq s \circ i$ and $g: B \to A$ by $g \coloneqq r \circ j$. In that case f and g are dpc maps. Furthermore,

$$f \circ g = s \circ (i \circ r) \circ j \sim_\Phi s \circ id_X \circ j = s \circ j = id_B.$$

Thus, $f \circ g \sim_\Phi id_B$. Moreover,

$$g \circ f = r \circ (j \circ s) \circ i \sim_\Phi r \circ id_X \circ i = r \circ i = id_A.$$

So, $g \circ f \sim_\Phi id_A$. Hence, $A \simeq_\Phi B$.

In the following sections, we will present some properties of the fixed set property of proximity and descriptive proximity spaces where these properties are stated for digital spaces in (Boxer et al., 2016), (Boxer, 2019) and (Ege & Karaca, 2015).

2.3. Approximate Proximal Fixed Set Property

Let X be a non-empty set. Then a map $f: X \to X$ has a fixed point provided there is $x \in X$ such that $f(x) = x$. Geometrically if $X \subseteq \mathbb{R}$ the fixed point of f is the common point between f and the line $y = x$. Fixed point theorems play a large role in sections of mathematics such as those used to iteratively approximate the roots of maps and while going through the spaces which has the fixed point property like $\mathbb{R}^n$ as Brouwer has shown in his fixed point theorem that every continuous map from $\mathbb{R}^n$ to itself has a fixed point (Spanier, 1966).

The following results are motivated by (Boxer et al., 2016), (Boxer, 2019) and (Ege & Karaca, 2015) in digital space.

Definition 2.3.1. Let (X, δ) be a Čech proximity space. Then we say that (X, δ) has the proximal fixed set property provided for every pc map $f: (X, \delta) \to (X, \delta)$ has a fixed subset.

Next we will introduce an approximate proximal fixed set property and prove some theorems which are related to the proximal retract and proximal isomorphism. We will see that this property corresponding to the proximal fixed set property is more interesting and we end up showing that this property is a proximal invariant property.

Definition 2.3.2. Let (X, δ) be a Čech proximity space. Then we say that (X, δ) has the approximate proximal fixed set property (APFSP), if every pc map $f: (X, \delta) \to (X, \delta)$ has an approximate fixed subset.

Definition 2.3.2 can be stated as follows: A Čech proximity (X, δ) space has the APFSP if and only if for every pc map $f: (X, \delta) \to (X, \delta)$, there is a subset $A \subset X$ such that $f(A) = A$ or $f(A)\, \delta\, A$. For a subset A of X we take the proper subset because otherwise if $A = X$, then $f(A)\, \delta\, A$ for every proximally continuous map which means that (X, δ) always has the APFSP which does not make any sense.

Example 2.3.3. Suppose $X = \{-1, 1\} \subset \mathbb{R}$ and δ is a metric proximity defined in Definition 1.1.3. on X. Then (X, δ) has not the APFSP.

Let $f: (X, \delta) \to (X, \delta)$ be a pc map, defined by

$$f(x) = \begin{cases} -1, & x = 1 \\ 1, & x = -1 \end{cases}$$

Then there is no proper subset A of X such that $f(A) = A$ or $f(A)\ \delta\ A$, because we have two choices for $A, i.e, A$ ={1},{-1}. So, we have two cases :

Case 1. If A ={1}, then $f(A) \neq A$ and $f(A)\ \underline{\delta}\ A$.

Case 2. If A ={-1}, then $f(A) \neq A$ and $f(A)\ \underline{\delta}\ A$.

Theorem 2.3.4. Let (X, δ_1) has the APFSP, and $h: (X, \delta_1) \to (Y, \delta_2)$ a proximal isomorphism map. Then (Y, δ_2) has the APFSP.

Proof: Suppose $f: (Y, \delta_2) \to (Y, \delta_2)$ is a pc map. Define $g := h^{-1} \circ f \circ h$, then $g: (X, \delta_1) \to (X, \delta_1)$ is a pc map by Proposition 1.1.31. In addition (X, δ_1) has the APFSP, so there is a subset $A \subset X$ such that $g(A) = A$ or $g(A)\ \delta_1\ A$, hence we have two cases

Case 1. If $g(A) = A$, then $h^{-1} \circ f \circ h(A) = A$. So, we have $f\big(h(A)\big) = h(A)$. Hence, $h(A) \subset Y$ is an approximate proximal fixed subset, then (Y, δ_2) has the APFSP.

Case 2. If $g(A)\ \delta_1\ A$, then $h^{-1} \circ f \circ h(A)\ \delta_1\ A$. So, $f\big(h(A)\big)\ \delta_1\ h(A)$. Hence, $h(A) \subset Y$ is an approximate fixed subset, then (Y, δ_2) has the APFSP.

Theorem 2.3.5. Let (X, δ_1) and (Y, δ_2) be Čech proximity spaces such that $X \approx_{\delta_1, \delta_2} Y$. If (X, δ_1) has the APFSP, then (Y, δ_2) also has the APFSP.

Proof: Given a pc map $g: (Y, \delta_2) \to (Y, \delta_2)$, we want to show that g has an approximate fixed subset. By assumption $X \approx_{\delta_1, \delta_2} Y$, there is a bijective map $h: (X, \delta_1) \to (Y, \delta_2)$ which is a pc map and $h^{-1}: (Y, \delta_2) \to (X, \delta_1)$ is also a pc map. From Theorem 2.3.4. we conclude that (Y, δ_2) has the APFSP.

Corollary 2.3.6. The APFSP is a proximal invariant property for proximity spaces.

In the next theorem we will show that how the APFSP is preserved under a proximal retract.

Theorem 2.3.7. Suppose (X, δ) is a proximity space and $A \subset X$ is a proximal retract of X. If (X, δ) has the APFSP, then (A, δ) has the APFSP.

Proof: Let $f: A \to A$ be a pc map. We will show that f has an approximate fixed subset. Furthermore, A is a proximal retract of X, then there is a pc map $r: X \to A$ such that $r(a) = a$ for all $a \in A$. For a proximal inclusion $i: A \hookrightarrow X$, define

$$g \coloneqq i \circ f \circ r.$$

Then $g: X \to X$ is a pc map by Proposition 1.1.31, from hypothesis (X, δ) has the APFSP, then there is $B \subset X$ such that $g(B) = B$ or $g(B)\ \delta\ B$. Letting $g(B) = D \subset A$, then $D = B$ or $D\ \delta\ B$.

Case 1. If $D = B$, and by assumption $D = g(B) = i \circ f \circ r\ (B) = i(f(r(D)) = i(f(D)) = f(D)$, so $D = f(D)$. This means that f has an approximate fixed subset D, and we can say that (A, δ) has the APFSP .

Case 2. If $D\ \delta\ B$, then $g(D)\ \delta\ g(B)$ since g is a pc map, and by assumption that $g(B) = D$ and $g(D) = i(f(r(D)) = i(f(D)) = f(D)$, so $f(D)\ \delta\ D$. This means that f has an approximate fixed subset D, then we can say that (A, δ) has the APFSP.

Next, we will introduce the proximal universal map and show how it is related to APFSP. The universal map is a map that can, in a specific way, imitate all other maps, or in other words it is an approximation of all other maps within a specified space. This type of maps is not limited to mathematics only, but is also included in other fields such as computer science and cryptography. In mathematics a universal map is one that contains subregions that approximate each holomorphic map of arbitrary precision.

Now we will study proximal universal map in a Čech proximity space. In addition, we will prove some theorems for spaces which has the APFSP.

Definition 2.3.8. Let (X, δ_1) and (Y, δ_2) be Čech proximity spaces. A pc map $f: (X, \delta_1) \to (Y, \delta_2)$ is proximal universal for (X, Y) if given a pc map $g: (X, \delta_1) \to (Y, \delta_2)$, there is $A \subseteq X$ such that $f(A) = g(A)$ or $f(A)\ \delta_2\ g(A)$.

Example 2.3.9. Let $(\mathbb{R}, \delta)$ be a Čech proximity space where δ is a metric proximity. Then $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not a proximal universal map.

Proof: f is a pc map, since for subsets A, B of $\mathbb{R}$ if $A\ \delta\ B, i.e., D(A, B) = 0$, then $f(A)\ \delta\ f(B)$ because

$$D\big(f(A), f(B)\big) = inf\big\{d\big(f(a), f(b)\big) : a \in A, b \in B\big\}$$

$$= \; inf\{|a^2 - b^2| : a \in A, b \in B\} = \; 0.$$

However f is not proximal universal. Consider the constant map on $\mathbb{R}$, c: $\mathbb{R} \to \mathbb{R}$, such that $c(x) = -1$ then c is a pc map, but there is no W subset of $\mathbb{R}$ such that $f(W) = c(W)$ or $f(W)\ \delta\ c(W)$. If we assume that we have such a subset W, then

Case 1. $f(W) = c(W)$ which is not true since $f(W) = \{w^2 : \ w \in W\}$.

Case 2. $f(W)\ \delta\ c(W)$, then $D(f(W), c(W)) = 0$, but

$$D(f(W), c(W)) = inf\{d(f(w), c(w)) : w \in W\}$$

$$= inf\{|w^2 + 1| : w \in W\}$$

$$= \; 1.$$

So, $f(W)\ \underline{\delta}\ W$. Thus f is not proximal universal.

The following definition is adapted from (Boxer et al., 2016) definition of digital dominating.

Definition 2.3.10. Suppose (X, δ) is a Čech proximity space and W is a subset of X. Then we say that W is a proximal dominating in X, if for every $A \subseteq X$ there exists $B \subseteq W$ such that $A = B$ or $A\ \delta\ B$.

Theorem 2.3.11. Assume (X, δ_1) and (Y, δ_2) are Čech proximity spaces and let $f: X \to Y$ be a proximal universal map for (X, Y). Then $f(X)$ is a proximal dominating in Y.

Proof: To show that $f(X)$ is a proximal dominating in Y, we need to prove that for any subset of Y, say B, there exists $D \subseteq f(X)$ such that $D = B$ or $D\ \delta_2\ B$. Given $y_0 \in B$ define $c_{y_0}: X \to Y$ where $c_{y_0}(x) = y_0, \forall\, x \in X$. Clearly c_{y_0} is a pc map and since $f: X \to Y$ is a proximal universal map for (X, Y), there is $A_{y_0} \subseteq X$ such that $f\big(A_{y_0}\big) = \{y_0\}$ or $f\big(A_{y_0}\big)\ \delta_2\ \{y_0\}$.

Case 1. If $f\big(A_{y_0}\big) = \{y_0\}$ and since y_0 is an arbitrary element of B, then for any $A = \cup_{y_0 \in B} A_{y_0}$, $f(A) = B$.

Case 2. If $f(A_{y_0})\ \delta_2\ \{y_0\}$ and since y_0 is an arbitrary point of B, then $f(A_{y_0})\ \delta_2\ B$. Letting $D = f(A_{y_0})$ completes the proof.

Therefore, $f(X)$ is a proximal dominating in Y.

For the proximal universal restricted map which has the proximal universal property causes the original map to have the same property as stated in the next theorem.

Theorem 2.3.12. Let (X,δ_1) and (Y,δ_2) be Čech proximity spaces, and $U \subset X$. If $f: X \to Y$ is a pc map and the restricted map $f|_U: (U,\delta_1) \to (Y,\delta_2)$ is a proximal universal map for (U,Y), then f is a proximal universal map for (X,Y).

Proof: Let $h: (X,\delta_1) \to (Y,\delta_2)$ be a pc map. Since $f|_U$ is a proximal universal map for (U,Y), there exists $V \subseteq U$ such that $f|_U(V) = h|_U(V)$ or $f|_U(V)\ \delta_2\ h|_U(V)$ where $V \subseteq U \subseteq X$ and $f|_U(V) = f(V)$, $h|_U(V) = h(V)$, f is a proximal universal map for (X,Y).

Theorem 2.3.13. Suppose (X,δ_1), (Y,δ_2) and (Z,δ_3) are Čech proximity spaces, $f: Z \to X$ and $g: X \to Y$ are pc maps. If $g \circ f: Z \to Y$ is a proximal universal map, then g is also a proximal universal map.

Proof: Let $h: X \to Y$ be a pc map. $h \circ f: Z \to Y$ is a pc map by Proposition 1.1.31. Since $g \circ f$ is a proximal universal map, there is $A \subseteq Z$ such that $h \circ f(A) = g \circ f(A)$ or $h \circ f(A)\ \delta_2\ g \circ f(A)$. Take $f(A) = W \subseteq X$, then $h(W) = g(W)$ or $h(W)\ \delta_2\ g\ (W)$. So g is a proximal universal map.

Theorem 2.3.14. Let $g: (Z,\delta_1) \to (X,\delta_2)$, and $h: (Y,\delta_3) \to (W,\delta_4)$ be proximal isomorphism maps. If $f: (X,\delta_2) \to (Y,\delta_3)$ is a pc map, then the following are equivalent.

i. f is a proximal universal map .
ii. $f \circ g$ is a proximal universal map.
iii. $h \circ f$ is a proximal universal map.

Proof:

(i)⟹(ii): Let $k: (Z,\delta_1) \to (Y,\delta_3)$ be a pc map, then $k \circ g^{-1}: (X,\delta_2) \to (Y,\delta_3)$ is a pc map by Proposition 1.1.31. Since f is a proximal universal map, then there is $A \subseteq X$ such that $k \circ g^{-1}(A) = f(A)$ or $k \circ g^{-1}(A)\ \delta_3\ f(A)$. Substituting by $A = g(g^{-1}(A))$, we get

$k \circ g^{-1}(g\ (g^{-1}(A))) = f\ (g(g^{-1}(A)))$ or $k \circ g^{-1}\left(g\left(\ g^{-1}(A)\right)\right)\ \delta_3\ f(g(g^{-1}(A)))$

So, $k\left(g^{-1}(A)\right) = f \circ g\left(g^{-1}(A)\right)$ or $k\left(g^{-1}(A)\right)\ \delta_3\ f \circ g\ \left(g^{-1}(A)\right)$ since $g^{-1}(A) \subseteq Z$. Hence $f \circ g$ is a proximal universal map.

(ii) $\Longrightarrow$ *(i)* : This statement was proved in the proof of Theorem 2.3.13.

(i)$\Longrightarrow$*(iii)* : Let $r: (X, \delta_2) \to (W, \delta_4)$ be a pc map, then $h^{-1} \circ r: (X, \delta_2) \to (Y, \delta_3)$ is a pc map by Proposition 1.1.31. Since f is a proximal universal map, there is $A \subseteq X$ such that $h^{-1} \circ r(A) = f(A)$ or $h^{-1} \circ r(A)\ \delta_3\ f(A)$. h is a proximal isomorphism map, so that

$$h(h^{-1}(r(A))) = h(f(A)) \text{ or } h(h^{-1}(r(A)))\ \delta_3\ h(f(A\ ))$$

So, $r(A) = h(f(A\ ))$ or $r(A)\ \delta_3\ h(f(A))$. Thus, $h \circ f$ is a proximal universal map.

(iii)$\Longrightarrow$*(i)* : Let $s: (X, \delta_2) \to (Y, \delta_3)$ be a pc map. Then $h \circ s: (X, \delta_2) \to (W, \delta_4)$ is a pc map by Proposition 1.1.31. Since $h \circ f$ is a proximal universal map, then there is $B \subseteq X$ such that $h \circ s(B) = h \circ f(B)$ or $h \circ s(B)\ \delta_4\ h \circ f(B)$. In addition, h^{-1} is a proximal isomorphism map, then

$$h^{-1}(h(s(B))) = h^{-1}(h(f(B\ ))) \text{ or } h^{-1}\left(h\left(s(B)\right)\right)\ \delta_3\ h^{-1}(h(f(B))).$$

So, $s(B) = f(B)$ or $s(B)\ \delta_3\ f(B\ )$. This shows that f is a proximal universal map.

Proposition 2.3.15. A proximity space (X, δ) has the APFSP if and only if the identity map $id_X: X \to X$ is a proximal universal map.

Proof: (X, δ) has the approximate proximal fixed set property APFSP if and only if given $f: X \to X$ is a pc map, then there is $A \subset X$ such that $f(A) = A$ or $f(A)\ \delta\ A$ since id_X is a pc map and $id_X(A) = A$, then $f(A) = id_X(A)$ or $f(A)\ \delta\ id_X(A)$ if and only if $id_X: X \to X$ is a proximal universal map.

Corollary 2.3.16. Assume $f: (X, \delta_1) \to (Y, \delta_2)$ is a proximal isomorphism map. Then f is a proximal universal for (X, Y) if and only if (X, δ_1) has the APFSP.

Proof: By Theorem 2.3.14, f is proximal universal for (X, Y) if and only if $f^{-1} \circ f = id_X$ is a proximal universal map and by Proposition 2.3.15, id_X is a proximal universal map if and only if (X, δ_1) has the APFSP.

Now we will present another type of proximal universal map called a weakly proximal universal map. Moreover, we will explain the relation between these two maps, and how an approximate proximal fixed set property is affected by weakly proximal universal map.

Definition 2.3.17. Let (X, δ_1) and (Y, δ_2) be proximity spaces. A pc map $f: (X, \delta_1) \to (Y, \delta_2)$ is a weakly proximal universal map for (X, Y) if given a pc map $g: (X, \delta_1) \to (Y, \delta_2)$, there is $A \subseteq X$ such that $f(A)\ \delta_2\ g(A)$.

Since a proximal universal map between proximity spaces is a weakly proximal universal map, the proof of the following theorems are directly obtained from the proof of the theorems related to proximal universal maps.

Theorem 2.3.18. Assume that (X, δ_1) and (Y, δ_2) are Čech proximity spaces, and $f: X \to Y$ is a weakly proximal universal map for (X, Y). Then $f(X)$ is a proximal dominating in Y.

Theorem 2.3.19. Let (X, δ) be a Čech proximity space, then (X, δ) has the APFSP if and only if the identity map $id_X: X \to X$ is a weakly proximal universal map.

Theorem 2.3.20. Suppose (X, δ_1), (Y, δ_2) and (Z, δ_3) are Čech proximity spaces, $f: Z \to X$ and $g: X \to Y$ are pc maps. If $g \circ f: Z \to Y$ is a weakly proximal universal map, then g is also a weakly proximal universal map.

Theorem 2.3.21. Let $g: (Z, \delta_1) \to (X, \delta_2)$, and $h: (Y, \delta_3) \to (W, \delta_4)$ be a proximal isomorphism maps. If $f: (X, \delta_2) \to (Y, \delta_3)$ is a pc map, then the following are equivalent.

i. f is a weakly proximal universal map.
ii. $f \circ g$ is a weakly proximal universal map.
iii. $h \circ f$ is a weakly proximal universal map.

Corollary 2.3.22. Suppose (X, δ_1) and (Y, δ_2) are Čech proximity spaces. If $f: (X, \delta_1) \to (Y, \delta_2)$ is proximal isomorphism, then the following are equivalent.

i. f is weakly proximal universal for (X, Y).

ii. (X, δ_1) has the APFSP.

iii. (Y, δ_2) has the APFSP.

Proof:

(i)⇔*(ii)*: f is weakly proximal universal for (X, Y) if and only if $f \circ f^{-1} = id_X$ is a weakly proximal universal map by Theorem 2.3.21. Furthermore, by Theorem 2.3.19, $id_X: X \to X$ is a weakly proximal universal map if and only if (X, δ_1) has the APFSP.

(i)⇔*(iii)*: f is weakly proximal universal for (X, Y) if and only if $f^{-1} \circ f = id_Y$ is weakly proximal universal map by Theorem 2.3.21. In addition, $id_Y: Y \to Y$ is a weakly proximal universal map if and only if (Y, δ_2) has the APFSP.

2.4. Approximate Descriptive Fixed Set Property

Now we can define the approximate descriptive fixed set property. As we mentioned in Chapter 1, an approximate descriptive fixed subset of a dpc map is a set that is descriptively near to its image.

Definition 2.4.1. Let (X, δ_Φ) be a descriptive proximity space, then we say that (X, δ_Φ) has the descriptive fixed set property provided every dpc map $f: (X, \delta_\Phi) \to (X, \delta_\Phi)$ has a descriptive fixed subset.

Definition 2.4.2. Let (X, δ_Φ) be a descriptive proximity space. Then we say that (X, δ_Φ) has the approximate descriptive fixed set property (ADFSP), if every dpc map $f: (X, \delta_\Phi) \to (X, \delta_\Phi)$ has an approximate descriptive fixed subset.

The approximate descriptive fixed property is preserved under a descriptive proximal isomorphism map as stated in the following theorem.

Theorem 2.4.3. Let (X, δ_{Φ_1}) has the approximate descriptive fixed set property, and let $h: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ be a descriptive proximal isomorphism, then (Y, δ_{Φ_2}) has the ADFSP.

Proof: Suppose $f: \left(Y, \delta_{\Phi_2}\right) \to \left(Y, \delta_{\Phi_2}\right)$ is a dpc map. Define

$$g := h^{-1} \circ f \circ h, \quad \text{so that } g: \left(X, \delta_{\Phi_1}\right) \to \left(X, \delta_{\Phi_1}\right)$$

is a dpc map by Proposition 1.1.41. Since, (X, δ_{Φ_1}) has the ADFSP, then there is $A \subset X$ such that $g(A) =_{des} A$ or $g(A)\ \delta_{\Phi_1}\ A$, so $h^{-1}(f\left(h(A)\right) =_{des} A$ or $h^{-1}(f\left(h(A)\right)\ \delta_{\Phi_1}\ A$. Moreover, from Lemma 1.1.66. $f\big(h(A)\big) =_{des} h(A)$ or $f\big(h(A)\big)\ \delta_{\Phi_2}\ h(A)$, since h is a descriptive proximal isomorphism map. Now take $h(A) = B$, so that there is $B \subset Y$ such that $f(B) =_{des} B$ or $f(B)\ \delta_{\Phi_2}\ B$. Hence (Y, δ_{Φ_2}) has the ADFSP.

Corollary 2.4.4. The approximate descriptive proximal fixed set property is a descriptive proximal invariant for descriptive proximity spaces**.**

Now we will show that the approximate descriptive fixed set property is preserved under descriptive proximal retract.

Theorem 2.4.5. Suppose (X, δ_Φ) is a descriptive proximity space and $A \subset X$ is a descriptive proximal retract of X. If (X, δ_Φ) has the ADFSP, then (A, δ_Φ) also has the ADFSP.

Proof: Let $f: (A, \delta_\Phi) \to (A, \delta_\Phi)$ be a dpc map. We will show that f has an approximate descriptive fixed subset. Since, A is a descriptive proximal retract of X, there is a dpc map $r: X \to A$ such that $r(a) = a$ for all $a \in A$. Let $i: A \hookrightarrow X$ be a descriptive proximal inclusion. Define $g \coloneqq i \circ f \circ r$, then $g: (X, \delta_\Phi) \to (X, \delta_\Phi)$ is a dpc map by Proposition 1.1.42, since (X, δ_Φ) has the ADFSP, then there is $B \subset X$ such that $g(B) =_{des} B$ or $g(B)\ \delta_\Phi\ B$. Let $g(B) = D$, where $D \subset A$, so that we have $D =_{des} B$ or $D\ \delta_\Phi\ B$.

Case 1. If $D =_{des} B$, then

$$D = g(B) = i \circ f \circ r(B) =_{des} i(f\big(r(D)\big) = i\big(f(D)\big) = f(D),$$

So, $D =_{des} f(D)$ means that f has an approximate descriptive fixed subset D.

Case 2. If $D \, \delta_\Phi \, B$, then $g(D) \, \delta_\Phi \, g(B)$, since g is a dpc map, we assumed $g(B) = D$ and $g(D) = i(f(r(D)) = i(f(D)) = f(D)$, so $f(D) \, \delta_\Phi \, D$ meaning that f has an approximate descriptive fixed subset D.

Next, we will present descriptive universal maps and prove some theorems for a descriptive proximity space which has the approximate descriptive fixed set property.

Definition 2.4.6. Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be descriptive proximity spaces. A dpc map $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is a descriptive proximal universal map for (X, Y) if given a dpc map $g: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$, then there is $A \subseteq X$ such that $f(A) =_{des} g(A)$ or $f(A) \, \delta_{\Phi_2} \, g(A)$.

The above definition tells us that a descriptive proximal universal map will imitate all descriptive proximally maps for some subsets.

Example 2.4.7. Let $X = \{rbE_1, rbE, rbE_2\}$ shown in Figure 5(a) and $Y = \{rbA, rbB\}$ shown in Figure 5(b) equipped with the same descriptive proximity relation

$$A \, \delta_\Phi \, B :\Leftrightarrow \Phi(A) = \beta_\alpha(A) = \beta_\alpha(B) = \Phi(B)$$

for a pair A, B subset of X or Y. Let $f: (X, \delta_\Phi) \to (Y, \delta_\Phi)$ be defined on the subsets of X by

$$f(rbE) = rbB, f(rbE_1) = rbA \text{ and } f(rbE_2) = rbA$$

Then as we showed in Example 1.1.54. f is a dpc map. Now we will show that f is a descriptive proximal universal map .We have three cases for dpc maps from X to Y.

Case 1. Define $k: (X, \delta_\Phi) \to (Y, \delta_\Phi)$ by $k(W) = rbA$ for all $W \subseteq X$, then k is a dpc map and if we consider rbE_1, then $f(rbE_1) =_{des} k(rbE_1)$, since $f(rbE_1) = k(rbE_1)$.

Case 2. Define $r: (X, \delta_\Phi) \to (Y, \delta_\Phi)$ by $r(V) = rbB$ for all $V \subseteq X$, then r is a dpc map and if we consider rbE, then $f(rbE) =_{des} r(rbE)$, since $f(rbE) = r(rbE)$.

Case 3. Let $g: (X, \delta_\Phi) \to (Y, \delta_\Phi)$ defined by

$$g(rbE) = rbA, g(rbE_1) = rbB \text{ and } g(rbE_2) = rbB.$$

Then g is a dpc map, since for $C, D \subseteq X$, $C\ \delta_\Phi\ D$. So $g(C)\ \delta_\Phi\ g(D)$. Take $U = \{rbE, rbE_1\}$, then $f(U) = \{rbB, rbA\}$ and $g(U) = \{rbA, rbB\}$. Therefore $f(U) =_{des} g(U)$.

Hence, for all dpc maps from X to Y, we can find a subset of X that satisfies the conditions of the descriptive map f to be a descriptive universal map. So, f is descriptive proximal universal map.

Definition 2.4.8. Suppose (X, δ_Φ) is a descriptive proximity space. We say that a subset W of X is a descriptive proximal dominating in X provided for every $A \subseteq X$ there exists $B \subseteq W$ such that $A =_{des} B$ or $A\ \delta_\Phi\ B$.

The image of a descriptive proximity space under a descriptive proximal universal map is a descriptive proximal dominating in the codomain.

Theorem 2.4.9. Assume (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) are descriptive proximity spaces and $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is a descriptive proximal universal map for (X, Y). Then $f(X)$ is a descriptive proximal dominating in Y.

Proof: To show that $f(X)$ is a descriptive proximal dominating in Y, we need to prove that for any subset of Y, say A, there exists $B \subseteq f(X)$ such that $A =_{des} B$ or $A\ \delta_{\Phi_2} B$. Let $y_0 \in A$. Define $c_{y_0}: X \to Y$ where $c_{y_0}(x) = y_0$, for all $x \in X$. Clearly c_{y_0} is a dpc map and since $f: X \to Y$ is a descriptive proximal universal map for (X, Y), there is $G_{y_0} \subseteq X$ such that $f(G_{y_0}) =_{des} \{y_0\}$ or $f(G_{y_0})\ \delta_{\Phi_2}\ \{y_0\}$.

Case 1. If $f(G_{y_0}) =_{des} \{y_0\}$, then for any arbitrary point $y_0 \in A$ we have $f(G_{y_0}) =_{des} \{y_0\}$. If we take $G = \cup_{y_0 \in A} G_{y_0}$, so that $f(G) =_{des} A$. This shows that $f(X)$ is a descriptive proximal dominating in Y.

Case 2. If $f(G_{y_\circ})\ \delta_{\Phi_2} \{y_0\}$, so is valid for any arbitrary $y_0 \in A$. Hence we have $f(G_{y_0})\ \delta_{\Phi_2} A$. This shows that $f(X)$ is a descriptive proximal dominating in Y.

Theorem 2.4.10. Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be descriptive proximity spaces and $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ a dpc map. For $U \subset X$ if the restricted map $f|_U: (U, \delta_{\Phi_1}) \to$

(Y,δ_{Φ_2}) is a descriptive proximal universal map for (U,Y), then f is also a descriptive proximal universal map for (X,Y).

Proof: Let $h:(X,\delta_{\Phi_1})\to(Y,\delta_{\Phi_2})$ be a dpc map and since $f|_U$ is a descriptive proximal universal map for (U,Y), there exists $V\subseteq U$ such that $f|_U(V)=_{des}h|_U(V)$ or $f|_U(V)\ \delta_{\Phi_2}\ h|_U(V)$. Moreover, $f|_U(V)=f(V)$ and $h|_U(V)=h(V)$ where $V\subseteq U\subseteq X$, and since f is a dpc map, f is a descriptive proximal universal map for (X,Y).

Theorem 2.4.11. Suppose (X,δ_{Φ_1}), (Y,δ_{Φ_2}) and (Z,δ_{Φ_3}) are descriptive proximity spaces, and $f:Z\to X$, $g:X\to Y$ are dpc maps. If $g\circ f$ is a descriptive proximal universal map, then so is g.

Proof : Let $h:X\to Y$ be a dpc map. We know that $h\circ f$ is a dpc map by Proposition 1.1.42. Since $g\circ f$ is a descriptive proximal universal map, then there is $A\subseteq Z$ such that

$$h\circ f(A)=_{des}g\circ f(A)\ \text{ or }\ h\circ f(A)\ \delta_{\Phi_2}\ g\circ f(A)$$

Take $f(A)=W\subseteq X$. In that case $h(W)=_{des}g(W)$ or $h\,(W)\ \delta_{\Phi_2}\ g\,(W)$. So, that g is a descriptive proximal universal map.

Theorem 2.4.12. Let $g:(Z,\delta_{\Phi_1})\to(X,\delta_{\Phi_2})$ and $h:(Y,\delta_{\Phi_3})\to(W,\delta_{\Phi_4})$ be descriptive proximal isomorphism maps. If $f:(X,\delta_{\Phi_2})\to(Y,\delta_{\Phi_3})$ is a dpc map, then the following are equivalent.

i. f is a descriptive proximal universal map.

ii. $f\circ g$ is a descriptive proximal universal map.

iii. $h\circ f$ is a descriptive proximal universal map.

Proof :

(i) $\Rightarrow$ *(ii)* : Let $k:(Z,\delta_{\Phi_1})\to(Y,\delta_{\Phi_3})$ be a dpc map, then $k\circ g^{-1}:(X,\delta_{\Phi_2})\to(Y,\delta_{\Phi_3})$ is a dpc map by Proposition 1.1.42. Furthermore, since f is a descriptive proximal universal map, there is $A\subseteq X$ such that $k\circ g^{-1}(A)=_{des}f(A)$ or $k\circ g^{-1}(A)\ \delta_{\Phi_3}\ f(A)$. Substituting by $A=g(g^{-1}(A))$, we get

$$k\circ g^{-1}\left(g\big(g^{-1}(A)\big)\right)=_{des}f\left(g\big(g^{-1}(A)\big)\right)$$

or

$$k \circ g^{-1}\left(g\left(g^{-1}(A)\right)\right) \delta_{\Phi_3} f(g(g^{-1}(A))).$$

then $k\left(g^{-1}(A)\right) =_{des} f \circ g\left(g^{-1}(A)\right)$ or $k\left(g^{-1}(A)\right) \delta_{\Phi_3} f \circ g\left(g^{-1}(A)\right)$. Since $g^{-1}(A) \subseteq Z$, then this shows that $f \circ g$ is a descriptive proximal universal map.

(ii) ⇒ *(i)* : The proof comes directly from Theorem 2.4.11.

(i) ⇒ *(iii)* : Let $r: (X, \delta_{\Phi_2}) \to \left(W, \delta_{\Phi_4}\right)$ be a dpc map. Then $h^{-1} \circ r: (X, \delta_{\Phi_2}) \to \left(Y, \delta_{\Phi_3}\right)$ is a dpc map by Proposition 1.1.42. In addition to this, since f is a descriptive proximal universal map, then there is $A \subseteq X$ such that $h^{-1} \circ r(A) =_{des} f(A)$ or $h^{-1} \circ r(A)\, \delta_{\Phi_3} f(A)$. Since h is a descriptive proximal isomorphism, we have

$$h(h^{-1}(r(A))) =_{des} h(f(A)) \text{ or } h(h^{-1}(r(A)))\, \delta_{\Phi_3} h(f(A))$$

In that case $r(A) =_{des} h(f(A))$ or $r(A)\ \delta_{\Phi_3} h(f(A))$. Thus $h \circ f$ is descriptive proximal universal.

(iii) ⇒ *(i)* : Let $s: (X, \delta_{\Phi_2}) \to \left(Y, \delta_{\Phi_3}\right)$ be a dpc map. Then $h \circ s: (X, \delta_{\Phi_2}) \to (W, \delta_{\Phi_4})$ is a dpc map by Proposition 1.1.42. Moreover $h \circ f$ is a descriptive proximal universal map, so that there is $B \subseteq X$ such that $h \circ s(B) =_{des} h \circ f(B)$ or $h \circ s(B)\, \delta_{\Phi_4} h \circ f(B)$. Since, h^{-1} is a descriptive proximal isomorphism, we have

$$h^{-1}(h(s(B))) =_{des} h^{-1}(h(f(B))) \text{ or } h^{-1}\left(h(s(B))\right)\ \delta_{\Phi_3} h^{-1}(h(f(B)))$$

So, $s(B) =_{des} f(B)$ or $s(B)\, \delta_{\Phi_3} f(B)$. This shows that f is a descriptive proximal universal map.

Proposition 2.4.13. Let (X, δ_Φ) be a descriptive proximity space. Then (X, δ_Φ) has the ADFSP if and only if the identity map $id_X: X \to X$ is descriptive proximal universal map.

Proof: (X, δ_Φ) has the ADFSP if and only if given $f: X \to X$ a dpc map, then there is $A \subseteq X$ such that $f(A) =_{des} A$ or $f(A)\ \delta_\Phi A$. Since $id_X(A) = A$, then $f(A) =_{des} id_X(A)$ or $f(A)\, \delta_\Phi\, id_X(A)$ if and only if $id_X: X \to X$ is a descriptive proximal universal map.

Corollary 2.4.14. Assume $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is a descriptive proximal isomorphism. Then f is a descriptive proximal universal for (X, Y) if and only if (X, δ_{Φ_1}) has the ADFSP.

Proof: By Theorem 2.4.12, f is a descriptive proximal universal for (X, Y) if and only if $f^{-1} \circ f = id_X$ is a descriptive proximal universal map and by Proposition 2.4.13, id_X is a descriptive proximal universal map if and only if (X, δ_{Φ_1}) has the ADFSP.

In the following, we will obtain a weakly descriptive proximal universal map by restricting the conditions of descriptive proximal universal map.

Definition 2.4.15. Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be descriptive proximity spaces. A dpc map $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is a weakly descriptive proximal universal map for (X, Y) if given a dpc map $g: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$, there is $A \subseteq X$ such that $f(A)\ \delta_{\Phi_2}\ g(A)$.

A descriptive proximal universal map between descriptive proximity spaces is a weakly descriptive proximal universal map. So, the following four theorems are the weaker version of Theorem 2.4.9, Proposition 2.4.13, Theorem 2.4.11. and Theorem 2.4.12, So that their proofs are straightforward.

Theorem 2.4.16. Assume (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) are descriptive proximity spaces and $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is a weakly descriptive proximal universal map for (X, Y). Then $f(X)$ is a descriptive proximal dominating in Y.

Theorem 2.4.17. A descriptive proximity space (X, δ_{Φ_1}) has the ADFSP if and only if the identity map on itself is a weakly descriptive proximal universal map.

Theorem 2.4.18. Suppose (X, δ_{Φ_1}), (Y, δ_{Φ_2}) and (Z, δ_{Φ_3}) are descriptive proximity spaces, and $f: Z \to X$ and $g: X \to Y$ are dpc maps. If $g \circ f$ is a weakly descriptive proximal universal map, then so is g.

Theorem 2.4.19. Let $g: (Z, \delta_{\Phi_1}) \to (X, \delta_{\Phi_2})$, and $h: (Y, \delta_{\Phi_3}) \to (W, \delta_{\Phi_4})$ be descriptive proximal isomorphism maps. If $f: (X, \delta_{\Phi_2}) \to (Y, \delta_{\Phi_3})$ is a dpc, then the following are equivalent.

i. f is a weakly descriptive proximal universal map.

ii. $f \circ g$ is a weakly descriptive proximal universal map.

iii. $h \circ f$ is a weakly descriptive proximal universal map.

Corollary 2.4.20. Suppose (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) are descriptive proximity spaces, $f: (X, \delta_{\Phi_1}) \to (Y, \delta_{\Phi_2})$ is a descriptive proximal isomorphism, then the following are equivalent.

i. f is a weakly descriptive proximal universal for (X, Y).

ii. (X, δ_{Φ_1}) has the ADFSP.

iii. (Y, δ_{Φ_2}) has the ADFSP.

Proof :

(i) $\Leftrightarrow$ *(ii)* : f is weakly descriptive proximal universal for (X, Y) if and only if $f \circ f^{-1} = id_X$ is a weakly descriptive proximal universal map by Theorem 2.4.19. Moreover, by Theorem 2.4.17 $id_X: X \to X$ is a weakly descriptive proximal universal map if and only if (X, δ_{Φ_1}) has the ADFSP.

(i) $\Leftrightarrow$ *(iii)* : f is weakly descriptive proximal universal for (X, Y) if and only if $f^{-1} \circ f = id_Y$ is a weakly descriptive proximal universal map by Theorem 2.4.19. In addition, $id_Y: X \to X$ is a weakly descriptive proximal universal map if and only if (Y, δ_{Φ_2}) has the ADFSP.

2.5. Proximal Fixed Set Property with Respect to the Proximity Space

In the previous section we studied (approximate) proximal fixed set property. Now, in this section we will analyze the proximal fixed set property with respect to proximity spaces and we will consider a special case of proximity spaces as the unit interval and we will call it a proximal homotopy fixed set property. However the name ‘homotopy’ was not chosen randomly because we would define this property concurrently with the homotopy definition of maps defined in Joseph Rotman's book (2009), and ended up showing that this property is a proximal invariant property. The following results are adapted and inspired from the work of (Szymik, 2015).

We will present the fixed set property in a Čech proximity space (X, δ_1) with respect to a different proximity space (T, δ_2), and then define the fixed set property respect to a unit interval $I = [0,1]$ which we will call proximal homotopy fixed set property.

Definition 2.5.1. Let (X, δ_1) be a Čech proximity space, then a proximal continuous family of self-maps of X is a pc map

$$f: T \times X \to X,$$

where (T, δ_2) is another Čech proximity space. A pc family $f: T \times X \to X$ of self-maps defines for each $A \subseteq T$ and $W \subseteq X$ a self-map defined by

$$f_a: X \to X \text{ such that } f_a(x) = f(a, x).$$

In that case we define

$$f_A(x) = \cup_{a\in A} f_a(x) = \cup_{a\in A} f(a, x)$$

and

$$f_A(W) = \cup_{a\in A}\cup_{w\in W} f_a(w) = \cup_{a\in A}\cup_{w\in W} f_a(a, w) = f(A, W)$$

Definition 2.5.2. Suppose $f: T \times X \to X$ is a pc family of self-maps of X, then a pc family of fixed sets of f is a pc map

$$p: T \to X,$$

such that

$$f_A(p(A)) = p(A)$$

for all $A \subseteq T$.

Definition 2.5.3. A proximity space (X, δ_1) has the proximal fixed set property with respect to a proximity space (T, δ_2) if for each pc families $f: T \times X \to X$ of self-maps of X, there exists at least one pc family $p: T \to X$ of fixed sets.

In other words, a proximity space (X, δ_1) has the proximal fixed set property with respect to a proximity space (T, δ_2) provided for any pc family of self-maps $f: T \times X \to X$ of X, there is a pc family $p: T \to X$ such that for all $A \subseteq T$, $f_A(p(A)) = p(A)$.

Remark 2.5.4. For example in (Szymik, 2015) that states "A topological space X has the fixed point property with respect to a singleton if and only if it has the fixed point property". However, unfortunately for a Čech proximity space (X, δ_1), only the necessary condition is satisfied, $i.e.$, if (X, δ_1) has the proximal fixed set property with respect to a singleton, then it has the proximal fixed set property. The sufficient condition is not necessarily true.

Proposition 2.5.5. If a Čech proximity space (X, δ_1) has the proximal fixed set property with respect to a singleton, then it has the proximal fixed set property.

Proof: Assume that (X, δ_1) has the proximal fixed set property with respect to a singleton. To show that (X, δ_1) has the proximal fixed set property, we need to find $W \subseteq X$ such that $g(W) = W$ where $g: X \to X$ be a pc map. Define

$$f := pr_X \circ \left(id_{\{a\}} \times g\right),$$

then $f: \{a\} \times X \to X$ is a pc family of self-maps on X according to Proposition 1.1.31. Further X has the proximal fixed set property with respect to a singleton, so that there is a pc family of fixed sets $p: \{a\} \to X$ of f such that $f\big(\{a\}, p(\{a\})\big) = p(\{a\})$. Since $f\big(\{a\}, p(\{a\})\big) = pr_X\Big(\big(\{a\}, g\big(p(\{a\})\big)\big)\Big) = g\big(p(\{a\})\big) = p(\{a\})$, g has a fixed set $W = p(\{a\})$. So X has the proximal fixed set property.

Proposition 2.5.6. Let (X, δ_1) has the proximal fixed set property with respect to a Čech proximity space (T, δ_2) and let S be a proximal retract of T. Then (X, δ_1) has the proximal fixed set property with respect to S.

Proof: Since S is a proximal retract of T, there are pc map $r: T \to S$ and the proximal inclusion map $i: S \hookrightarrow T$ such that $r \circ i = id_S$. To show that X has the proximal fixed set property with respect to S, let $f: S \times X \to X$ be a pc family of self-maps of X. Then define

$$g := f \circ (r \times id_X).$$

So, $g: T \times X \to X$ is a pc family of self-maps on X by Proposition 1.1.31. Furthermore (X, δ_1) has the proximal fixed set property with respect to T, so that there is a pc family $q: T \to X$ of fixed sets. Now we can define a pc family $p: S \to X$ by $p := q \circ i$. This leads us to $p(A) = q \circ i(A)$ for $A \subseteq S$. Then p has a fixed subset of f_A, since

$$f_A\big(p(A)\big) = f\big(A, p(A)\big) = f\Big(r(A), id_X\big(p(A)\big)\Big) = g\big(A, p(A)\big) = g(A, q\big(i(A)\big) =$$
$$g\big(A, q(A)\big) = q(A) = q\big(i(A)\big) = p(A).$$

Then $p(A)$ is a fixed subset of f_A, hence X has the proximal fixed set property with respect to S.

Example 2.5.7. If a proximity space (X, δ) has the proximal fixed set property with respect to the unit interval $I = [0,1]$, then it has the proximal fixed set property respect to a singleton.

This is because a singleton $\{t\}$ for $t \in I$ is a proximal retract of I because there is a proximal retraction $r: I \to \{t\}$, $r(s) = \{t\}$, $\forall s \in I$. By Proposition 2.5.6, X has the proximal fixed set property with respect to $\{t\}$.

In the following proposition, we will show how the proximal fixed set property is preserved under a proximal retract.

Proposition 2.5.8. If A is a proximal retract of (X, δ_1) which has the proximal fixed set property with respect to a Čech proximity space (T, δ_2), then (A, δ_1) has also the proximal fixed set property with respect to T.

Proof: Since A is a proximal retract of X, there are proximally continuous retraction $r: X \to A$ and the proximal inclusion map $j: A \hookrightarrow X$ such that $r \circ j = id_A$. To show that A has the proximal fixed set property with respect to T, let $f: T \times A \to A$ be a pc family of self-maps of A and define

$$g \coloneqq j \circ f \circ (id_T \times r).$$

Then $g: T \times X \to X$ is a pc family of self-maps of X according to Proposition 1.1.31. In addition X has the proximal fixed set property with respect to T, so there is a pc family of fixed sets $q: T \to X$ of g. We claim that a pc family $p: T \to A$ defined by $p := r \circ q$ has a fixed subset of f_A. Since q is a pc family of fixed sets of g, we have

$$g_W(q(W)) = g\big(W, q(W)\big) = j(f\left(W, r\big(q(W)\big)\right)) = q(W).$$

Now we apply the proximal retraction r to the both sides of the last equality two expressions in the previous equation, $i.e.$,

$$r(j(f\left(W, r\big(q(W)\big)\right)) = r(q(W)),$$

Since $r \circ j = id_A$, we have $f\big(W, p(W)\big) = p(W)$. So, $p(W)$ is a fixed subset of f_W. Hence, A has the proximal fixed set property with respect to T.

Now we will consider a special case when (X, δ) has the proximal fixed set property with respect to a Čech proximity space $T = I$, where $I = [0,1]$ is the unit interval.

Definition 2.5.9. Let (X, δ) be a Čech proximity space, then X has the proximal homotopy fixed set property, if it has the fixed set property with respect to the unit interval $I = [0,1]$.

Notice that from Definition 2.5.9, for any pc family of self-maps of X, $f: I \times X \to X$, here we will call f a proximal homotopy map, there is a pc path $p: I \to X$ such that $p(A)$ is a fixed subset of f_A for all $A \subseteq [0,1]$.

Example 2.5.10. Assume $X = \{x_0\}$ with a Čech proximity δ, then (X, δ) has the proximal homotopy fixed set property. For any pc family of self-maps $f: I \times \{x_0\} \to \{x_0\}$ of X, we have a pc path $p: I \to \{x_0\}$ so that $f_A\big(p(A)\big) = f\big(A, p(A)\big) = p(A) = \{x_0\}$ for all $A \subseteq I$.

Proposition 2.5.11. Suppose (X, δ_1) and (Y, δ_2) are Čech proximity spaces such that $X \approx_{\delta_1, \delta_2} Y$. If X has the proximal homotopy fixed set property, then Y also has the proximal homotopy fixed set property.

Proof: To show that (Y, δ_2) has the proximal homotopy fixed set property, for a pc family of self-maps $g: I \times Y \to Y$ of Y we need to find a pc path $q: I \to Y$ such that $q(A)$ is a fixed subset of g_A for all $A \subseteq [0,1]$. Since $X \approx_{\delta_1, \delta_2} Y$, there exists $h: (X, \delta_1) \to (Y, \delta_2)$ such that h is bijective, pc, and h^{-1} is also a pc map. Define

$$f \coloneqq h^{-1} \circ g \circ (id_I \times h).$$

Then $f: I \times X \to X$ is a pc family of self-maps of X. We know that X has the proximal homotopy fixed set property, so that there is a pc path $p: I \to X$ such that $p(B)$ is a fixed subset of f_B for all $B \subseteq [0,1]$. Define $q \coloneqq h \circ p$, then $q: I \to Y$ is a pc path and q has fixed subsets of g_B, $i.e.$,

$$g_B(q(B)) = g\big(B, q(B)\big) = g\big(B, h(p(B))\big)$$

$$= g\left(id_I \times h\big(B, p(B)\big)\right) = h \circ f\big(B, p(B)\big)$$

$$= h\big(p(B)\big) = q(B).$$

Thus, we get the required result.

Corollary 2.5.12. The proximal homotopy fixed set property is a proximal invariant property for proximity spaces.

Now we will give another type of fixed set property with respect to all Čech proximity spaces.

Definition 2.5.13. Let (X, δ_1) be a Čech proximity space, then (X, δ_1) has the proximal universal fixed set property provided it has the proximal fixed set property with respect to all Čech proximity spaces.

Definition 2.5.13. can be restated as the follows: Suppose (X, δ_1) is a Čech proximity space, then (X, δ_1) has the proximal universal fixed set property if and only if for all Čech proximity space (T, δ_2), and for every pc family of self-maps of X, there is a pc family of fixed sets of f, say $p: T \to X$ such that for all $A \subseteq T$, $p(A)$ is a fixed subset of f_A.

Lemma 2.5.14. If (X, δ_1) has the proximal universal fixed set property, then it has the homotopy fixed set property.

Proof: It is straight forward from the Definition 2.5.13. and Definition 2.5.9.

Corollary 2.5.15. The proximal universal fixed set property is a proximal invariant property for Čech proximity spaces.

Proof: The proof is obtained directly from Proposition 2.5.11.

Example 2.5.16. Let $X = \{x_0\}$ and δ be a Čech proximity on X, then (X, δ) has the proximal universal fixed set property.

2.6. Descriptive Fixed Set Property with Respect to a Descriptive Proximity Space

In this section for the descriptive proximity space (X, δ_{Φ_1}), we will define the fixed set property with respect to another descriptive proximity space (T, δ_{Φ_2}). In other words, we will show that any dpc family of self-maps of X has a dpc family of fixed sets and here we are looking for the fixed sets according to fixed descriptions, $i.e.$, we look for descriptive fixed sets.

Definition 2.6.1. Let $\left(X, \delta_{\Phi_1}\right)$ be a descriptive proximity space. Then a dpc family of self-maps of X is a dpc map

$$f: T \times X \to X,$$

where $\left(T, \delta_{\Phi_2}\right)$ is another descriptive proximity space. A dpc family $f: T \times X \to X$ of self-maps define for each $A \subseteq T$ and $W \subseteq X$ a self-map defined by

$$f_a: X \to X \text{ such that } f_a(x) =_{des} f(a, x).$$

In that case we define

$$f_A(x) =_{des} \cup_{a \in A} f_a(x) =_{des} \cup_{a \in A} f(a, x)$$

and

$$f_A(W) =_{des} \cup_{a \in A} \cup_{w \in W} f_a(w) =_{des} \cup_{a \in A} \cup_{w \in W} f_a(a, w) =_{des} f(A, W).$$

This means that it is not necessary that $f_A(W) = f(A, W)$ for $W \subseteq X$, but they must have the same descriptions under the probe map Φ_1, $i.e.$, $\Phi_1(f_A(W)) = \Phi_1(f(A, W))$.

Definition 2.6.2. Suppose $f: T \times X \to X$ is a dpc family of self-maps of X. Then a dpc family of descriptive fixed sets of f is a dpc map $p: T \to X$,

such that $$f_A(p(A)) = f\big(A, p(A)\big) =_{des} p(A)$$

for all $A \subseteq T$.

Definition 2.6.3. A descriptive proximity space (X, δ_{Φ_1}) has the descriptive proximal fixed set property with respect to a descriptive proximity space (T, δ_{Φ_2}) if for all descriptive continuous families of self-maps $f: T \times X \to X$ of X, there exists at least one dpc family of descriptive fixed sets $p: T \to X$ of f.

(X, δ_{Φ_1}) has the descriptive proximal fixed set property with respect to (T, δ_{Φ_2}) if and only if for all descriptive proximally continuous families of self-maps $f: T \times X \to X$ of X, there is a dpc family of descriptive fixed sets $p: T \to X$ of f, $i.e.$, for all $A \subseteq T$, $\Phi_1(f(A, p(A))) = \Phi_1(p(A))$.

Proposition 2.6.4. A descriptive proximity space (X, δ_Φ) has the descriptive proximal fixed set property provided (X, δ_Φ) has the descriptive proximal fixed set property with respect to a singleton.

Proof: Assume that (X, δ_Φ) has the descriptive proximal fixed set property with respect to a singleton. We want to show that (X, δ_Φ) has the descriptive proximal fixed set property, so let $g: X \to X$ be a dpc map. We want to find a subset $W \subseteq X$ such that $g(W) =_{des} W$. For a fixed element a, define

$$f := pr_X \circ (id_{\{a\}} \times g).$$

Then $f: \{a\} \times X \to X$ is a dpc family of self-maps on X. Furthermore X has the descriptive proximal fixed set property with respect to $\{a\}$, so that there is a dpc family of descriptive fixed sets $p: \{a\} \to X$ of f such that $f(\{a\}, p(\{a\})) =_{des} p(\{a\})$. Since

$$f(\{a\}, p(\{a\})) = pr_X\left(\{a\}, g(p(\{a\}))\right) = g(p(\{a\})) =_{des} p(\{a\}),$$

g has a descriptive fixed set $W = p(\{a\})$. Hence, X has the descriptive proximal fixed set property.

Proposition 2.6.5. Assume that (X, δ_{Φ_1}) has the descriptive proximal fixed set property with respect to a descriptive proximity space (T, δ_{Φ_2}). If S is a descriptive retract of T, then X has the descriptive proximal fixed set property with respect to S.

Proof: Since S is a descriptive retract of T, there are descriptive continuous retraction $r: T \to S$ and the descriptive proximally inclusion map $i: S \hookrightarrow T$ such that $r \circ i = id_S$. To show that X has the descriptive proximal fixed set property with respect to S, take a dpc family of self-maps $f: S \times X \to X$ of X and define $g := f \circ (r \times id_X)$ so that $g: T \times X \to$

X is a dpc family of self-maps on X. Since X has the descriptive proximal fixed set property with respect to T, then there is a dpc family of descriptive fixed sets $q: T \to X$ of g. Now we can define a dpc family of descriptive fixed sets $p: S \to X$ by $p(A) = q \circ i(A)$, for $A \subseteq S$. Then p has a descriptive fixed subset of f, since

$$f\big(A, p(A)\big) = f\Big(r(A), id_X\big(p(A)\big)\Big) = g\big(A, p(A)\big) = g(A\,, q\big(i(A)\big))$$
$$= g\big(A, q(A)\big) =_{des} q(A) = q\big(i(A)\big) = p(A).$$

In that case $p(A)$ is a descriptive fixed subset of f. Hence (X, δ_{Φ_1}) has the descriptive proximal fixed set property with respect S.

Proposition 2.6.6. If (X, δ_{Φ_1}) has the descriptive proximal fixed set property with respect to a descriptive proximity space (T, δ_{Φ_2}) and (A, δ_{Φ_1}) is a descriptive retract of X, then (A, δ_{Φ_1}) also has the descriptive proximal fixed set property with respect to T.

Proof: Since A is a descriptive retract of X, then there are descriptive continuous retraction $r: X \to A$ and the descriptive proximally inclusion map $j: A \hookrightarrow X$ such that $r \circ j = id_A$. To show that A has the descriptive proximal fixed set property with respect to T, take a dpc family of self-maps $f: T \times A \ \to A$ of A. Define

$$g \coloneqq j \circ f \circ (id_T \times r).$$

Then $g: T \times X \to X$ is a dpc family of self-maps of X. Since X has the descriptive proximal fixed set property with respect to T, there is a dpc family of descriptive fixed sets $q: T \to X$ of g. We claim that a dpc family

$$p: T \to A, \text{ defined by } \ p = r \circ q$$

has a descriptive fixed subset of f. Since q is a dpc family of fixed set of g, then we have $g\big(G, q(G)\big) =_{des} q(G)$ where

$$g\big(G, q(G)\big) = j(f\Big(G, r\big(q(G)\big)\Big) =_{des} q(G).$$

Now apply the descriptive retraction r to the both sides of the last equality in the above equation, $i.e.$, $r(j(f\Big(G, r\big(q(G)\big)\Big)) =_{des} r(q(G))$. Since $r \circ j = id_A$, we have

$f\big(G, p(G)\big) =_{des} p(G)$, so $p(G)$ is a descriptive fixed subset of f_G. Hence, (A, δ_{Φ_1}) has the descriptive proximal fixed set property with respect to T.

Definition 2.6.7. Let (X, δ_Φ) be a descriptive proximity space, then X has the descriptive homotopy fixed set property, if it has the descriptive fixed set property with respect to the unit interval $I = [0,1]$.

In other words, for any dpc family of self-maps $f: I \times X \to X$ of X, also known as a descriptive proximal homotopy, f here called a descriptive proximal homotopy map there is a dpc path $p: I \to X$ such that $p(A)$ is a descriptive fixed subset of f_A for all $A \subseteq [0,1]$, $i.e.$, $f\big(A, p(A)\big) =_{des} p(A)$.

Example 2.6.8. Assume $X = \{x_0\}$ with a descriptive proximity δ_Φ, then (X, δ_Φ) has the descriptive homotopy fixed set property. Take a dpc family of self-maps $f: I \times \{x_0\} \to \{x_0\}$ of X. For every dpc path $p: I \to \{x_0\}$ and $A \subseteq [0,1]$ we have

$$f_A\big(p(A)\big) = f\big(A, p(A)\big) =_{des} p(A) =_{des} \{x_0\}.$$

Proposition 2.6.9. Suppose (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) are descriptive proximity spaces such that $X \approx_{\Phi_1,\Phi_2} Y$. If X has the descriptive homotopy fixed set property, then Y has the descriptive homotopy fixed set property.

Proof : To show that Y has the descriptive homotopy fixed set property, we want to find a dpc path $q: I \to Y$ such that $q(A)$ is a descriptive fixed subset of g_A for all $A \subseteq [0,1]$ where $g: I \times Y \to Y$ be a dpc family of self-maps of Y. Since $X \approx_{\Phi_1,\Phi_2} Y$, there exists $h: \big(X, \delta_{\Phi_1}\big) \to (Y, \delta_{\Phi_2})$ such that h is bijective, dpc, and h^{-1} is a dpc map. Define $f \coloneqq h^{-1} \circ g \circ (id_I \times h)$. Then $f: I \times X \to X$ is a dpc family of self-maps of X. Moreover X has the descriptive homotopy fixed set property, so that there is a dpc path $p: I \to X$ such that $p(B)$ is a descriptive fixed subset of f_B for all $B \subseteq [0,1]$. Define $q \coloneqq h \circ p$, then $q: I \to Y$ is a dpc path and $q(B)$ is a descriptive fixed subsets of g_B, $i.e.$, $g\big(B, q(B)\big) = g\big(B, h(p(B))\big)$, and

$$g(B, h(p(B))) = g(id_I \times h(B, p(B))) = h \circ f(B, \big(p(B)\big) =_{des} h\big(p(B)\big) = q(B).$$

Hence, Y has the descriptive homotopy fixed set property.

Corollary 2.6.10. The descriptive homotopy fixed set property is a descriptive invariant property for descriptive proximity spaces.

Now we will present the descriptive fixed set property with respect to all descriptive proximity spaces called the descriptive proximal universal fixed set property.

Definition 2.6.11. Let (X, δ_{Φ_1}) be a descriptive proximity space, then X has the descriptive proximal universal fixed set property, if it has the descriptive proximal fixed set property with respect to all descriptive proximity spaces.

Definition 2.6.11. can be restated as the follows: Suppose (X, δ_{Φ_1}) is a descriptive proximity space, then X has the descriptive proximal universal fixed set property if and only if for all descriptive proximity space (T, δ_{Φ_2}), and for every dpc family of self-maps of X there is a dpc family of fixed sets of f, say $p: T \to X$, such that for all $A \subseteq T$ $p(A)$ is a descriptive fixed subset of f_A, $i.e.$ $f_A(p(A)) =_{des} p(A)$.

Proposition 2.6.12. If (X, δ_Φ) has the descriptive proximal universal fixed set property, then it has the descriptive homotopy fixed set property.

Proof: It is clear from the Definition 2.6.7. and Definition 2.6.11.

Corollary 2.6.13. The descriptive universal fixed set property is a descriptive invariant property for descriptive proximity spaces.

Proof: We can show it directly from Proposition 2.6.9.

Example 2.6.14. Let $X = \{x_0\}$ and δ_Φ be a descriptive proximity, then X has the descriptive proximal universal fixed set property.

3. CONCLUSION

In this work, firstly we gave some definitions and theorems about the spatial and descriptive proximity, proximal and descriptive conjugacy, proximal and descriptive connectedness, and recall the simplicial complex and CW$-$structure. Then, in Chapter 2 we have presented the proximal homotopy and discussed the homotopy between two proximal paths. Furthermore, we introduced the proximal deformation retraction. Next, we have studied the homotopic relation between two descriptive continuous maps and two descriptive paths and introduced the descriptive deformation retraction. Finally, we have obtained an application of proximal property for a fixed subset of proximity space which is approximate fixed set property and the relation between the universal map and the proximal approximate fixed set property. In addition, we gave the proximal fixed set property with respect to a proximity space and consider a special case of the proximity space which is the unit interval. On the top of that, we have attended the descriptive approximate fixed set property and the descriptive fixed set property with respect to a descriptive proximity space. At the end of that section, we gave a case of descriptive fixed set property with respect to the unit interval where is called the descriptive homotopy.

4. SUGGESTIONS

In our study, by using the simplicial complexes we can define the proximal and descriptive singular homology. Furthermore, we can define the proximal and descriptive covering map as a tool to find the proximal and descriptive fundamental group for some special proximity spaces.

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RESUME

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The conferences made within the scope of the thesis are given below in order of date.

1. Al-mahariq M. with Vergili, T., On Proximal Homotopy And Proximal Path Homotopy In Computational Proximity, 2nd International E-conference On Mathematical And Statistical Sciences: A SELCUK MEETING, Türkiye, 5-7 June 2023, pp.221

2. Al-mahariq M. with Vergili, T., Proximity Relations In Topological Spaces, Karadeniz Technical University, Department of Mathematics, internal seminar.Türkiye, 2022.