

ANALYTICAL AND EXPERIMENTAL STUDY OF IMPERFECTIONS, STRESS, AND TEMPERATURE EFFECTS IN CIRCULAR MEMS GYROSCOPES

A DISSERTATION SUBMITTED TO
THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN
ELECTRICAL AND ELECTRONICS ENGINEERING

By
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February 2024

Analytical and Experimental Study of Imperfections, Stress, and
Temperature Effects in Circular MEMS Gyroscopes

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February 2024

We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

ANALYTICAL AND EXPERIMENTAL STUDY OF IMPERFECTIONS, STRESS, AND TEMPERATURE EFFECTS IN CIRCULAR MEMS GYROSCOPES

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Ph.D. in Electrical and Electronics Engineering

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February 2024

Inspired by the outstanding performance of the hemispherical resonator gyroscopes (HRG), MEMS gyroscopes with circularly symmetric structures are a promising candidate for the next generation of high-performance, cost-effective gyroscopes. However, scaling down to MEMS poses certain performance-limiting challenges: increased sensitivity to inevitable fabrication imperfections and environmental variations, chiefly stress and temperature, that perturb the gyroscope's ideal modal space and result in quadrature/in-phase errors and, more importantly, long-term drift. Understanding these limiting factors is imperative for harnessing the full potential of circular MEMS gyroscopes. This thesis approaches this objective through an analytical modeling viewpoint. Here, “analytical” is meant to connote an approach based on the physics underlying the gyroscopes' operation as described by the variational principles of solid mechanics. For the experimental evaluations, we use our fabricated double-ring vibrating ring gyroscope (VRG) (3.2 mm-diameter, 57-59 kHz) and 10-ring disk resonator gyroscope (DRG) (3.4 mm-diameter, 41 kHz).

We start by calculating the mode shapes of the entire structure of multi-ring gyroscopes in the presence of structural imperfections and elastic anisotropy. By deriving the gyroscope's nonideal drive-sense transfer function matrix, we provide rigorous definitions for the quadrature and in-phase outputs, highlighting the role of angular gain, frequency split, mode shape rotations, and quadrature leakage into the in-phase due to the sense mode's phase error.

Next, we present a model for the effects of mechanical stresses leading to the concept of stress stiffness, an additional stiffness induced by such stresses through geometric nonlinearity. We carry out an eigenvalue perturbation analysis to obtain the frequency shifts, mode shape rotations, and quadrature/in-phase errors generated by the stress stiffness. Taking advantage of the circular geometry, we have equipped our ring gyroscopes with 16 capacitive stress sensors located

45°-apart (eight inside and eight outside the main ring), which pick up the local stress at the substrate level. We present an interpolation scheme to reconstruct the substrate's stress field using the outputs of the stress sensors, providing us with the mechanical stresses responsible for the stress stiffness in the silicon layer. We validate the model based on PCB bending tests.

We finally set out a modeling framework for temperature effects in ring gyroscopes. Our temperature experiments gave temperature coefficient of frequencies (TCFs), such as -10 ppm/°C and -14 ppm/°C, that are considerably different than the TCF value $\sim$ -30 ppm/°C expected from the $\sim$ -60 ppm/°C temperature dependency of Young's modulus of silicon. The model revolves around the engendered stiffness and opposing interaction of two fundamental mechanisms of temperature effects: changes in material properties and thermal stresses. The model demonstrates remarkable efficacy in accurately predicting the TCF and sheds light on residual stresses' role in forming frequency-temperature hysteresis loops.

Considering the great potential of integrating stress with temperature for the long-term performance improvement of MEMS gyroscopes, the results of this thesis serve as a building block toward physics-informed drift compensation algorithms.

Keywords: Circular Symmetry, Fabrication Imperfections, Long-Term Drift, Quadrature Error, Ring MEMS Gyroscope, Stress Effects, Temperature Effects.

ÖZET

YAPISAL KUSURLAR, STRES VE SICAKLIĞIN DAİRESEL MEMS DÖNÜ-ÖLÇERLERE ETKİLERİNİN ANALİTİK VE DENEYSEL OLARAK ÇALIŞILMASI

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Şubat 2024

Yarı küresel rezonatör dönü-ölçerin (HRG) üstün performansından esinlenen dairesel MEMS dönü-ölçerler, yeni nesil yüksek performanslı ve düşük maliyetli duyargalar olarak umut verici adaylardır. Fakat boyut MEMS düzeylerine indiğinde performansı limitleyen zorluklar ortaya çıkmaktadır: kaçınılmaz üretim kusurlarına ve çevresel değişimlere karşı daha fazla duyarlılık, başta stres ve sıcaklık ideal dönü-ölçer uzayını bozarak ofset hatalarına ve en önemlisi uzun dönem kaymaya sebep olmaktadır. MEMS dönü-ölçerin tam potansiyelinden yararlanmak için bu sınırlayıcı faktörleri anlamak önemlidir. Bu tez bu amaca analitik modelleme yöntemiyle yaklaşmaktadır. Burada “analitik” kelimesi katı hal mekaniği prensiplerini kullanarak dönü-ölçerin çalışmasındaki fiziksel prensipleri anlamayı ifade etmektedir. Deneysel çalışmalarda, çift halkadan oluşan dönü-ölçer (3.2 mm çap, 57-59 kHz cınlama frekansı) ve 10 halkalı disk rezonatör dönü-ölçer (3.4 mm çap, 45 kHz cınlama frekansı) kullanılmıştır.

Analize yapısal kusurları ve elastik anizotropiyi dikkate alıp çok halkalı dönü-ölçerlerin mod şekillerini hesaplayarak başlıyoruz. Dönü-ölçerin ideal olmayan sürüş-algılama transfer matrisini bulup titizlikle dönü-ölçer ofset hatalarını tanımlıyoruz. İzlediğimiz bu metod frekans farkının, mod şekli kaymasının ve algılama modu faz hataları kaynaklı ofset hatalarının dönü-ölçer üzerindeki etkilerini ayırt etmemizi sağlamıştır. Sonrasında, mekanik stres modellemesi için geliştirdiğimiz model neticesinde stres direngenliği konseptini tanımlıyoruz: stresin geometrik doğrusalsızlık ile etkileşimi neticesinde oluşan ek bir direngenlik. Stres direngenliği sebebiyle oluşan frekans kaymalarını, mod şekli dönüşlerini ve ofset hatalarını öz değer perturbasyon yöntemiyle buluyoruz. Dairesel geometriden faydalanarak, dairesel dönü-ölçerimizi 45°’lik açılarla yerleştirilmiş 16 adet alt taştaki stresi ölçen kapasitif stress duyargaları ile (8 adet içeride 8 adet ana halkanın dışında) entegre ettik. Sunduğumuz interpolasyon yöntemi ile stres

duyarga çıkışlarını kullanarak alt taşın stres alanını buluyoruz ve bu bize silikon tabakasındaki stres direngenliğine sebep olan mekanik stresi veriyor. Sunduğumuz modeli PCB bükme testleri ile doğruluyoruz.

Son olarak halka dönü-ölçerlerde sıcaklık etkilerini anlamak için bir modelleme metodu öneriyoruz. Sıcaklık deneylerimiz $-10 \text{ ppm}/^{\circ}\text{C}$ ve $-14 \text{ ppm}/^{\circ}\text{C}$ gibi frekans sıcaklık katsayısı (TCF) verdi, bu değerler silikonun $\sim 60 \text{ ppm}/^{\circ}\text{C}$ Young katsayısı sebebiyle beklenen $\sim 30 \text{ ppm}/^{\circ}\text{C}$ TCF'ten oldukça farklıdır. Model sıcaklıktan etkilenen ve birbirine karşı koyan iki mekanizmanın direngenliği üzerine kuruludur: malzeme özelliklerinde ve termal stresteeki değişiklikler. Model TCF'yi hassasiyetle tahmin ederek dikkate değer bir performans sergilemekte ve frekans-sıcaklık histerezis döngülerinde kalıntı stresin rolüne ışık tutmaktadır.

MEMS dönü-ölçerlerin uzun dönem performans iyileştirmesinde stres ve sıcaklığı beraber kullanmanın büyük potansiyeli değerlendirildiğinde, bu tezin sonuçları fizik-temelli bir kayma giderme algoritması için temel bir yapıtaşı olacaktır.

Anahtar sözcükler: Dairesel Simetri, Üretim Kusurları, Uzun-Dönem Kayma, Ofset Hatası, Halkasal MEMS Dönü-Ölçer, Stres Etkileri, Sıcaklık Etkileri.

Acknowledgement

I would like to express my deepest gratitude to my advisor, Asst. Prof. Erdiñç Tatar, for his support and guidance that made this thesis possible. I have learned a lot from him and gained valuable insights from his expertise during my Ph.D. I would like to thank the members of my thesis committee, Prof. Yusuf Ziya İder, Prof. Abdullah Atalar, Assoc. Prof. Kıvanç Azgıñ, and Prof. Haluk K  lah for their invaluable time and helpful comments that improved the quality of the thesis. Prof. İder and Assoc. Prof. Azgıñ were also the T  K members for my Ph.D. thesis. I am grateful for their support and insightful discussions throughout the T  K meetings.

I would like to thank the Scientific and Technological Research Council of Turkey (T  B  TAK) for the financial support of this work under the 2232 Program (grant no: 118C247).

To my beloved family:

My parents, Hossein and Mitra, and my brother, Mehrdad.



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Chapter 1

Introduction

Microelectromechanical systems (MEMS) gyroscopes are silicon-based devices that are manufactured by the well-established micromachining processes inherited from the integrated circuits (IC) industry. These gyroscopes can be integrated with MEMS accelerometers to form cost-effective inertial measurement units (IMUs) suitable for mid-range navigation purposes [1]. MEMS gyroscopes are now widely used in numerous application areas, such as consumer electronics (e.g., image stabilization in smartphones and orientation detection in interactive gaming controllers), robotics, biomechanics, and vehicle stability control systems [2]. Large-scale production with low per-device cost, small footprint size, and lower power consumption are among the notable advantages of MEMS gyroscopes in competition with mechanical and optical gyroscopes [3]. In the following, we provide a brief overview of the working principles of these sensors.

1.1 Basics of MEMS Gyroscopes

A gyroscope is a device that measures the rotation—in the form of either angle or rate—of a body with respect to an inertial reference frame. There are various types of gyroscopes that operate based on different physical principles. For example,

conventional mechanical gyroscopes are based on the conservation of angular momentum, and fiber-optic gyroscopes work based on the Sagnac effect [4]. In that respect, the operation basis of MEMS gyroscopes is the Coriolis effect, a fictitious force (i.e., a force that does not comply with Newton's third law) observed in a non-inertial rotating reference frame [5]. Assume that the angular velocity vector Ω characterizes the rotation of such a reference frame with respect to an inertial frame; the Coriolis force for a point mass m , which has the velocity of V with respect to the rotating frame, is given by

$$F_c = -2m\Omega \times V, \quad (1.1)$$

where $\times$ denotes the vector cross product. We observe from Equation (1.1) that the magnitude of the Coriolis force is proportional to the magnitude of Ω , and its direction is orthogonal to that of Ω . Therefore, to detect the angular rate about a particular axis, we can generate a velocity vector perpendicular to the axis and then measure the resultant Coriolis force. Realizing this idea is the main goal of a MEMS gyroscope in its various forms and designs. As Figure 1.1 shows, the core of a MEMS gyroscope is a two-degrees-of-freedom (2-DOF) oscillator, capable of vibrating along two orthogonal modes, x and y . We can model the oscillator as a proof mass with dampers and springs, having the natural frequencies and quality factors of w_x, Q_x and w_y, Q_y along the x - and y -axes, respectively. First, we drive the x -mode (referred to as the *drive mode*) using an AC force with the amplitude F_0 and frequency ω :

$$\ddot{x} + \frac{w_x}{Q_x}\dot{x} + \omega_x^2 x = F_0 \sin(\omega t). \quad (1.2)$$

In the presence of a rotation rate Ω_z about the z -axis (orthogonal to both x and y), the (normalized) Coriolis force

$$F_c = -2\Omega_z \dot{x}, \quad (1.3)$$

excites the y -mode (referred to as the *sense mode*):

$$\ddot{y} + \frac{w_y}{Q_y}\dot{y} + \omega_y^2 y = -2\Omega_z \dot{x}. \quad (1.4)$$

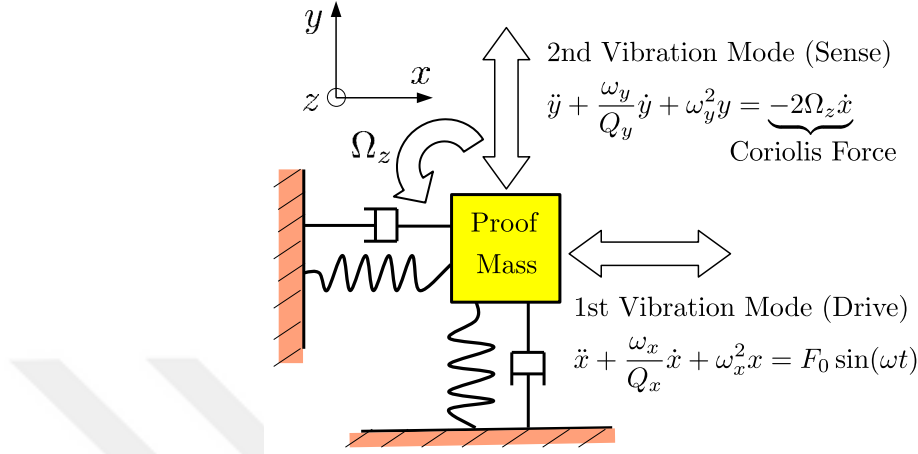


Figure 1.1: Working principle of a MEMS gyroscope.

Let

$$\begin{aligned} H_x(j\omega) &= \frac{1}{\omega_x^2 - \omega^2 + j\frac{\omega\omega_x}{Q_x}}, \\ H_y(j\omega) &= \frac{1}{\omega_y^2 - \omega^2 + j\frac{\omega\omega_y}{Q_y}}, \end{aligned} \quad (1.5)$$

be the transfer functions of the x - and y -modes, respectively, and y_{ss} be the steady-state amplitude of the sense mode; we have

$$|\Omega_z| = SF \cdot y_{ss}, \quad (1.6)$$

where

$$SF = \frac{1}{|H_x(j\omega)| \cdot |H_y(j\omega)| \omega F_0} \quad (1.7)$$

is the (mechanical) scale factor from the sense mode to the rotation rate. For large quality factors Q_x, Q_y (e.g., in the order of 10^3), SF is maximized if $\omega_x \approx \omega$ and $\omega_y \approx \omega_x$. The above explanation reveals the two fundamental structural requirements for the operation of an ideal MEMS gyroscope:

1. Two orthogonal mode shapes that enable Coriolis-based momentum interchange, forming the drive and sense modes.
2. Frequency matching between these two modes to maximize the scale factor for accurate rotation rate detection.

1.2 Performance Metrics of MEMS Gyroscopes

We provide a brief description of the key metrics that are used to assess and quantify the performance of MEMS gyroscopes. Detailed and comprehensive discussions of this topic, along with practical guidelines, can be found in [6], [7], and [8].

Scale factor. We introduced the mechanical scale factor in Equation (1.7). The overall scale factor is the conversion gain from the gyroscope's output (in V) to the input rate (in $^{\circ}/\text{s}$), determined by the series connection of the transfer functions of the read-out electronics and the mechanical resonator. In practice, the scale factor can be obtained as the slope of the least-squares fitted line of the input-output data. Deviations of the data points from this straight line are an estimate of the gyroscope's *nonlinearity*.

Range. The range of rotation rates that the gyroscope can detect without falling to the saturation limits of electronics.

Resolution. The slightest change in the input rate that the gyroscope responds to and picks up. More technically, the resolution is the smallest, larger than noise level rate increment that generates an appreciable output as a predefined percentage of the nominal scale factor.

Frequency split. The difference between the resonance frequencies of the drive and sense modes, i.e., $\Delta\omega = \omega_x - \omega_y$. As discussed in Section 1.1, frequency matching between the two modes is ideal for maximizing the sensitivity. However, we will show in Chapter 2 that structural imperfections and elastic anisotropy cause frequency split. Moreover, because of high quality factors of MEMS gyroscopes, introducing an intentional frequency split can improve the scale factor's stability against frequency changes.

Bandwidth. The range of frequencies of input rates that is detectable by the gyroscope. In the mode-matched case, $\Delta\omega = 0$, this range coincides with the

sense mode's bandwidth, and, in the mode-mismatched case, $|\Delta\omega| \gg \frac{w_y}{2Q_y}$, it is around the same order as the frequency split.

Quadrature and in-phase errors. The output signal picked up from the sense mode is demodulated into two channels: in-phase with and orthogonal to the Coriolis force. Any outputs (apart from the rate) observed in these two channels correspond to the in-phase and quadrature errors, respectively. We will give more accurate definitions in Chapter 2. The long-term in-phase error is also known as *bias* or *zero-rate output*.

Acceleration sensitivity. For an ideal Coriolis-based operation, an implicit assumption is that the rotating reference frame to which the gyroscope is attached has a constant linear velocity. Whenever this is not the case, the drive and sense modes will be perturbed by acceleration terms that appear as external inputs. The response to constant accelerations can be minimized by carefully designing the gyroscope's rigid body mode and using differential sense electrodes. In general, however, external accelerations can affect the scale factor or even the output bias depending on their frequency and noise characteristics.

1.2.1 Allan variance analysis

Long-term drift is a ubiquitous problem in all MEMS sensors. For example, Figure 1.2 depicts the output of one of our gyroscopes for 16 days under the zero-rate condition. Not only is the output nonzero, but it also grows over time while exhibiting unexpected patterns. This phenomenon, which is mainly associated with temperature and stress [9, 10], poses a fundamental challenge in pushing the performance of MEMS gyroscopes beyond the current limits, especially for navigation-grade applications. Allan variance, originally developed for the stability analysis of atomic clocks, is a method to characterize the long-term performance of MEMS gyroscopes. For this task, first, the zero-rate output of the gyroscope is collected over a reasonably long period, and then, for incrementally increasing averaging times, the Allan variance is calculated in the following way.

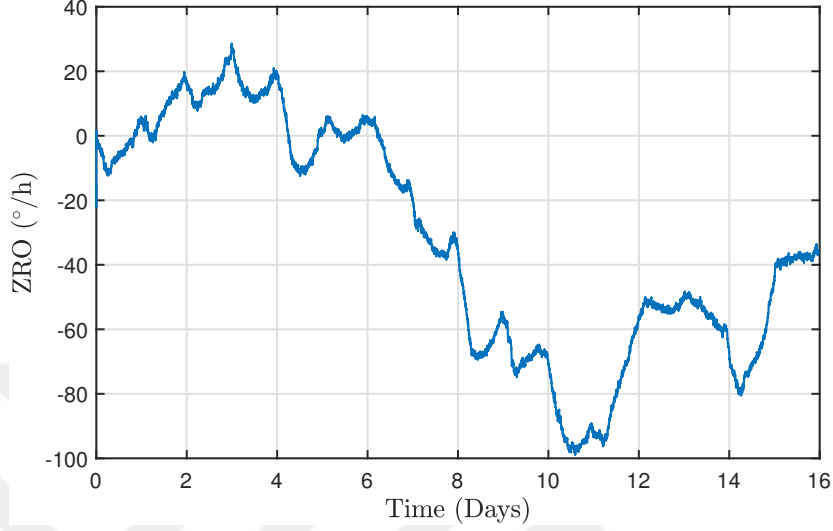


Figure 1.2: Sample zero-rate output of one of our MEMS gyroscopes over 16 days [10].

For a given averaging time T , the rate data is partitioned into clusters of length T , and averages over these clusters are calculated; the Allan variance $\sigma^2(T)$ is the variance of this sequence of averages. Allan variance, unlike power spectral analysis, is a time-domain method and is easier to compute. The distinct advantage of Allan variance analysis is that it reveals the long-term patterns in the gyroscope's output by pulling the short-term effects and noises aside [11]. Figure 1.3 shows a typical Allan variance graph for standard deviation values plotted in log-log axes. This plot illustrates several key performance parameters that we define in the following (see [4, 11] for a detailed discussion and [12] for practical guidelines):

- **Quantization noise (slope = -1):** This is caused by the analog-to-digital conversion (ADC) of the gyroscope's output.
- **Angle random walk (ARW) (slope = -1/2):** It is the integrated white noise in the rate. The ARW coefficient W_{ARW} , expressed in $^\circ/\sqrt{\text{h}}$ or $^\circ/\sqrt{\text{s}}$, gives us the standard deviation as $\sigma = W_{\text{ARW}}\sqrt{t}$.
- **Bias instability (slope = 0):** After filtering out short-term fluctuations, at a particular averaging time, we have the bias instability (in $^\circ/\text{h}$ or $^\circ/\text{s}$) as a

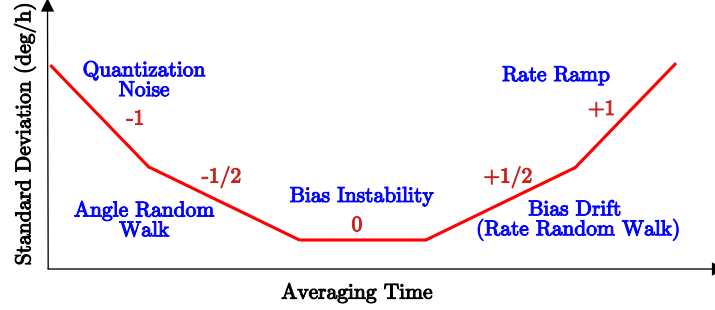


Figure 1.3: Typical Allan deviation (log-log) plot; the numerical values are the slopes over different regions.

critical point that signifies the onset of long-term effects. In terms of frequency, the bias instability is determined by the $1/f$ flicker noise of electronics.

- **Bias drift (slope = $+1/2$):** Also called rate random walk, bias drift quantifies parts of the long-term effects that are of stochastic origin.
- **Rate ramp (slope = $+1$):** This part of the drift is affiliated with the nonstationary variations over long time scales (e.g., hours or days).

In practice, other components—arising from colored or exogenous environmental noises—might also emerge in the Allan variance plot.

1.3 Circular MEMS Gyroscopes

This section explains why circularly symmetric structures are particularly suited for a Coriolis-based gyroscope. Circular symmetry in a mechanical structure imposes certain periodicity conditions on its mode shapes. More specifically, let the polar angle θ be the spatial field variable describing the structure's configuration and $u(\theta)$ be a kinematically admissible displacement field that defines a mode shape; the circular symmetry implies that $u(\theta + 2\pi) = u(\theta)$, i.e., $u(\theta)$ is (at least) 2π -periodic. By the Fourier expansion theorem, the trigonometric pairs $\{\cos(n\theta), \sin(n\theta)\}$, $n = 1, 2, \dots$, form a complete basis for the space of 2π -periodic, non-constant, continuous functions. Since $\cos(n\theta)$ and $\sin(n\theta)$

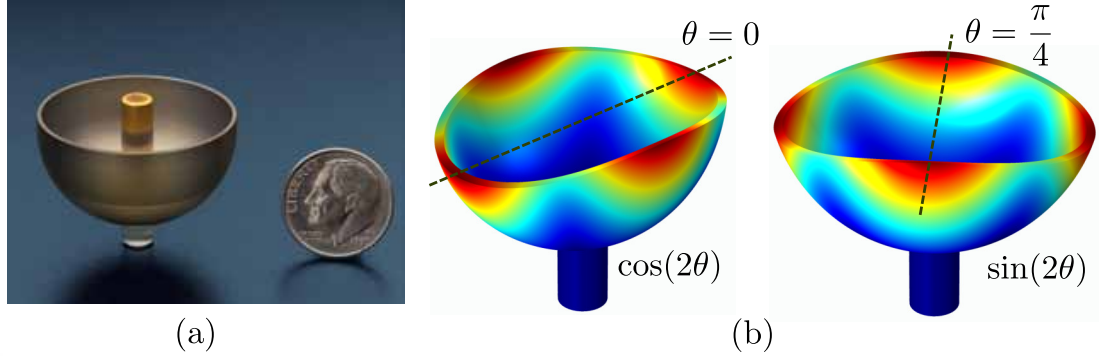


Figure 1.4: HRG: resonator [14] (a) and wineglass mode shapes (b).

are orthogonal to each other, for each $n = 1, 2, \dots$, we have a *degenerate pair* of independent mode shapes that share the same effective mass, stiffness, and consequently, frequency.¹ Hence, this type of structure naturally satisfies the two fundamental requirements (that we pointed out in Section 1.1) of an ideal MEMS gyroscope. We should note that the circular symmetry we are interested in is not purely geometric but should also exist in the structure’s mass and stiffness distributions. For example, although an elliptical structure’s geometry satisfies the circular symmetry condition $u(\theta + 2\pi) = u(\theta)$, it has higher stiffness along its major axis.

The hemispherical resonator gyroscope (HRG) perfectly embodies the advantages of circular symmetry for Coriolis-based rate detection. HRG operates based on the two so-called *wineglass* mode shapes (corresponding to $\{\cos(2\theta), \sin(2\theta)\}$) of a hemispherical shell, serving as the drive and sense modes (see Figure 1.4). As an example, the performance specifications of Hubble telescope’s HRG are $0.000080^\circ/\text{h}$ bias instability, $0.000010^\circ/\sqrt{\text{h}}$ angle random walk, $0.000150 \text{ arcsec}/\sqrt{\text{Hz}}$ angle white noise for the maximum rate of $0.5^\circ/\text{s}$ [14]. This exceptional performance has prompted the inertial MEMS community to seek answers to an intriguing question: Can we replicate HRG on the MEMS scale? The answer is a potential avenue toward the next generation of high-performance, cost-effective MEMS gyroscopes, which—compared to the HRG—would be more

¹We provide an informal proof of a ring’s mode shape degeneracy in Appendix A. A more rigorous treatment for general symmetric structures can be established using the representation theory of symmetry groups [13].

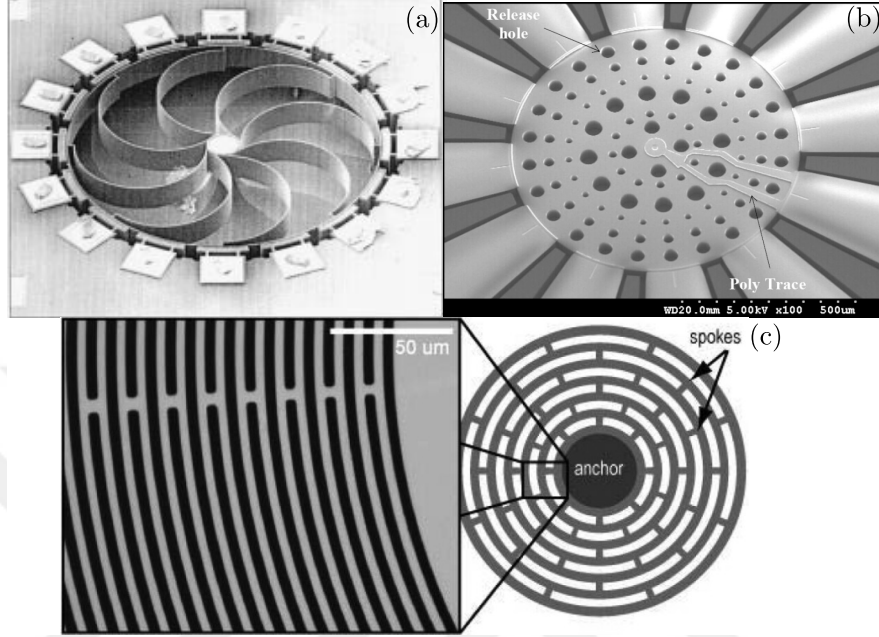


Figure 1.5: SEMs of three types of circular MEME gyroscopes: single ring [16] (a), BAW [17] (b), and DRG [18] (c).

accessible to a broader range of applications.

1.4 Literature Review

Here, we give a summary of the reported circular MEMS gyroscopes; for a more comprehensive survey of the topic, we refer to [15].

From a microfabrication point of view, replicating the HRG's three-dimensional structure is a challenging and complex task. Hence, MEMS gyroscopes predominantly use planar circular structures. Putty [19] reported the first case of a fabricated ring gyroscope. The $19\text{ }\mu\text{m}$ thick structure comprised a 1 mm diameter ring with eight semi-circular supporting beams to operate at 33.850 kHz in the $n = 2$ mode and was fabricated by an electroforming process from nickel on a silicon substrate. This gyroscope was redesigned and manufactured by the high aspect-ratio combined poly- and single-crystal silicon (HARPSS) technology in 1.1 mm diameter and $80\text{ }\mu\text{m}$ thickness [16] (see Figure 1.5-(a)). The goal was

to overcome several limitations associated with using nickel, including large electrostatic gaps, significant mismatch between the coefficients of thermal expansion (CTE) of nickel and silicon, and nickel’s less-desirable material proprieties that increase the risk of mechanical fatigue, creep, and long-term drift [16]. A commercially available example of single-ring gyroscopes is Silicon Sensing’s CRS39A gyroscope that uses a magnetically actuated 6 mm diameter ring and has a bias instability of $0.03^\circ/\text{h}$ [20].

Besides the ring, the disk is another circularly symmetric structure utilized in a group of MEMS gyroscopes known as the bulk acoustic wave (BAW) gyroscopes [21]. In [17], an 800 μm diameter, 50 μm thick perforated disk was used to operate at 5.88 MHz in the $n = 3$ mode (Figure 1.5-(b)). The device was fabricated from silicon using the HARPSS technology and had small electrostatic gaps of 250 nm due to its high stiffness. The BAW gyroscope in [22] has an annulus with the outer and inner diameters of 1400 μm and 800 μm , respectively, operates at 2.98 MHz in the $n = 2$ mode, and is fabricated from 60 μm doped (100) silicon coated with 1 μm aluminum nitride (AlN) piezoelectric layer. A similar structure was used in [23] as a thick-ring with 5.5 mm outer diameter and 1 mm width, operating at 426 kHz in the $n = 3$ mode, exhibiting a record quality factor of 950k. The BAW gyroscope reported in [24] operates at 4.9 MHz in the $n = 3$ mode, and its design implements suitable backside trenches to separate the silicon region in the handle layer and eliminate possible gyro-package couplings. Shock resistance, vibration rejection, and high quality factor are among the main advantages of BAW gyroscopes.

Multi-ring structures are a popular design choice for circular MEMS gyroscopes. This type of design includes a set of concentric rings interconnected by beams placed at the nodal points of the operating mode shapes. The collection of rings resembles a disk; hence, these devices are often called *disk resonator gyroscopes (DRGs)*. The Boeing DRG had an outer diameter of 8 mm, operated in the $n = 2$ mode with 14 kHz frequency, 80k quality factor, and demonstrated the remarkable bias instability of $0.012^\circ/\text{h}$ [26]. The DRG reported in [18] had 60 rings within 2 mm diameter and 35 μm thickness, operated at 78 kHz in the $n = 2$ mode, and was fabricated from (111) silicon (Figure 1.5-(c)). In [27],

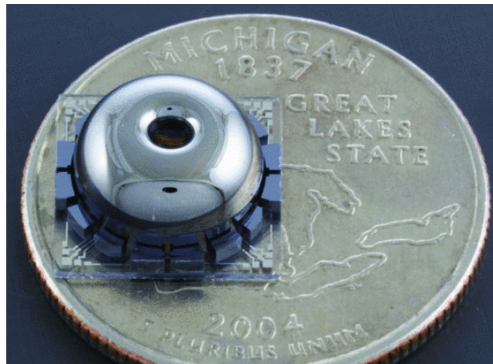


Figure 1.6: PSI gyroscope [25].

lumped masses were added between the certain rings to tune the mass distribution, resulting in a high quality factor (510k) DRG and improved bias instability of $0.04^\circ/\text{h}$. The honeycomb DRG in [28] employed a mesh-like compliant structure interlacing 10 rings with the outermost diameter of 8 mm, operated at 4 kHz in the $n = 2$ mode with 650k quality factor.

The performance of circular MEMS gyroscopes has been steadily improving. A perfect example is the precision shell integrating (PSI) gyroscope [25] (see Figure 1.6) with $0.0014^\circ/\text{h}$ bias instability that demonstrates a successful replication of HRG in a 1 cm diameter birdbath shape geometry. This trend is expected to continue as circular MEMS gyroscopes are the subject of ongoing intense research.

1.5 Overview of Our MEMS Gyroscopes

Throughout this thesis, we will focus on two types of multi-ring MEMS gyroscopes as the basis of our modeling and experiments. The SEM and $n = 2$ wineglass mode shapes of these gyroscopes are given in Figure 1.7. The first type, vibrating ring gyroscopes (VRGs), has two rings—with the outer one having 3.2 mm diameter—and 16 interconnecting beams. The outer ring is surrounded by 16 pairs of differential electrodes for drive, sense, and tuning purposes. The resonance frequencies of VRGs are in the 57-59 kHz range and exhibited $\sim 1^\circ/\text{h}$ bias instability and $\sim 0.233^\circ/\sqrt{\text{h}}$ angle random walk. The second type is 10-ring

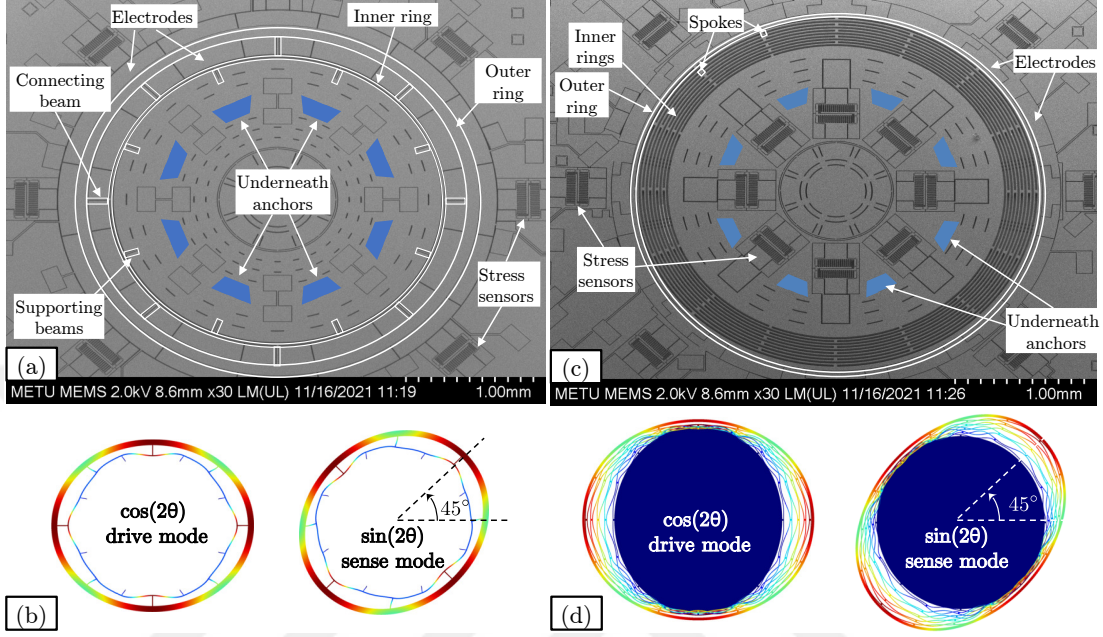


Figure 1.7: Our fabricated gyroscopes with their operating mode shapes: double-ring VRG (a), (b) and 10-rings DRG (c), (d).

DRGs, where 16 electrodes encircle the outermost ring (3.4 mm in diameter), and eight spokes connect any two adjacent rings. These DRGs have a frequency of ~ 41 kHz. In the design of these gyroscopes, we followed two important guidelines: 1) the stiffness of the rigid body mode should be high enough to lower the acceleration sensitivity; 2) the frequency zone around the wineglass modes should be clear of any other torsional or out-of-plane modes. We will provide the parameters of these gyroscopes in the next chapter. As the SEMs in Figure 1.7 show, our gyroscopes are equipped with capacitive stress sensors. The area enclosed by a ring is particularly advantageous to stress sensing since it allows placing multiple stress sensors in different directions. We will discuss the role and function of the stress sensors in Chapter 3.

The fabrication of our devices was outsourced and carried out by a Silicon-on-glass (SOG) process with wafer-level vacuum packaging and vertical feed-through [29]. Figure 1.8 provides a visual overview of this fabrication process. Detailed description of the process steps can be found in [29]; here, we give a brief summary:

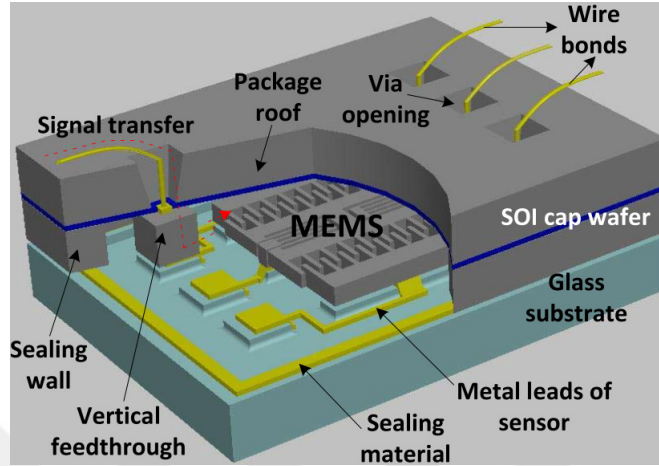


Figure 1.8: Overview of SOG fabrication process and encapsulation [29].

1. Wet etching of the glass substrate and deposition of the first Cr-Au metalization layer.
2. Anchor formation on the substrate by wet etching and second metalization for electrical connections and pads.
3. Fabrication (lithography and deep reactive-ion etching (DRIE)) of the device layer on a silicon-on-insulator (SOI) wafer followed by anodic bonding with the glass substrate.
4. Hermetic encapsulation by anodic bonding and, finally, wire bonding for electrical connection to the outside.

1.6 Contribution

As discussed in Section 1.3, high performance with low to moderate cost is the promise of circular MEMS gyroscopes. However, shrinking the size of a device to the micro-scale imposes certain prohibitive limits on its attainable performance. First, the device would be more sensitive to geometric imperfections in its structure. Although the microfabrication processes enable the mass production of MEMS devices, the ubiquitous tolerances and errors—especially during

lithography and etching—make these imperfections inevitable. Additionally, single crystalline silicon exhibits elastic anisotropy, which is particularly significant for (100) oriented wafers. Second, compared to the macro-scale, environmental factors—notably, stress and temperature—have more pronounced effects on the MEMS structure’s thermo-mechanical behavior. We should highlight that stress and temperature have strong correlations with the long-term drift [9, 30]; hence, understanding their effects is paramount for the performance improvement of circular MEMS gyroscopes. Toward this end, our approach will be mainly through the lens of analytical modeling. Variational principles of solid mechanics will be the basis of our modeling, and we will use the Ritz method as a direct solution technique [31, 32]. Variational formulations are particularly suited for our analysis since they provide a convenient framework for handling large numbers of interconnection kinematic constraints arising in multi-structures. Moreover, the kinetic and strain energy functionals directly give us the mass and stiffness matrices, respectively.

The first section of each chapter contains a detailed discussion of the relevant contributions. In the following, we briefly go through the key points:

Chapter 2. We present a method for calculating the mode shapes of ring gyroscopes’ entire moving structure, which enables us to model the effects of imperfections and material anisotropy on the mass and stiffness matrices. We use this model to identify the critical parts of the gyroscope’s structure by examining how their defects affect the overall mass and stiffness matrices. We use perturbation analysis to derive the gyroscope’s 2×2 transfer matrix that reveals the effects of nonidealities on the coupling between the drive and sense modes. We provide rigorous definitions of the key performance indices such as angular gain, quadrature, and in-phase errors. These concepts often have loose definitions in the existing literature, which are not geared toward computationally tractable models.

Chapter 3. Stress-based calibration holds great potential for improving the long-term performance of MEMS gyroscopes, and our goal in this chapter is to establish a physics-based analytical framework to pave the way for harnessing this potential. We explicate a new approach for modeling stress effects in

ring gyroscopes. We show that external stress perturbs the gyroscope’s stiffness distribution via a *stress stiffness* matrix, resulting in frequency shifts and quadrature/in-phase errors. We discuss interpolating the stress field across the gyroscope’s substrate from the outputs of the stress sensors and then using this interpolation to calculate the stress stiffness.

Chapter 4. In practice, the source of undesired stresses in MEMS gyroscopes is often thermal rather than mechanical. Such stresses result from conflict between thermal deformation and structural constraints as well as CTE mismatches at the die-attach level. Additionally, temperature affects the silicon’s thermo-mechanical properties. In this chapter, we extend the stress modeling in Chapter 3 to account for temperature effects and thermal stresses. We set out a framework to calculate both stress and *thermal stiffness* matrices using the measurements of stress sensors. This model enables us to explain and predict

1. The countering role of thermal stresses against elastic modulus variations in the gyroscope’s temperature coefficient of frequency (TCF);
2. Frequency-temperature hysteresis loops arising from residual stresses over temperature cycles.

1.7 Organization

The rest of this thesis is organized as follows. Chapter 2 lays out the main results on modeling ring-based gyroscopes in the presence of imperfections. Chapter 3 presents a new mathematical model that enables us to predict the effects of mechanical stress on frequencies, mode shapes, and quadrature/in-phase errors. Chapter 4 extends the modeling to include the temperature effects—in terms of material properties and thermal stresses—on the gyroscope’s frequencies. Chapter 5 concludes the thesis with suggestions for future research. Appendices A and B contain complementary and additional technical results.

Chapter 2

Modeling Circular MEMS Gyroscope in the Presence of Imperfections¹

2.1 Overview

As explained in Chapter 1, a circularly symmetric structure inherently possesses the essential properties for a Coriolis-based operation: a pair of orthogonal, frequency-matched mode shapes. However, scaling down such structures to the MEMS domain poses challenges to realizing an ideal gyroscope as it aggravates sensitivity to the inevitable fabrication tolerances. Two factors affect a ring gyroscope's given pair of mode shapes: structural imperfections and material anisotropy [36]. Without these factors, the ring is ideal, the mode shapes share the same natural frequency, and their orientation is indeterminate, depending only on the directions of the external actuation. In the presence of the above factors, however, the ring is nonideal; a frequency split may exist between the mode shapes, and their orientation becomes determinate. The implications of

¹The material presented in this chapter is based on the author's publications [33], [34], and [35].

these nonidealities are

1. in terms of topology and modal space: misalignment between the nominal actuation/sensing directions and the actual mode shapes;
2. in terms of performance: mode mismatch and quadrature/in-phase errors.

A detailed mathematical model, in turn, is imperative for analyzing the effects of imperfections. Such a model, compared to the commonly used finite element method (FEM), can provide further insight into the underlying physics as a crucial step toward a thorough evaluation and enhancement of the devices' performance.

Related literature. Perturbation analysis—within a variational formulation—has been used in the literature to study imperfection-related eigenvalue splitting and vibration instability in rings [37, 38]. The adverse effects of imperfections and environmental variables have also been analyzed in the literature for ring-based gyroscopes. Gallacher et al. [36] studied mass/stiffness coupling and mismatch in a VRG and presented a method to identify and electrostatically compensate for the ensuing effects. However, the considered error model does not cover all possible imperfection scenarios of our VRG's suspension design. For the Boeing DRG, Challoner [26] presented an error model that accounts for frequency/damping mismatch between the modes and misalignment in the drive-sense axes. They derived the in-phase and quadrature errors under the force-rebalanced control. The anisotropic effects of single-crystalline Silicon with (100) and (111) planes on a ring's mode shapes have been examined [39, 40]. Different design schemes were proposed in [41] to eliminate the frequency split in a (100) Silicon DRG. In [42, 43], rigid body modes of the ring have been incorporated to study VRGs' sensitivity to low-frequency external vibrations and examine frequency split caused by imperfections. For a VRG subject to severe shock inputs, Chouvion et al. [44] modeled the coupling and frequency splitting effects of nonlinear electrostatic forces on the modes. Ma et al. [45] also investigated the same effects for a varying width profile with a uniform sidewall angle.

Contribution. Figure 2.1 shows the SEM of our 59 kHz, $n = 2$ wineglass modes

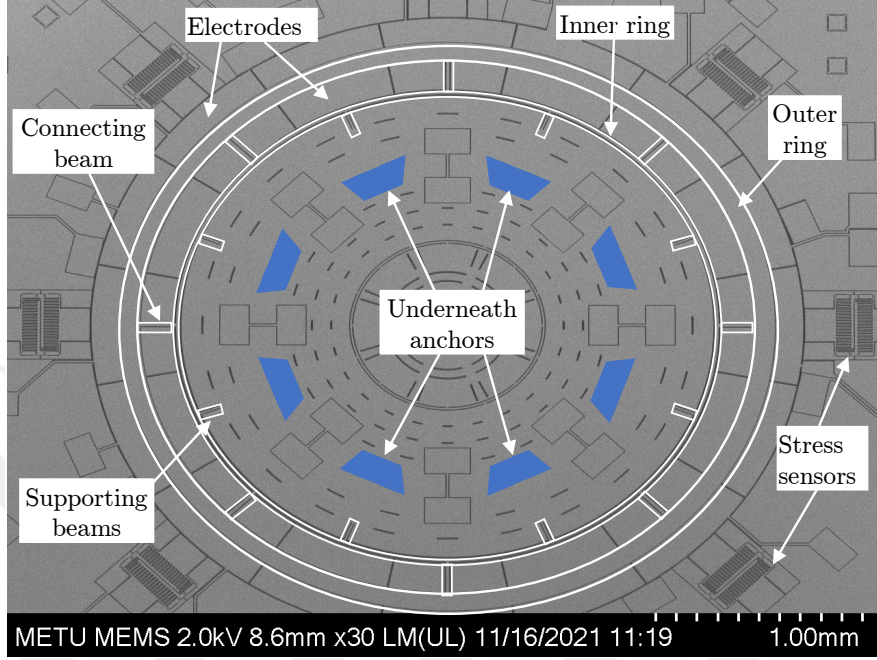


Figure 2.1: SEM of fabricated VRG.

VRG having two concentric, outer and inner rings. The device is fabricated from (111) wafers using a silicon-on-glass (SOG) process [29]. The outer ring is the primary vibrating component, mounted on a suspension provided by the inner ring and a set of 16 beams. As the main contribution, we present a comprehensive mathematical model for this VRG and analyze the effect of structural imperfections. The main steps of our analysis are as follows:

1. We express the displacement fields of the VRG's components—including rings and suspension beams—as the linear combination of dual degenerate mode shapes with the respective generalized coordinate. Considering the boundary conditions imposed by the ring-beam connections, we parametrize the mode shapes by suitable basis functions. Applying the Ritz method, we then solve an optimization problem to find the coefficients of the basis functions and obtain the mode shapes of the whole structure. This step distinguishes our approach from the common method in the VRG literature where massless springs model the ring-suspension interaction [42, 45]; this approach provides more insight into the different components' contributions to the overall mass and stiffness distribution.

2. Using the calculated mode shapes, we parametrize the kinetic and potential energies of the VRG in terms of the generalized coordinates. By applying Lagrange's equations, we obtain the equations of motion and, subsequently, the mass and stiffness matrices. In these matrices, elastic anisotropy and structural imperfections manifest as off-diagonal and mismatches between the diagonal entries.
3. We study the eigenvalue problem associated with the mass and stiffness matrices. The eigenvalues give us the natural frequencies and the frequency split. Additionally, we observe that the imperfections could lead to (possibly non-orthogonal) eigenvectors that do not match the standard basis of $\mathbb{R}^2$ (i.e., $[1, 0]^\top, [0, 1]^\top$). Equivalently, the mode shapes' orientation is determinate, rotating with respect to the nominal drive/sense directions.
4. We show that, because of the rotation of the mode shapes, any mismatch between their transfer functions (including effective mass, damping, and stiffness) translates into quadrature error in the VRG's output, which can further leak into the in-phase channel. Damping coupling directly appears in the in-phase error.

We provide detailed simulations along with experimental data to validate our mathematical modeling. To put its advantages in perspective, we compare our method with the FEM. We also integrate an imperfect ring model into our simulations to examine etching-related errors.

Notation. Throughout this chapter, $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n \times m}$ denote the set of real numbers, the n -dimensional Euclidean space, and the space of $n \times m$ real matrices, respectively. $[\cdot]^\top$ and $\otimes$ stand for matrix transpose and Kronecker product, respectively. $\text{col}(x_i)_{i=0}^n$ returns the column vector $[x_0, \dots, x_n]^\top$. $\text{diag}(x_i)_{i=n_1}^{n_2}$, $n_2 \geq n_1$, is the diagonal matrix having the entries $x_{n_1}, \dots, x_{n_2}$. $I_{n \times n}$ is the $n \times n$ identity matrix, and 0_n is the n -dimensional zero vector. For integers $n_2 > n_1$, $i = n_1 : n_2$ means that the index i takes all the values $n_1, n_1 + 1, \dots, n_2$.

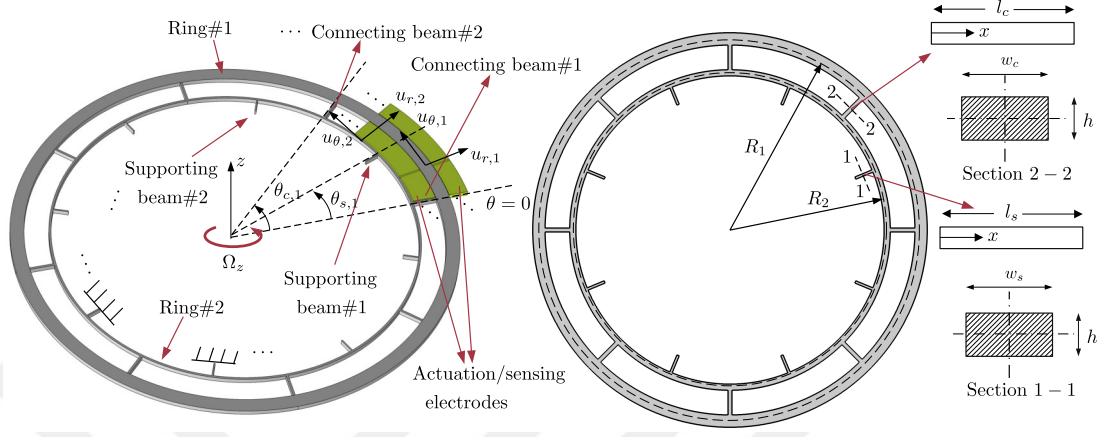


Figure 2.2: Schematics of vibrating ring gyroscope.

2.2 Analysis of Ring Gyroscopes

Figure 2.2 shows the gyroscope's mechanical model, and Table 1 provides the definitions and numerical values of the pertinent symbols. We consider a general design in which a set of N_b equispaced *connecting beams* interconnect the outer and inner rings, and another set of N_b equispaced *supporting beams* join the inner ring to an anchored internal structure. We introduce the cylindrical coordinate system, $\{r, \theta, z\}$ as a gyroscope-fixed reference frame. The connecting beams are located at the angles $\theta_{c,i} = \frac{2\pi}{N_b}(i - 1)$, $i = 1 : N_b$, and, with a $\frac{\pi}{N_b}$ shift, the supporting beams have the angular positions $\theta_{s,i} = \frac{\pi}{N_b}(2i - 1)$, $i = 1 : N_b$. Our analysis is based on the following key assumptions: thin ring, inextensibility, and the Euler-Bernoulli beam theory. Appendix A contains a detailed discussion of these assumptions and their implications in the modeling of ring mechanics

Remark 2.1. Kirkhope [46] has reported that for the width-to-mean radius ratios of below 0.15 the first four natural frequencies—calculated by the thin ring and inextensibility assumptions—are in good agreement with the experimental data. In our VRG, we are interested in the second and third modes of vibration and, in addition, we have $\frac{w_1}{R_1} \approx 0.0645$, $\frac{w_2}{R_2} \approx 0.0121$; therefore, the thin ring and inextensibility assumptions are justified for our application.

Table 2.1: Parameters of vibrating ring gyroscope

Symbol	Definition	Value
R_1	Mean radius of outer ring	1550 μm
R_2	Mean radius of inner ring	1317 μm
w_1	Width of outer ring	100 μm
w_2	Width of inner ring	16 μm
A_1	Outer ring's cross-section area	3500 μm^2
A_2	Inner ring's cross-section area	560 μm^2
I_1	Second moment of area of outer ring's cross-section	$2.9167 \times 10^6 \mu\text{m}^4$
I_2	Second moment of area of inner ring's cross-section	$1.1947 \times 10^4 \mu\text{m}^4$
l_c	Length of each connecting beam	175 μm
w_c	Width of each connecting beam	16 μm
l_s	Length of each supporting beam	110 μm
w_s	Width of each supporting beam	10 μm
g_0	Electrode-ring gap	3 μm
h	Thickness of the whole structure	35 μm
ρ	Mass density	2329 kg/m^3
E	Young's modulus	170 GPa

2.2.1 Kinematics

To describe the in-plane motion of the structure, we consider the displacement fields $[u_r(\theta, t), u_\theta(\theta, t)]^\top \in \mathbb{R}^2$ and $[v_r(\theta, t), v_\theta(\theta, t)]^\top \in \mathbb{R}^2$ defined at the center-lines of the outer and the inner rings, respectively. Here, the subscripts r and θ denote the radial and tangential directions, respectively, and $t \geq 0$ is time. As a result of the inextensibility, the radial and tangential displacements are differentially coupled as [47]

$$\begin{aligned}
u_r(\theta, t) + \frac{\partial u_\theta(\theta, t)}{\partial \theta} &= 0, \\
v_r(\theta, t) + \frac{\partial v_\theta(\theta, t)}{\partial \theta} &= 0.
\end{aligned} \tag{2.1}$$

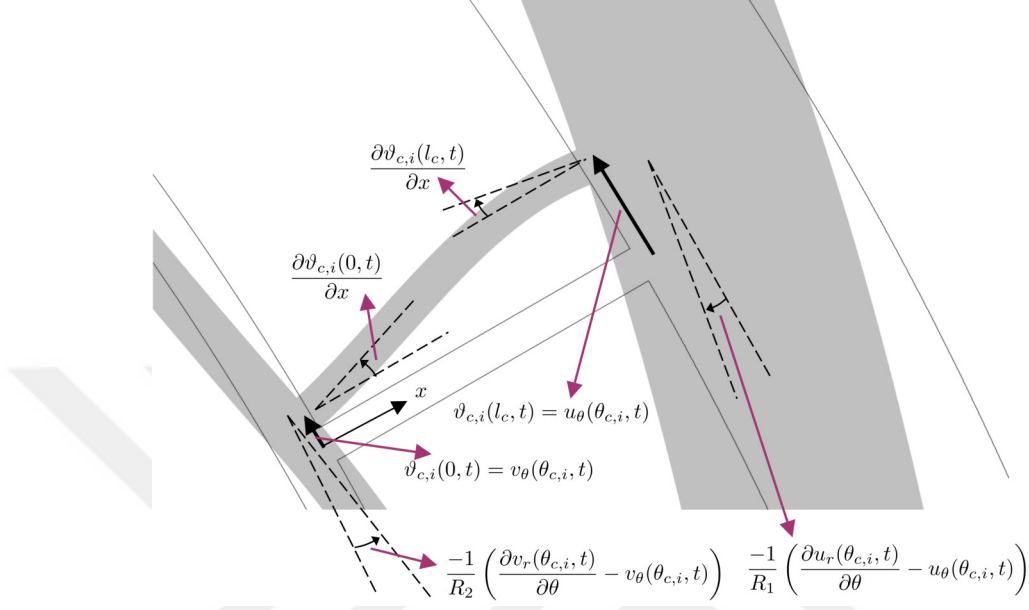


Figure 2.3: Illustration of the i -th connecting beam's boundary conditions.

Since the connecting and supporting beams are highly stiff along their longitudinal directions, we consider them all to be axially rigid with negligible radial deformation. Let x be the axial coordinate of the beams. We denote the tangential displacement field of the i -th connecting beam and the i -th supporting beam by $\vartheta_{c,i}(x,t)$, $x \in [0, l_c]$, and $\vartheta_{s,i}(x,t)$, $x \in [0, l_s]$, respectively. The ring-beam connections impose the following boundary conditions on the displacement fields:

1. *Connecting beams:* By the axial rigidity of these beams, both rings have the same radial displacement at the connection points:

$$u_r(\theta_{c,i},t) = v_r(\theta_{c,i},t), \quad i = 1 : N_b. \quad (2.2)$$

Additionally, the beams $i = 1 : N_b$ satisfy the following deflection/slope boundary conditions (see Figure 2.3):

$$\begin{aligned} \vartheta_{c,i}(0,t) &= v_\theta(\theta_{c,i},t), \\ \vartheta_{c,i}(l_c,t) &= u_\theta(\theta_{c,i},t), \\ \frac{\partial \vartheta_{c,i}(0,t)}{\partial x} &= \frac{-1}{R_2} \left(\frac{\partial v_r(\theta_{c,i},t)}{\partial \theta} - v_\theta(\theta_{c,i},t) \right), \\ \frac{\partial \vartheta_{c,i}(l_c,t)}{\partial x} &= \frac{-1}{R_1} \left(\frac{\partial u_r(\theta_{c,i},t)}{\partial \theta} - u_\theta(\theta_{c,i},t) \right). \end{aligned} \quad (2.3)$$

2. *Supporting beams:* The axial rigidity of these beams implies that the radial displacement of the inner ring is zero at the points of connection; that is,

$$v_r(\theta_{s,i}, t) = 0, \quad i = 1 : N_b. \quad (2.4)$$

Similar to the connecting beams, we have the following deflection/slope boundary conditions for the supporting beams $i = 1 : N_b$:

$$\begin{aligned} \vartheta_{s,i}(0, t) &= 0, \\ \vartheta_{s,i}(l_s, t) &= v_\theta(\theta_{s,i}, t), \\ \frac{\partial \vartheta_{s,i}(0, t)}{\partial x} &= 0, \\ \frac{\partial \vartheta_{s,i}(l_s, t)}{\partial x} &= \frac{-1}{R_2} \left(\frac{\partial v_r(\theta_{s,i}, t)}{\partial \theta} - v_\theta(\theta_{s,i}, t) \right). \end{aligned} \quad (2.5)$$

These beams have the fixed boundary condition at $x = 0$, i.e., at their junction with the anchored internal structure.

We note that the slope of a beam at its connection point with a ring is equivalent to the ring's rotation angle at that point. As illustrated in Figure 2.3, this angle is the superposition of two rotations: the first is caused by the gradient of the radial displacement, and the second is the result of the tangential displacement. According to (2.2)-(2.5), admissible displacement fields must satisfy a total number of $10N_b$ boundary conditions.

Velocity field. Assume that the gyroscope has a constant angular velocity of Ω_z about the z -axis with respect to an inertial frame fixed at the ground. Then, the velocity fields $\nu_1(\theta, t)$, $\nu_2(\theta, t)$, $\nu_{c,i}(x, t) \in \mathbb{R}^2$, and $\nu_{s,i}(x, t) \in \mathbb{R}^2$ of the outer ring, the inner ring, the i -th connecting beam, and the i -th supporting beam, respectively, are given by

$$\begin{aligned} \nu_1(\theta, t) &= \begin{bmatrix} \frac{\partial}{\partial t} u_r(\theta, t) - \Omega_z u_\theta \\ \frac{\partial}{\partial t} u_\theta(\theta, t) + \Omega_z (R_1 + u_r(\theta, t)) \end{bmatrix}, \\ \nu_2(\theta, t) &= \begin{bmatrix} \frac{\partial}{\partial t} v_r(\theta, t) - \Omega_z v_\theta \\ \frac{\partial}{\partial t} v_\theta(\theta, t) + \Omega_z (R_2 + v_r(\theta, t)) \end{bmatrix}, \\ \nu_{c,i}(x, t) &= \begin{bmatrix} -\Omega_z \vartheta_{c,i}(x, t) \\ \frac{\partial}{\partial t} \vartheta_{c,i}(x, t) + \Omega_z \left(R_2 + \frac{w_2}{2} + x \right) \end{bmatrix}, \end{aligned} \quad (2.6)$$

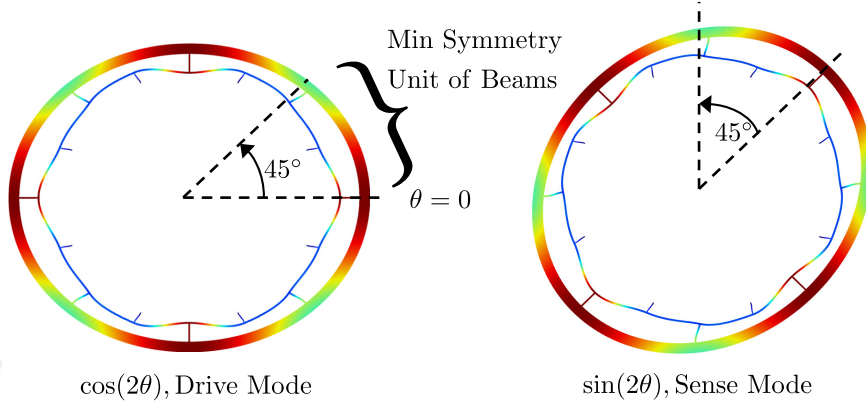


Figure 2.4: Wineglass ($n = 2$) mode shapes of the VRG.

$$\nu_{s,i}(x, t) = \begin{bmatrix} -\Omega_z \vartheta_{s,i}(x, t) \\ \frac{\partial}{\partial t} \vartheta_{s,i}(x, t) + \Omega_z \left(R_2 - l_s - \frac{w_2}{2} + x \right) \end{bmatrix}.$$

Here, we have used the relation $D(\cdot)/Dt = d(\cdot)/dt + \Omega \times (\cdot)$ between the time derivative operators D/Dt and d/dt in an inertial reference frame and a noninertial rotating frame with the angular velocity vector Ω , respectively (see [11] for more details).

2.2.2 Assumed modes

The VRG's mechanical structural is a continuous system with infinite degrees-of-freedom (DOF). We adopt the *assumed modes* method to obtain a finite DOF approximation. This method represents a given displacement field by the linear combination of a set of basis functions, decoupling the spatial and temporal variables [47, 48]. The basis functions should be compatible with the boundary conditions and closely capture the mode shapes. Because of the circular symmetry of a ring, the displacement $u_\theta(\theta, t)$ —at any given time, t —belongs to the space of 2π -periodic, continuous, nonconstant functions.² As in the Fourier series, the functions $\cos(n\theta)$ and $\sin(n\theta)$ provide an orthogonal complete basis for this function space [49]. Accordingly, taking into account the inextensibility condition (2.1), we consider the following assumed modes approximation for the outer

²Constant functions are excluded because of the inextensibility condition.

ring's displacement field:

$$\begin{aligned} u_\theta(\theta, t) &= -Q_1(t) \frac{\sin(n\theta)}{n} + Q_2(t) \frac{\cos(n\theta)}{n}, \\ u_r(\theta, t) &= Q_1(t) \cos(n\theta) + Q_2(t) \sin(n\theta), \end{aligned} \quad (2.7)$$

where $Q_1(\cdot), Q_2(\cdot) \in \mathbb{R}$ are the generalized coordinates corresponding to the contributions of the n -th mode shapes, $\{\cos(n\theta), \sin(n\theta)\}$, to the displacement field. For a freely vibrating ring, the mode shapes occur as the degenerate pairs, $\{\cos(n\theta), \sin(n\theta)\}$ for $n = 1, 2, \dots$ [50, 47]; the $n = 2$ mode shapes are depicted in Figure 2.4. This is also the case in Equation (2.7) as $\cos(n\theta)$ and $\sin(n\theta)$ should be considered in pair so that they form a complete orthogonal basis for the underlying function space. Similar to (2.7), we consider the following assumed modes approximation for the inner ring's displacement field:

$$\begin{aligned} v_\theta(\theta, t) &= -Q_1(t)f_n(\theta) - Q_2(t)g_n(\theta), \\ v_r(\theta, t) &= Q_1(t)f'_n(\theta) + Q_2(t)g'_n(\theta), \end{aligned} \quad (2.8)$$

where $f_n(\cdot), g_n(\cdot) \in \mathbb{R}$ are the n -th mode shapes to be determined. Again, because of the ring's circular symmetry, the functions $\{f_n(\theta), g_n(\theta)\}$, $n = 1, 2, \dots$, should constitute a basis for the space of 2π -periodic, continuous, non constant functions and satisfy the orthogonality condition, $\int_0^{2\pi} f_n(\theta)g_n(\theta)d\theta = 0$. The assumed modes approximations corresponding to the beams are

$$\begin{aligned} \vartheta_{c,i}(x, t) &= \varpi_{c,i}(x; n)Q_1(t) + \varrho_{c,i}(x; n)Q_2(t), \\ \vartheta_{s,i}(x, t) &= \varpi_{s,i}(x; n)Q_1(t) + \varrho_{s,i}(x; n)Q_2(t), \end{aligned} \quad (2.9)$$

where $\varpi_{c(s),i}(x; n)$ and $\varrho_{c(s),i}(x; n)$ are the beam mode shapes associated with the rings' n -th mode.

Remark 2.2. The assumed modes approximations (2.7)-(2.9) transform the infinite-DOF VRG into a 2-DOF system described by the generalized coordinates, $(Q_1, Q_2) \in \mathbb{R}^2$. This approximation adequately captures the n -th mode dynamics of the VRG. Higher-DOF approximations are also possible by incorporating multiple modes; for example, see [42, 44].

2.2.3 Parametrization of mode shapes

We now focus on a VRG that is designed to operate at the n -th vibration mode ($n \geq 2$) and the $N_b = 4n$ connecting/supporting beams are placed at the mode's nodal points, i.e., $\theta_{c,i} = \frac{\pi}{2n}(i-1)$ and $\theta_{s,i} = \frac{\pi}{4n}(2i-1)$ for $i = 1 : 4n$. For this type of design, we simplify calculation of the mode shapes of the inner ring and the beams by the following observations:

Observation 1. The outer ring, as the primary vibrating element, is driven by electrostatic actuation. The inner ring is in the linear elastic region of its material behavior because of small deformations. Hence, we can consider the inner ring as a linear system subjected to the inputs $\{\cos(n\theta), \sin(n\theta)\}$, i.e., the mode shapes of the outer ring. As a results of linearity, the functions $f_n(\cdot)$ and $g_n(\cdot)$ are $\frac{2\pi}{n}$ -periodic and need to satisfy the boundary conditions only for one period, $\theta \in [0, \frac{2\pi}{n}]$.

Observation 2. In addition to periodicity, the inner rings also preserve key symmetry axes of the input mode shapes. This property has the following important implications:

1. $g_n(\theta)$ is the dual mode shape of $f_n(\theta)$ obtained by its $\frac{\pi}{2n}$ rotation; that is,

$$g_n(\theta) = f_n\left(\theta - \frac{\pi}{2n}\right). \quad (2.10)$$

Hence, we need to calculate only $f_n(\theta)$.

2. In terms of the radial displacement, $f'_n(\theta)$ preserves the evenness of its corresponding input, $\cos(n\theta)$. As a result, $f'_n(\theta)$ lies in the span of the cosine basis functions $\{\cos(jn\theta) : j = 1, 2, \dots\}$.
3. We have $f'_n(0) + f'_n(\pi/n) = 0$; that is, $f'_n(\theta)$ exhibits a consecutive pair of maximum-minimum at the same angles that $\cos(n\theta)$ does. This further confines the set of cosine basis functions to $\{\cos((2j-1)n\theta) : j = 1, 2, \dots\}$. Therefore, using N basis functions, we parametrize $f_n(\theta)$ as

$$f_n(\theta) = \chi_r^\top(\theta)\alpha, \quad (2.11a)$$

where

$$\chi_r(\theta) = \text{col} \left(\frac{\sin((2j-1)n\theta)}{(2j-1)n} \right)_{j=1}^N \in \mathbb{R}^N \quad (2.11b)$$

is the vector of basis functions and $\alpha \in \mathbb{R}^N$ is the corresponding coefficients vector to be determined.

Observation 3. When the outer ring assumes the $\cos(n\theta)$ mode shape, the odd-numbered connecting beams undergo no deflection as they have pure axial translational motion; hence, $\varpi_{c,2i-1}(x) = 0$ for $i = 1 : 2n$. Moreover, the period and symmetry-preserving properties of the inner ring imply symmetries of the form $\varpi_{c,4i-2}(x) = \varpi_{c,2}(x)$, $i = 2 : n$, and $\varpi_{c,4i}(x) = -\varpi_{c,2}(x)$, $i = 1 : n$, between the deflection curves of the even-numbered beams. According to (2.10), by a $\pi/2n$ rotation, the case of the outer ring's $\sin(n\theta)$ mode shape is equivalent to the $\cos(n\theta)$ case. Therefore, we have the relations $\varrho_{c,1}(x, t) = \varpi_{c,4n}(x)$ and $\varrho_{c,i}(x, t) = \varpi_{c,i-1}(x)$, $i = 2 : 4n$, between the dual deflection curves induced by the two mode shapes of the outer ring. Applying the same rationale to the supporting beam reveals that $\varpi_{s,4i-3}(x) = \varpi_{s,4i-2}(x) = \varpi_{s,1}(x)$, $\varpi_{s,4i-1}(x) = \varpi_{s,4i}(x) = -\varpi_{s,1}(x)$, $i = 1 : n$, and their dual deflection curves satisfy $\varrho_{s,1}(x, t) = \varpi_{s,4n}(x)$ and $\varrho_{s,i}(x, t) = \varpi_{c,i-1}(x)$, $i = 2 : 4n$.

In conclusion, we need to calculate only the deflection curves $\varpi_{c,2}(x)$ and $\varpi_{s,1}(x)$ of the second connecting beam and the first supporting beam, respectively. Figure 2.4 visually illustrates the above analysis as the *minimal symmetry unit* from which, we can construct all the beam mode shapes.

Observation 4. Both connecting and supporting beams are subject to loads only at the junctions, and, by the Euler-Bernoulli theory, a cubic polynomial captures their deflection curves. On this basis, we set forth the parametrizations

$$\varpi_{c,2}(x; n) = \chi_c^\top(x) \beta_c, \quad (2.12a)$$

$$\varpi_{s,1}(x; n) = \chi_s^\top(x) \beta_s, \quad (2.12b)$$

$$\chi_c(x) = \text{col} \left(P_j \left(2 \left(\frac{x}{l_c} \right) - 1 \right) \right)_{j=0}^3 \in \mathbb{R}^4, \quad (2.12c)$$

$$\chi_s(x) = \text{col} \left(P_j \left(2 \left(\frac{x}{l_s} \right) - 1 \right) \right)_{j=0}^3 \in \mathbb{R}^4, \quad (2.12d)$$

where $P_j(\cdot)$ is the j -th Legendre polynomial, and $\beta_c, \beta_s \in \mathbb{R}^4$ are the coefficients vectors to be determined. We note that orthogonality of the Legendre polynomials gives them computational advantage over the monomials $\{1, x, x^2, \dots\}$ [49].

2.2.4 Simplifying boundary conditions

In view of Observations 1-4, we identify the independent boundary conditions as follows:

- Substituting (2.11a) into the radial displacement boundary conditions (2.2) and (2.4) reveals that there exist only two independent equations: the radial displacements of the first connecting beam and the second supporting beam; i.e.,

$$\begin{bmatrix} \chi_r'^\top(0) \\ \chi_r'^\top(\frac{\pi}{4n}) \end{bmatrix} \alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (2.13a)$$

- Boundary conditions of the second connecting beam:

$$\begin{bmatrix} \chi_c^\top(0) \\ \chi_c'^\top(0) \\ \chi_c^\top(l_c) \\ \chi_c'^\top(l_c) \end{bmatrix} \beta_c = \begin{bmatrix} -\chi_r^\top(\frac{\pi}{2n}) \\ \frac{-1}{R_2} (\chi_r^\top(\frac{\pi}{2n}) + \chi_r''^\top(\frac{\pi}{2n})) \\ 0_4^\top \\ 0_4^\top \end{bmatrix} \alpha + \begin{bmatrix} 0 \\ 0 \\ \frac{-1}{n} \\ \frac{n^2-1}{R_1 n} \end{bmatrix}. \quad (2.13b)$$

- Boundary conditions of the first supporting beam:

$$\begin{bmatrix} \chi_s^\top(0) \\ \chi_s'^\top(0) \\ \chi_s^\top(l_s) \\ \chi_s'^\top(l_s) \end{bmatrix} \beta_s = \begin{bmatrix} 0_4^\top \\ 0_4^\top \\ -\chi_r^\top(\frac{\pi}{4n}) \\ \frac{-1}{R_2} (\chi_r^\top(\frac{\pi}{4n}) + \chi_r''^\top(\frac{\pi}{4n})) \end{bmatrix} \alpha. \quad (2.13c)$$

2.2.5 Calculation of mode shapes

Upon parametrizing the unknown mode shapes by suitable basis functions, based on the Ritz method [31], we seek to minimize

$$\begin{aligned} \mathcal{V}_{\text{susp}} = & \frac{EI_2}{2R_2^3} \int_0^{2\pi} (f'_n(\theta) + f'''_n(\theta))^2 d\theta + 2n \frac{EI_c}{2} \int_0^{l_c} \varpi_{c,2}''^2(x; n) dx \\ & + 4n \frac{EI_s}{2} \int_0^{l_s} \varpi_{s,1}''^2(x; n) dx, \end{aligned} \quad (2.14)$$

which is the total strain potential energy of the inner ring and the beams (i.e., the suspension structure) as the outer ring assumes the $\cos(n\theta)$ mode shape. By substituting (2.11a), (2.12a), and (2.12b) into (2.14), we obtain the quadratic form

$$\mathcal{V}_{\text{susp}}(\alpha, \beta_c, \beta_s) = \frac{1}{2} \alpha^\top \Pi_1 \alpha + \frac{1}{2} \beta_c^\top \Pi_2 \beta_c + \frac{1}{2} \beta_s^\top \Pi_3 \beta_s, \quad (2.15)$$

where

$$\begin{aligned} \Pi_1 &= \frac{EI_2}{R_2^3} \int_0^{2\pi} (\chi'_r(\theta) + \chi'''_r(\theta)) (\chi'_r(\theta) + \chi'''_r(\theta))^\top d\theta, \\ \Pi_2 &= 2nEI_c \int_0^{l_c} \chi_c''(x) \chi_c''^\top(x) dx, \\ \Pi_3 &= 4nEI_s \int_0^{l_s} \chi_s''(x) \chi_s''^\top(x) dx. \end{aligned}$$

The Ritz method is now equivalent to the following optimization problem in the coefficients vectors α , β_c , and β_s :

$$\begin{aligned} \min_{\alpha, \beta_c, \beta_s} \quad & \mathcal{V}_{\text{susp}}(\alpha, \beta_c, \beta_s) \\ \text{subject to} \quad & (2.13a), (2.13b), \text{ and } (2.13c). \end{aligned} \quad (2.16)$$

We note that (2.16) is a standard *quadratic programming* (QP) problem with linear equality constraints, which can be solved with a desired tolerance in numerical computing software³. A careful examination, however, reveals that we can calculate the optimal coefficient vectors α , β_c , and β_s rapidly and efficiently without resorting to any recursive optimization algorithm. For this, by using (2.13b) and (2.13c), we first eliminate β_c and β_s ; then, we use (2.13a) to eliminate two redundant variables in the vector α . In this manner, we convert (2.16) into an unconstrained optimization problem that admits a unique minimizing solution.

³This can be done in MATLAB using the `quadprog` command.

2.2.6 Dynamics of VRG

Building upon the assumed modes approximation, we use the Lagrangian formalism to derive the equations of motion governing the VRG's dynamics. We use the mode shapes that we calculated in the previous section to obtain the VRG's kinetic and potential energies and then apply Lagrange's equations. Moreover, our formulation accommodates material anisotropy and geometric imperfections by considering Young's modulus, area, and second moment of area of the rings' cross-sections as functions of θ . Similarly, we consider the cross-section areas and second moment of areas of the beams as functions of x .

2.2.6.1 Kinetic Energy

Based on the velocity fields (2.6), VRG's total kinetic energy, $\mathcal{T}$ is given by

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4, \quad (2.17)$$

$$\mathcal{T}_1 = \frac{1}{2}\rho R_1 \int_0^{2\pi} \nu_1^\top(\theta, t) \nu_1(\theta, t) A_1(\theta) d\theta,$$

$$\mathcal{T}_2 = \frac{1}{2}\rho R_2 \int_0^{2\pi} \nu_2^\top(\theta, t) \nu_2(\theta, t) A_2(\theta) d\theta,$$

$$\mathcal{T}_3 = \sum_{i=1}^{4n} \frac{1}{2}\rho \int_0^{l_c} \nu_{c,i}^\top(x, t) \nu_{c,i}(x, t) A_{c,i}(x) dx,$$

$$\mathcal{T}_4 = \sum_{i=1}^{4n} \frac{1}{2}\rho \int_0^{l_s} \nu_{s,i}^\top(x, t) \nu_{s,i}(x, t) A_{s,i}(x) dx,$$

where $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, and $\mathcal{T}_4$ are the kinetic energies of the outer ring, inner rings, connecting beams, and supporting beams, respectively. Using (2.7) and (2.8), and defining $Q(t) := [Q_1(t), Q_2(t)]^\top \in \mathbb{R}^2$ as the vector of generalized coordinates, we obtain the kinetic energy in terms of Q and $\dot{Q}$ as follows:

$$\begin{aligned} \mathcal{T}(\dot{Q}, Q) &= \frac{1}{2}\dot{Q}^\top M \dot{Q} + m_c \Omega_z Q^\top S_c \dot{Q} + \frac{1}{2}\Omega_z^2 Q^\top M Q + \Omega_z f_{Co}^\top \dot{Q} + \Omega_z^2 f_{Ce}^\top Q \\ &\quad + \frac{1}{2}\Omega_z^2 \rho \int_0^{2\pi} (R_1^3 A_1(\theta) + R_2^3 A_2(\theta)) d\theta, \end{aligned} \quad (2.18)$$

$$M = M_1 + M_2 + M_3 + M_4,$$

$$m_c = m_{c_1} + m_{c_2},$$

$$S_c = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Here, $M_1, M_2, M_3 \in \mathbb{R}^{2 \times 2}$, and $M_4 \in \mathbb{R}^{2 \times 2}$ are the mass matrices of the outer ring, inner ring, connecting beams, and supporting beams, respectively; $m_{c_1} \in \mathbb{R}$ and $m_{c_2} \in \mathbb{R}$ are the Coriolis masses of the outer and inner rings, respectively; $S_c \in \mathbb{R}^{2 \times 2}$ is a skew-symmetric matrix representing cross product with the angular velocity, and $f_{Co} \in \mathbb{R}^2$ and $f_{Ce} \in \mathbb{R}^2$ are terms associated with the Coriolis and centripetal accelerations, respectively:

$$M_1 = \rho R_1 \left(\frac{n^2 + 1}{2n^2} \right) \int_0^{2\pi} A_1(\theta) d\theta I_{2 \times 2} + \rho R_1 \left(\frac{n^2 - 1}{2n^2} \right) \int_0^{2\pi} \begin{bmatrix} \cos(2n\theta) & \sin(2n\theta) \\ \sin(2n\theta) & -\cos(2n\theta) \end{bmatrix} A_1(\theta) d\theta, \quad (2.19a)$$

$$M_2 = \rho R_2 \int_0^{2\pi} \begin{bmatrix} f_n^2(\theta) + f_n'^2(\theta) & (f_n(\theta)g_n(\theta) + f_n'(\theta)g_n'(\theta)) \\ \star & g_n^2(\theta) + g_n'^2(\theta) \end{bmatrix} A_2(\theta) d\theta, \quad (2.19b)$$

$$M_3 = \rho \sum_{i=1}^{4n} \int_0^{l_c} \begin{bmatrix} \varpi_{c,i}^2(x) & 0 \\ 0 & \varrho_{c,i}^2(x) \end{bmatrix} A_{c,i}(x) dx, \quad (2.19c)$$

$$M_4 = \rho \sum_{i=1}^{4n} \int_0^{l_s} \begin{bmatrix} \varpi_{s,i}^2(x) & \varpi_{s,i}(x)\varrho_{s,i}(x) \\ \star & \varrho_{s,i}^2(x) \end{bmatrix} A_{s,i}(x) dx, \quad (2.19d)$$

$$m_{c_1} = \frac{\rho R_1}{n} \int_0^{2\pi} A_1(\theta) d\theta, \quad (2.19e)$$

$$m_{c_2} = \rho R_2 \int_0^{2\pi} (f_n(\theta)g_n'(\theta) - f_n'(\theta)g_n(\theta)) A_2(\theta) d\theta, \quad (2.19f)$$

$$f_{Co} = \rho \int_0^{2\pi} \begin{bmatrix} -R_1^2 \frac{\sin(n\theta)}{n} A_1(\theta) - R_2^2 f_n(\theta) A_2(\theta) \\ R_1^2 \frac{\cos(n\theta)}{n} A_1(\theta) - R_2^2 g_n(\theta) A_2(\theta) \end{bmatrix} d\theta + \rho \sum_{i=1}^{4n} \int_0^{l_c} \begin{bmatrix} \varpi_{c,i}(x) \\ \varrho_{c,i}(x) \end{bmatrix} \left(R_2 + \frac{w_2}{2} + x \right) A_{c,i}(x) dx + \rho \sum_{i=1}^{4n} \int_0^{l_s} \begin{bmatrix} \varpi_{s,i}(x) \\ \varrho_{s,i}(x) \end{bmatrix} \left(R_2 + \frac{w_2}{2} - l_s + x \right) A_{s,i}(x) dx, \quad (2.19g)$$

$$f_{Ce} = \rho \int_0^{2\pi} \begin{bmatrix} R_1^2 \cos(n\theta) A_1(\theta) + R_2^2 f_n'(\theta) A_2(\theta) \\ R_1^2 \sin(n\theta) A_1(\theta) + R_2^2 g_n'(\theta) A_2(\theta) \end{bmatrix} d\theta. \quad (2.19h)$$

2.2.6.2 Mechanical Potential Energy

The mechanical potential energy of the VRG, $\mathcal{V}_m$ is the sum of bending strain energies $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$, and $\mathcal{V}_4$ of the outer ring, inner ring, connecting beams, and supporting beams, respectively. That is,

$$\mathcal{V}_m = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 + \mathcal{V}_4, \quad (2.20)$$

$$\begin{aligned} \mathcal{V}_1 &= \frac{1}{2R_1^3} \int_0^{2\pi} E(\theta) I_1(\theta) \left(u_r(\theta, t) + \frac{\partial^2 u_r(\theta, t)}{\partial \theta^2} \right)^2 d\theta, \\ \mathcal{V}_2 &= \frac{1}{2R_2^3} \int_0^{2\pi} E(\theta) I_2(\theta) \left(v_r(\theta, t) + \frac{\partial^2 v_r(\theta, t)}{\partial \theta^2} \right)^2 d\theta, \\ \mathcal{V}_3 &= \sum_{i=1}^{4n} \frac{1}{2} E(\theta_{c,i}) \int_0^{l_c} \left(\frac{\partial^2 \vartheta_{c,i}(x, t)}{\partial x^2} \right)^2 I_{c,i}(x) dx, \\ \mathcal{V}_4 &= \sum_{i=1}^{4n} \frac{1}{2} E(\theta_{s,i}) \int_0^{l_s} \left(\frac{\partial^2 \vartheta_{s,i}(x, t)}{\partial x^2} \right)^2 I_{s,i}(x) dx. \end{aligned}$$

Details of bending strain energies of rings and beams can be found in [31, 32] (see also Appendix A). By substituting (2.7), (2.8) and (2.9) into (2.20), we obtain

$$\mathcal{V}_m(Q) = \frac{1}{2} Q^\top K_m Q, \quad (2.21)$$

$$K_m = K_1 + K_2 + K_3 + K_4,$$

where K_1 , K_2 , $K_3 \in \mathbb{R}^{2 \times 2}$ and $K_4 \in \mathbb{R}^{2 \times 2}$ are the stiffness matrices of the outer ring, inner ring, connecting beams, and supporting beams, respectively:

$$\begin{aligned} K_1 &= \frac{(n^2 - 1)^2}{2R_1^3} \int_0^{2\pi} E(\theta) I_1(\theta) d\theta I_{2 \times 2} \\ &\quad + \frac{(n^2 - 1)^2}{2R_1^3} \int_0^{2\pi} \begin{bmatrix} \cos(2n\theta) & \sin(2n\theta) \\ \sin(2n\theta) & -\cos(2n\theta) \end{bmatrix} E(\theta) I_1(\theta) d\theta, \end{aligned} \quad (2.22a)$$

$$K_2 = \frac{1}{R^3} \int_0^{2\pi} \begin{bmatrix} (f'_n(\theta) + f'''_n(\theta))^2 & (f'_n(\theta) + f'''_n(\theta))(g'_n(\theta) + g'''_n(\theta)) \\ \star & (g'_n(\theta) + g'''_n(\theta))^2 \end{bmatrix} E(\theta) I_2(\theta) d\theta, \quad (2.22b)$$

$$K_3 = \sum_{i=1}^{4n} E(\theta_{c,i}) \int_0^{l_c} \begin{bmatrix} \varpi_{c,i}''^2(x) & 0 \\ 0 & \varrho_{c,i}''^2(x) \end{bmatrix} I_{c,i}(x) dx, \quad (2.22c)$$

$$K_4 = \sum_{i=1}^{4n} E(\theta_{s,i}) \int_0^{l_s} \begin{bmatrix} \varpi_{s,i}''^2(x) & \varpi_{s,i}''(x) \varrho_{s,i}''(x) \\ \star & \varrho_{s,i}''^2(x) \end{bmatrix} I_{s,i}(x) dx. \quad (2.22d)$$

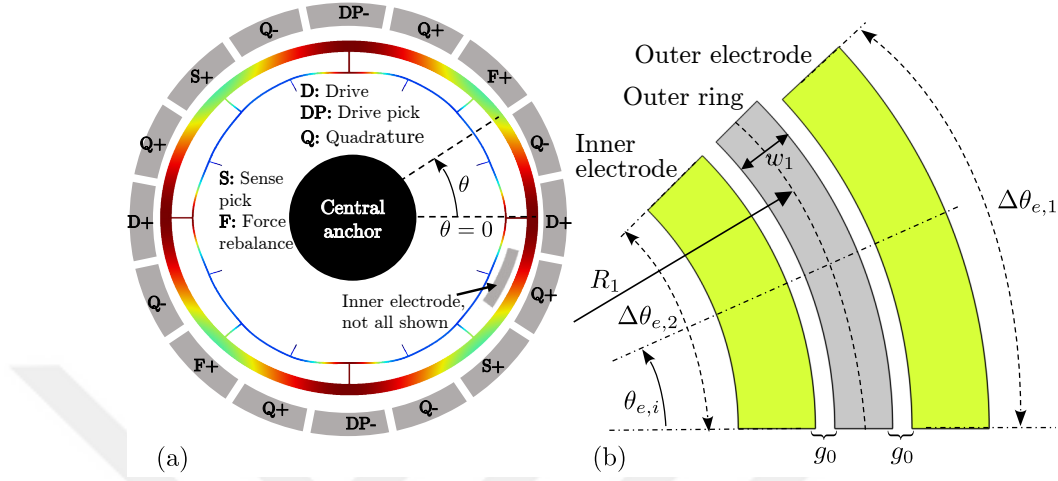


Figure 2.5: Configuration of electrodes for VRG's $n = 2$ mode (a) and modeling schematics for a pair of electrodes (b).

Remark 2.3. Equations (2.19a)-(2.19d) and (2.22a)-(2.22d) show that—assuming material isotropy—the ideal suspension system induces no off-diagonal nor mismatch between the diagonal entries of the mass and stiffness matrices, and hence, it preserves the symmetry of the outer ring.

2.2.6.3 Electrostatic Potential Energy

On both inside and outside, the outer ring is surrounded by N_e pairs of differential electrodes (see Figure 2.5). The i -th electrode pair is centered at the angle $\theta_{e,i} = \frac{2\pi}{N_e}(i - 1)$ with the angular spans of $\Delta\theta_{e,1}$ and $\Delta\theta_{e,2}$ on the outside and inside, respectively. The capacitances $C_{1,i}$ and $C_{2,i}$ between the ring and the outer and inner electrodes, respectively, are given by

$$C_{1,i} = \epsilon h \int_{\theta_{e,i} - \frac{\Delta\theta_{e,1}}{2}}^{\theta_{e,i} + \frac{\Delta\theta_{e,1}}{2}} \frac{d\theta}{\ln \left(\frac{R_o + g_0}{R_o + u_r(\theta, t)} \right)}$$

$$\approx \frac{\epsilon h}{g_0} \int_{\theta_{e,i}-\frac{\Delta\theta_{e,1}}{2}}^{\theta_{e,i}+\frac{\Delta\theta_{e,1}}{2}} \left(1 + \left(\frac{u_r(\theta, t)}{g_0} \right) + \left(\frac{u_r(\theta, t)}{g_0} \right)^2 \right) R_o(\theta) d\theta, \quad (2.23a)$$

$$C_{2,i} = \epsilon h \int_{\theta_{e,i}-\frac{\Delta\theta_{e,2}}{2}}^{\theta_{e,i}+\frac{\Delta\theta_{e,2}}{2}} \frac{d\theta}{\ln \left(\frac{R_i + u_r(\theta, t)}{R_i - g_0} \right)}$$

$$\approx \frac{\epsilon h}{g_0} \int_{\theta_{e,i}-\frac{\Delta\theta_{e,2}}{2}}^{\theta_{e,i}+\frac{\Delta\theta_{e,2}}{2}} \left(1 - \left(\frac{u_r(\theta, t)}{g_0} \right) + \left(\frac{u_r(\theta, t)}{g_0} \right)^2 \right) R_i(\theta) d\theta, \quad (2.23b)$$

where $R_o(\theta) = R_1 + \frac{w_1(\theta)}{2}$ and $R_i(\theta) = R_1 - \frac{w_1(\theta)}{2}$ are the outer and inner radii of the outer ring, respectively, g_0 is the nominal gap, and $\epsilon = 8.8542 \times 10^{-12}$ F/m is the permittivity of vacuum; note that we have calculated $C_{1,i}$ and $C_{2,i}$ using the cylindrical capacitance formula [51]. For the i -th electrodes pair, let $V_{1,i}$ and $V_{2,i}$ be the voltage differences between the ring and the outer and inner electrodes, respectively. The total electrostatic potential energy of the VRG is

$$\mathcal{V}_e = \sum_{i=1}^{N_e} \frac{1}{2} C_{1,i} V_{1,i}^2 + \frac{1}{2} C_{2,i} V_{2,i}^2. \quad (2.24)$$

By substituting (2.7) into (2.24) and using the Taylor expansions given in (2.23a) and (2.23b), we express the potential function $\mathcal{V}_e$ in terms of the generalized coordinates:

$$\mathcal{V}_e(Q) \approx \mathcal{V}_{e,0} + F_e^\top Q + \frac{1}{2} Q^\top K_e Q, \quad (2.25)$$

where $\mathcal{V}_{e,0} \geq 0$ is a constant term, $F_e = [f_{e,1}, f_{e,2}]^\top \in \mathbb{R}^2$ is a vector of electrostatic forces, and $K_e \in \mathbb{R}^{2 \times 2}$ is the electrostatically-induced stiffness matrix that accounts for the spring softening. We have

$$\mathcal{V}_{e,0} = \frac{\epsilon h}{2g_0} \sum_{i=1}^{N_e} V_{1,i}^2 \int_{\theta_{e,i}-\frac{\Delta\theta_{e,1}}{2}}^{\theta_{e,i}+\frac{\Delta\theta_{e,1}}{2}} R_o(\theta) d\theta + V_{2,i}^2 \int_{\theta_{e,i}-\frac{\Delta\theta_{e,2}}{2}}^{\theta_{e,i}+\frac{\Delta\theta_{e,2}}{2}} R_i(\theta) d\theta, \quad (2.26a)$$

$$f_{e,1} = \frac{\epsilon h}{2g_0^2} \sum_{i=1}^{N_e} V_{1,i}^2 \int_{\theta_{e,i}-\frac{\Delta\theta_{e,1}}{2}}^{\theta_{e,i}+\frac{\Delta\theta_{e,1}}{2}} R_o(\theta) \cos(n\theta) d\theta - V_{2,i}^2 \int_{\theta_{e,i}-\frac{\Delta\theta_{e,2}}{2}}^{\theta_{e,i}+\frac{\Delta\theta_{e,2}}{2}} R_i(\theta) \cos(n\theta) d\theta, \quad (2.26b)$$

$$f_{e,2} = \frac{\epsilon h}{2g_0^2} \sum_{i=1}^{N_e} V_{1,i}^2 \int_{\theta_{e,i} - \frac{\Delta\theta_{e,1}}{2}}^{\theta_{e,i} + \frac{\Delta\theta_{e,1}}{2}} R_o(\theta) \sin(n\theta) d\theta - V_{2,i}^2 \int_{\theta_{e,i} - \frac{\Delta\theta_{e,2}}{2}}^{\theta_{e,i} + \frac{\Delta\theta_{e,2}}{2}} R_i(\theta) \sin(n\theta) d\theta, \quad (2.26c)$$

$$K_e = \begin{bmatrix} k_{e,11} & k_{e,12} \\ k_{e,12} & k_{e,22} \end{bmatrix}, \quad (2.26d)$$

$$k_{e,11} = \frac{\epsilon h}{2g_0^3} \sum_{i=1}^{N_e} V_{1,i}^2 \int_{\theta_{e,i} - \frac{\Delta\theta_{e,1}}{2}}^{\theta_{e,i} + \frac{\Delta\theta_{e,1}}{2}} R_o(\theta) (1 + \cos(2n\theta)) d\theta + V_{2,i}^2 \int_{\theta_{e,i} - \frac{\Delta\theta_{e,2}}{2}}^{\theta_{e,i} + \frac{\Delta\theta_{e,2}}{2}} R_i(\theta) (1 + \cos(2n\theta)) d\theta, \quad (2.26e)$$

$$k_{e,22} = \frac{\epsilon h}{2g_0^3} \sum_{i=1}^{N_e} V_{1,i}^2 \int_{\theta_{e,i} - \frac{\Delta\theta_{e,1}}{2}}^{\theta_{e,i} + \frac{\Delta\theta_{e,1}}{2}} R_o(\theta) (1 - \cos(2n\theta)) d\theta + V_{2,i}^2 \int_{\theta_{e,i} - \frac{\Delta\theta_{e,2}}{2}}^{\theta_{e,i} + \frac{\Delta\theta_{e,2}}{2}} R_i(\theta) (1 - \cos(2n\theta)) d\theta, \quad (2.26f)$$

$$k_{e,12} = \frac{\epsilon h}{2g_0^3} \sum_{i=1}^{N_e} V_{1,i}^2 \int_{\theta_{e,i} - \frac{\Delta\theta_{e,1}}{2}}^{\theta_{e,i} + \frac{\Delta\theta_{e,1}}{2}} R_o(\theta) \sin(2n\theta) d\theta + V_{2,i}^2 \int_{\theta_{e,i} - \frac{\Delta\theta_{e,2}}{2}}^{\theta_{e,i} + \frac{\Delta\theta_{e,2}}{2}} R_i(\theta) \sin(2n\theta) d\theta. \quad (2.26g)$$

2.2.6.4 Damping

The energy loss in VRGs, similar to other MEMS devices, is attributable to viscous damping of the surrounding atmosphere, thermoelastic damping in the mechanical structure, material defects, and anchor losses [52, 53]. We model the combined effect of these factors by a Rayleigh dissipation function of the form

$$\mathcal{F}(\dot{Q}) = \frac{1}{2} \dot{Q}^\top D \dot{Q}, \quad (2.27)$$

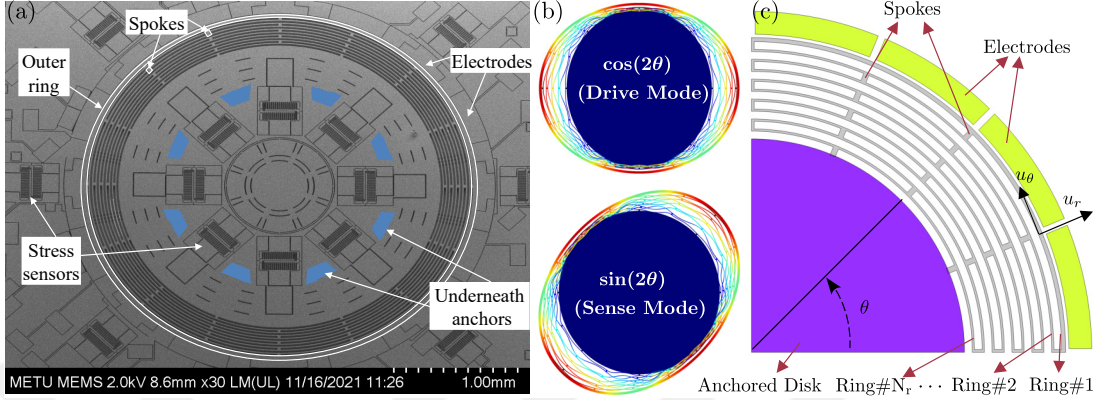


Figure 2.6: Fabricated DRG's SEM (a), operating mode shapes (b), and schematics (c).

where $D \in \mathbb{R}^{2 \times 2}$ is a symmetric positive semidefinite damping matrix.

2.3 Extension to Higher Number of Rings: DRGs

This section elaborates on a natural extension of the proposed analysis method for a double-ring structure to gyroscopes that employ a large collection of concentric rings, i.e., DRGs. Our fabricated, 10-rings, 41 kHz, (111) silicon DRG is shown in Figure 2.6, and its key parameters are provided in Table 2.2. We, in effect, follow the same procedure as Section 2.2. Given the mode shapes of the outermost ring (the wineglass modes in Figure 2.6-(b)), we formulate the boundary conditions at the interconnection of the rings by spokes (i.e., thick beams) and parametrize the mode shapes of the inner rings in terms of suitable basis functions. Following the Ritz method, we solve a least-squares optimization problem to calculate the mode shapes and obtain the corresponding modal mass, Coriolis mass, and stiffness. For the simplicity of exposition, we do not incorporate θ -dependent geometric properties and Young's modulus in our formulation. Such imperfections can be easily implemented, similar to the double-ring case.

The literature on the modeling of multi-ring DRGs is scarce. For mode shape

Table 2.2: Parameter of fabricated DRG

Parameter	Value
Number of rings (N_r)	10
Mean radius of Ring#1 (R_1)	1699 μm
Width of Ring#1 (w_1)	41 μm
Average width of Rings#2-10*	11.44 μm
Spoke length (l_{sp})	17.5 μm
Spoke width	40 μm
Structural thickness	35 μm

*We have a varying width design for the suspension rings.

calculation and lumped parameter modeling of DRGs, Li et al. [54] used the component mode synthesis method, and Lin et al. [55] adopted the ray tracing method from the wave propagation theory. The advantages of our method over these works are its simplicity and computational efficiency. Moreover, we present a general setting for the n -th mode of the rings, which includes a $n = 3$ device. Additionally, as pointed out above, our approach can handle the geometrical/-material imperfections.

Following Figure 2.6-(c), we consider a general DRG whose core mechanical structure consists of N_r concentric rings, numbered consecutively from outer to inner as Ring#1, Ring#2, ..., and Ring# N_r . We use the polar coordinates (r, θ) to study the in-plane vibration of the rings. We denote by R_i , w_i , A_i , and I_i the mean radius, width, cross-section area, and second moment of the cross-section area of Ring# i , respectively. We assume the rings to be thin (i.e., $w_i/R_i \ll 1$), inextensible, and governed by the Euler-Bernoulli beam theory. For $i = 1 : N_r - 1$, Ring# i and Ring# $i + 1$ are connected by $4n$ spokes located at the angles $\theta_{i,j}$, $j = 1 : 4n$, and each has the length l_{sp} ; here, $n \geq 2$ is the mode number for the operation of the DRG. The angular positions of the spokes alternate between two configurations with $\pi/4n$ relative angle shift; more specifically,

$$\theta_{i,j} = \begin{cases} \frac{\pi}{2n}(j-1) & \text{if } i \text{ is odd,} \\ \frac{\pi}{4n}(2j-1) & \text{if } i \text{ is even,} \end{cases} \quad (2.28)$$

where $j = 1 : 4n$. The last set of spokes connects Ring# N_r to a rigid anchored

disk. We note that Ring#1 is the main vibrating component that we electrostatically drive into resonance and detect its motion; the rest of the rings along with the spokes have the role of a supporting suspension system for Ring#1. We denote by $u_{r,i}(\theta, t)$ and $u_{\theta,i}(\theta, t)$ the radial and tangential components of the displacement field of Ring#i, respectively, where $t \geq 0$ is time.

2.3.1 Boundary conditions

Because of their sizable width-to-length ratio, we consider the spokes as rigid connections. This consideration, in turn, implies the following set of boundary conditions:

1. *Radial motion.* At the ring-spoke connections, Ring# i and Ring# $i + 1$, $i = 1 : N_r - 1$, have the same radial displacements, and Ring# N_r has zero radial displacement because of its connection to the innermost anchored disk:

$$\begin{aligned} u_{r,i+1}(\theta_{i,j}, t) &= u_{r,i}(\theta_{i,j}, t), \quad i = 1 : N_r - 1, j = 1 : 4n, \forall t \geq 0, \\ u_{r,N_r}(\theta_{N_r,j}, t) &= 0, \quad j = 1 : 4n, \forall t \geq 0. \end{aligned} \quad (2.29)$$

2. *Tangential motion.* The j -th spoke rigidly transmits the tangential displacement of Ring# i to Ring# $i + 1$, generating a tangential displacement and a local rotation at Ring# $i + 1$. This rotation is the superposition of two separate rotations caused by the gradient of radial displacement and the tangential motion. Mathematically,

$$\begin{aligned} u_{\theta,i+1}(\theta_{i,j}, t) - \frac{l_{sp}}{R_{i+1}} \left(\frac{\partial u_{r,i+1}}{\partial \theta}(\theta_{i,j}, t) - u_{\theta,i+1}(\theta_{i,j}, t) \right) &= u_{\theta,i}(\theta_{i,j}, t), \\ u_{\theta,N_r}(\theta_{N_r,j}, t) &= 0, \end{aligned} \quad (2.30)$$

for $i = 1 : N_r - 1$, $j = 1 : 4n$, and all $t \geq 0$.

2.3.2 Basis functions for mode shapes

As a result of rings' circular symmetry, their mode shapes occur in the form of degenerate pairs with a single natural frequency. Combining this property with the inextensibility of the rings' centerlines for the n -th mode, we have

$$\begin{aligned} u_{\theta,i}(\theta, t) &= -Q_1(t) \int_0^\theta f_i(\tau; n) d\tau - Q_2(t) \int_0^\theta g_i(\tau; n) d\tau, \\ u_{r,i}(\theta, t) &= Q_1(t) f_i(\theta; n) + Q_2(t) g_i(\theta; n), \end{aligned} \quad (2.31)$$

where the mode shapes $f_i(\cdot), g_i(\cdot) \in \mathbb{R}$ are (at least) 2π -periodic continuous functions that satisfy the orthogonality condition $\int_0^{2\pi} f_i(\theta; n) g_i(\theta; n) d\theta = 0$, and $Q_1(\cdot) \in \mathbb{R}$ and $Q_2(\cdot) \in \mathbb{R}$ are the generalized coordinates associated with the displacements of $f_i(\cdot)$ and $g_i(\cdot)$, respectively. We assume that the suspension system is compliant enough to not significantly distort the modal space of Ring#1 in its free-standing case. Accordingly, the n -th mode shapes of Ring#1 are (see Figure 2.6-(b))

$$f_1(\theta; n) = \cos(n\theta), \quad g_1(\theta; n) = \sin(n\theta). \quad (2.32)$$

Now, our objective is to calculate the mode shapes of Rings#2 : N_r , given the mode shapes (2.32) of Ring#1. Similar to the VRG, we consider the inner suspension as a linear system subjected to $f_1(\theta; n)$ and $g_1(\theta; n)$ as its inputs and proceed with the following observations:

1. The suspension system, due to its linearity, preserves the period of its inputs, and therefore, $f_i(\theta; n)$ and $g_i(\theta; n)$ are $\frac{2\pi}{n}$ -periodic. Moreover, $f_i(\theta; n)$ and $g_i(\theta; n)$ will have the same relative phase shift of $\frac{\pi}{2n}$ as $f_1(\theta; n)$ and $g_1(\theta; n)$. This means that we only need to calculate the responses $f_i(\theta; n)$ to $f_1(\theta; n)$ and g_i 's can be obtained as $g_i(\theta; n) = f_i(\theta - \frac{\pi}{2n}; n)$, $i = 2 : N_r$.
2. A careful inspection reveals that the suspension system preserves the evenness of $f_1(\theta; n)$ and—same as $f_1(\theta; n)$ —has a pair of consecutive maximum-minimum at the angles 0 and $\frac{\pi}{2n}$.

These two observations give us $\{\cos((2j-1)n\theta) : j = 1, 2, \dots\}$ as the underlying set of basis functions for the mode shapes $f_i(\theta; n)$. Accordingly, we form the

truncated expansions

$$f_i(\theta; n) \approx \phi^\top(\theta; n)\alpha_i, \quad i = 2 : N_r, \quad (2.33)$$

where

$$\phi(\theta; n) = [\cos(n\theta), \cos(3n\theta), \dots, \cos((2N-1)n\theta)]^\top \in \mathbb{R}^N \quad (2.34)$$

is the vector of basis functions and $\alpha_i \in \mathbb{R}^N$ is the vector of their coefficients; N is the number of basis of functions that we use to approximate the mode shapes. Let $\varphi(\theta; n) := \int_0^\theta \phi(\tau; n) d\tau$; the basis functions satisfy the following properties over the spoke angles:

1. If ring number i is odd:

$$\begin{aligned} \phi(\theta_{i,j}; n) &= \begin{cases} 0, & \text{if } j \text{ is even,} \\ (-1)^{\frac{j-1}{2}} \phi(\theta_{i,1}; n), & \text{if } j \text{ is odd,} \end{cases} \\ \phi'(\theta_{i,j}; n) &= \begin{cases} (-1)^{\frac{j-2}{2}} \phi'(\theta_{i,2}; n), & \text{if } j \text{ is even,} \\ 0, & \text{if } j \text{ is odd,} \end{cases} \\ \varphi(\theta_{i,j}; n) &= \begin{cases} (-1)^{\frac{j-2}{2}} \varphi(\theta_{i,2}; n), & \text{if } j \text{ is even,} \\ 0, & \text{if } j \text{ is odd.} \end{cases} \end{aligned} \quad (2.35)$$

2. If ring number i is even:

$$\begin{aligned} \phi(\theta_{i,j}; n) &= \begin{cases} \phi(\theta_{i,1}; n), & \text{if } j \equiv 0, 1 \pmod{4}, \\ -\phi(\theta_{i,1}; n), & \text{if } j \equiv 2, 3 \pmod{4}, \end{cases} \\ \phi'(\theta_{i,j}; n) &= \begin{cases} \phi'(\theta_{i,2}; n), & \text{if } j \equiv 1, 2 \pmod{4}, \\ -\phi'(\theta_{i,2}; n), & \text{if } j \equiv 0, 3 \pmod{4}, \end{cases} \\ \varphi(\theta_{i,j}; n) &= \begin{cases} \varphi(\theta_{i,2}; n), & \text{if } j \equiv 1, 2 \pmod{4}, \\ -\varphi(\theta_{i,2}; n), & \text{if } j \equiv 0, 3 \pmod{4}. \end{cases} \end{aligned} \quad (2.36)$$

In view of the properties (2.35) and (2.36), expansion (2.33) allows us to simplify the $8nN_r$ equations in the boundary conditions (2.29) and (2.30) to the following

set of $2N_r$ equations:

$$\begin{aligned}
\phi^\top(\theta_{1,1}; n)\alpha_2 &= \cos(n\theta_{1,1}), \\
\phi^\top(\theta_{i,1}; n)(\alpha_{i+1} - \alpha_i) &= 0, \quad i = 2 : N_r - 1, \\
\phi^\top(\theta_{N_r,1}; n)\alpha_{N_r} &= 0, \\
\left(\left(1 + \frac{l_{sp}}{R_2} \right) \varphi^\top(\theta_{1,2}; n) + \frac{l_{sp}}{R_2} \phi'^\top(\theta_{1,2}; n) \right) \alpha_2 &= \frac{\sin(n\theta_{1,2})}{n}, \\
\left(\left(1 + \frac{l_{sp}}{R_{i+1}} \right) \varphi^\top(\theta_{i,2}; n) + \frac{l_{sp}}{R_{i+1}} \phi'^\top(\theta_{i,2}; n) \right) \alpha_{i+1} \\
&\quad - \varphi^\top(\theta_{i,2}; n)\alpha_i = 0, \quad i = 2 : N_r - 1, \\
\varphi^\top(\theta_{N_r,2}; n)\alpha_{N_r} &= 0.
\end{aligned} \tag{2.37}$$

Defining $\alpha := [\alpha_2^\top, \dots, \alpha_{N_r}^\top]^\top \in \mathbb{R}^{N(N_r-1)}$ as the augmented vector of unknown coefficients, we put the equations in (2.37) into the following matrix form:

$$A\alpha = b, \tag{2.38}$$

where $A \in \mathbb{R}^{2N_r \times N(N_r-1)}$ and $b \in \mathbb{R}^{2N_r}$.

2.3.3 Mode shape calculation

Similar to the procedure used in Section 2.2.5, we minimize the total potential energy

$$\mathcal{V}_{susp} = \sum_{i=2}^{N_r} \frac{EI_i}{2R_i^3} \int_0^{2\pi} (f_i(\theta; n) + f_i''(\theta; n))^2 d\theta, \tag{2.39}$$

corresponding to the strain energy stored in the suspension structure due to the n -th mode shapes. Substituting (2.33) into (2.39) results in the following least-squares optimization problem:

$$\alpha^* := \arg \min_{\alpha \in \mathbb{R}^{N(N_r-1)}} \mathcal{V}_{susp} \approx \frac{1}{2} \alpha^\top P_k \alpha \quad \text{subject to (2.38)}, \tag{2.40}$$

where $\alpha^* \in \mathbb{R}^{N(N_r-1)}$ is the vector of optimal coefficients and

$$\begin{aligned} P_k &= \text{diag} \left(\frac{EI_i}{R_i^3} \right)_{i=2}^{N_r} \otimes \Pi_k, \\ \Pi_k &= \int_0^{2\pi} (\phi(\theta; n) + \phi''(\theta; n)) (\phi(\theta; n) + \phi''(\theta; n))^\top d\theta \\ &= \pi \text{diag} \left((1 - (2j-1)^2 n^2)^2 \right)_{j=1}^N. \end{aligned} \quad (2.41)$$

To ensure that we not only satisfy the boundary conditions but also minimize the potential function, we select the number of basis functions such that $N > \lfloor \frac{2N_r}{N_r-1} \rfloor$ to render the system (2.38) underdetermined. We note that problem (2.40) can be solved accurately and efficiently using MATLAB's `lsqr` command.

2.3.4 Modal properties

Upon obtaining α^* , we calculate the modal properties:

1. Effective mass:

$$\begin{aligned} m &= \sum_{i=1}^{N_r} \rho R_i A_i \int_0^{2\pi} \left\{ f^2(\theta; n) + \left(\int_0^{2\pi} f_i(\tau; n) d\tau \right)^2 \right\} d\theta \\ &= \pi \rho R_1 A_1 \frac{n^2 + 1}{n^2} + \alpha^{*\top} P_m \alpha^*, \end{aligned} \quad (2.42)$$

$$P_m = \text{diag} (\rho R_i A_i)_{i=2}^{N_r} \otimes \Pi_m,$$

$$\Pi_m = \int_0^{2\pi} \phi(\theta; n) \phi^\top(\theta; n) + \varphi(\theta; n) \varphi^\top(\theta; n) d\theta = \pi \text{diag} \left(1 + \frac{1}{(2j-1)^2 n^2} \right)_{j=1}^N.$$

2. Coriolis mass:

$$\begin{aligned} m_c &= \sum_{i=1}^{N_r} \rho R_i A_i \int_0^{2\pi} \int_0^\theta \{ f_i(\tau; n) g_i(\theta; n) - f_i(\theta; n) g_i(\tau; n) \} d\tau d\theta \\ &= 2\pi \rho R_1 A_1 \frac{1}{n} + \alpha^{*\top} P_c \alpha^*, \end{aligned} \quad (2.43)$$

$$P_c = \text{diag} (\rho R_i A_i)_{i=2}^{N_r} \otimes \Pi_c,$$

$$\Pi_c = \int_0^{2\pi} \varphi(\theta; n) \phi^\top(\theta - \frac{\pi}{2n}; n) - \phi(\theta; n) \varphi^\top(\theta - \frac{\pi}{2n}; n) d\theta = \frac{2\pi}{n} \text{diag} \left(\frac{(-1)^{j+1}}{2j-1} \right)_{j=1}^N.$$

3. Effective stiffness:

$$k = \sum_{i=2}^{N_r} \frac{EI_i}{R_i^3} \int_0^{2\pi} (f_i(\theta; n) + f_i''(\theta; n))^2 d\theta = \frac{\pi EI_1}{R_1^3} (n^2 - 1)^2 + \alpha^{*\top} P_k \alpha^*. \quad (2.44)$$

2.4 Equations of Motion

The total potential energy of the gyroscope, combining the mechanical and electrostatic parts, is

$$\mathcal{V}(Q) = \mathcal{V}_m(Q) - \mathcal{V}_e(Q). \quad (2.45)$$

Now, the equations of motion are given by the Lagrange's equation [47]

$$\frac{d}{dt} \left(\frac{\partial \mathcal{T}(\dot{Q}, Q)}{\partial \dot{Q}} \right) - \frac{\partial \mathcal{T}(\dot{Q}, Q)}{\partial Q} + \frac{\partial \mathcal{F}(\dot{Q})}{\partial \dot{Q}} + \frac{\partial \mathcal{V}(Q)}{\partial Q} = 0 \quad (2.46)$$

that results in

$$M\ddot{Q} + D\dot{Q} + (K - \Omega_z^2 M) Q = F_e + 2m_c \Omega_z S_c \dot{Q} + \Omega_z^2 f_{Ce}, \quad (2.47)$$

where $K = K_m - K_e$ is the total stiffness matrix.

2.5 Eigenvalues and Eigenvectors

We now study the modal properties associated with the equations of motion (2.47). For this purpose, we consider the undamped, zero input rate $\Omega_z = 0$, free vibration of the VRG governed by the equation

$$M\ddot{Q} + KQ = 0. \quad (2.48)$$

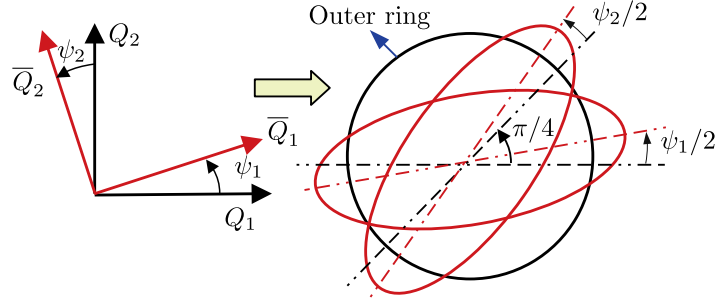


Figure 2.7: Principal coordinate system and rotated mode shapes for $n = 2$.

Substituting $Q(t) = \eta e^{j\omega t}$ into (2.48) results in the following generalized eigenvalue problem:

$$K\eta_i = \omega_i^2 M\eta_i, \quad i = 1 : 2, \quad (2.49)$$

where ω_i^2 are the eigenvalues and $\eta_i \in \mathbb{R}^2$ are the eigenvectors. Defining $A := M^{-1}K$ as the *dynamic matrix*, we have the following results:

1. The eigenvalues give us the two natural frequencies $\omega_{1,2}$ associated with the n -th vibration mode; that is,

$$\omega_{1,2} = \sqrt{\frac{\text{tr}(A)}{2} \left(1 \pm \sqrt{1 - 4 \frac{\det(A)}{\text{tr}^2(A)}} \right)}, \quad (2.50)$$

where the trace and the determinant of A are given by $\text{tr}(A) = a_{11} + a_{22}$ and $\det(A) = a_{11}a_{22} - a_{12}a_{21}$, respectively. According to (2.50), we obtain the frequency split as

$$\Delta\omega = |\omega_1 - \omega_2| = \sqrt{\text{tr}(A) - 2\sqrt{\det(A)}}. \quad (2.51)$$

2. The eigenvectors η_1, η_2 determine the orientation of the true mode shapes in the two-dimensional plane of the generalized coordinates, $(Q_1, Q_2) \in \mathbb{R}^2$. Because of the structural imperfection, the mass and stiffness matrices deviate from their nominal counterparts which are scalar multiples of the identity matrix. This, in turn, results in the rotation of the mode shapes with respect to $\{\cos(n\theta), \sin(n\theta)\}$ as shown in Figure w. To explain this point, assume that η_1 and η_2 are normalized, ψ_1 is the angle between the direction of η_1 and the Q_1 -axis, and ψ_2 is the angle

between η_2 's direction and the Q_2 -axis. Let $T := [\eta_1, \eta_2] \in \mathbb{R}^{2 \times 2}$; this matrix can be written as

$$T = \begin{bmatrix} \cos(\psi_1) & -\sin(\psi_2) \\ \sin(\psi_1) & \cos(\psi_2) \end{bmatrix}. \quad (2.52)$$

Let $(\bar{Q}_1, \bar{Q}_2) \in \mathbb{R}^2$ be the principal coordinate system whose directions are defined by the eigenvectors. The linear transformation

$$Q = T\bar{Q} \quad (2.53)$$

defines the generalized coordinates vector $\bar{Q} = [\bar{Q}_1, \bar{Q}_2] \in \mathbb{R}^2$ that corresponds to the orientations of the true mode shapes. This situation has been graphically illustrated in Figure 2.7.

Remark 2.4. The eigenvectors η_1 and η_2 are orthogonal with respect to the mass and stiffness matrices [48]; that is,

$$\eta_i^\top M \eta_j = \bar{m}_i \delta_{i,j}, \quad \eta_i^\top K \eta_j = \bar{k}_i \delta_{i,j}, \quad (2.54)$$

where $\bar{m}_i$ and $\bar{k}_i$ are the modal mass and stiffness corresponding to the i -th eigenvector, respectively, and $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. In general, the eigenvectors are not orthogonal to each other, i.e., $\psi_1 \neq \psi_2$. However, if the dynamic matrix A is symmetric (for example, when the mass matrix is a scalar multiple of the identity matrix), then we have orthogonal eigenvectors and $\psi_1 = \psi_2$ [48]. In the latter case, T is a unitary matrix that defines a transformation between the two Cartesian coordinate systems (Q_1, Q_2) and $(\bar{Q}_1, \bar{Q}_2)$.

Transforming the assumed modes expression (2.7) into the principal coordinate system gives us

$$u_r(\theta, t) = \bar{Q}_1(t) \cos(n\theta - \psi_1) + \bar{Q}_2(t) \sin(n\theta - \psi_2). \quad (2.55)$$

Thereby, the nominal mode shapes $\cos(n\theta)$ and $\sin(n\theta)$ are rotated by the angles $\frac{\psi_1}{n}$ and $\frac{\psi_2}{n}$, respectively. Because of the linear behavior of the inner suspension, the mode shapes of the inner rings rotate by $\frac{\psi_{1,2}}{n}$ as well. In view of Remark 2.4, both mode shapes rotate with the same angle if the mass matrix has matched diagonal entries and zero off-diagonal elements.

3. Using the transformation (2.53) and by premultiplying by $T^\top$, we transform the equations of motion (2.47) into the principal coordinate system:

$$\bar{M}\ddot{\bar{Q}} + \bar{D}\dot{\bar{Q}} + \bar{K}\bar{Q} = \bar{F}_e + \Omega_z^2 \bar{f}_{Ce} + 2\bar{m}_c \Omega_z S_c \dot{\bar{Q}} + \Omega_z^2 \bar{M}\bar{Q}, \quad (2.56)$$

where

$$\begin{aligned} \bar{M} &= T^\top M T = \begin{bmatrix} \bar{m}_1 & 0 \\ 0 & \bar{m}_2 \end{bmatrix}, \\ \bar{D} &= T^\top D T = \begin{bmatrix} \bar{d}_{11} & \bar{d}_{12} \\ \bar{d}_{12} & \bar{d}_{22} \end{bmatrix}, \\ \bar{K} &= T^\top K T = \begin{bmatrix} \bar{k}_1 & 0 \\ 0 & \bar{k}_2 \end{bmatrix}, \\ \bar{m}_c &= m_c \cos(\psi_1 - \psi_2), \\ \bar{F} &= T^\top F = \begin{bmatrix} f_1 \cos(\psi_1) + f_2 \sin(\psi_1) \\ -f_1 \sin(\psi_2) + f_2 \cos(\psi_2) \end{bmatrix}, \\ \bar{f}_{Ce} &= T^\top f_{Ce}. \end{aligned}$$

We note that the diagonal forms of $\bar{M}$ and $\bar{K}$ follow from the orthogonality property (2.54). However, the damping matrix $\bar{D}$ in the principal coordinate systems is not necessarily diagonal.

2.6 Fourier Analysis of Imperfections

Mathematically, any imperfection—whether geometric or stemming from material anisotropy—is a 2π -periodic continuous function of θ . Hence, we can invoke the Fourier series to analyze the effects of imperfections, which—as shown in the following—leads to interesting yet important observations. We define the Fourier series coefficient of a 2π -periodic, continuous function $f(\theta) \in \mathbb{R}$ as

$$\begin{aligned} a_\ell(f(\theta)) &:= \int_0^{2\pi} f(\theta) \cos(\ell\theta) d\theta, \\ b_\ell(f(\theta)) &:= \int_0^{2\pi} f(\theta) \sin(\ell\theta) d\theta, \end{aligned} \quad (2.57)$$

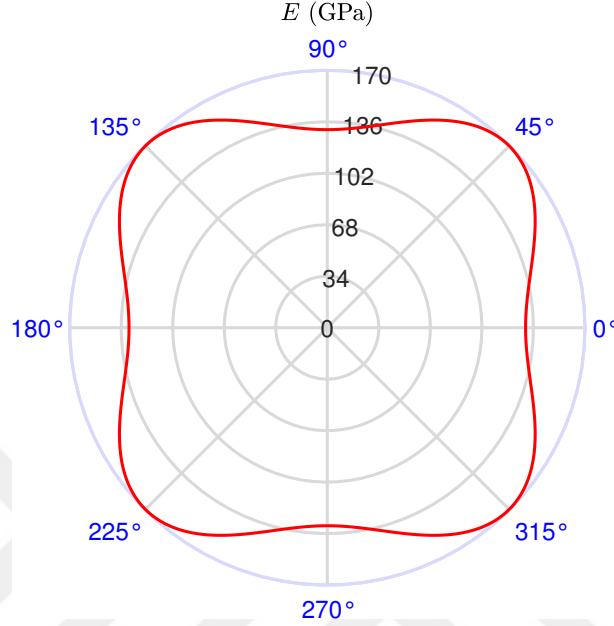


Figure 2.8: (100) silicon's Young's modulus (GPa) as a function of the polar angle (deg).

where $\ell = 0, 1, 2, \dots$. By comparing (2.57) with (2.19b) and (2.22b), we can rewrite the mass and stiffness matrices of the outer ring as

$$M_1 = \rho R_1 \left(\frac{n^2 + 1}{2n^2} \right) a_0(A_1(\theta)) I_{2 \times 2} + \rho R_1 \left(\frac{n^2 - 1}{2n^2} \right) \begin{bmatrix} a_{2n}(A_1(\theta)) & b_{2n}(A_1(\theta)) \\ b_{2n}(A_1(\theta)) & -a_{2n}(A_1(\theta)) \end{bmatrix}, \quad (2.58a)$$

$$K_1 = \frac{(n^2 - 1)^2}{2R_1^3} a_0(E(\theta)I_1(\theta)) I_{2 \times 2} + \frac{(n^2 - 1)^2}{2R_1^3} \begin{bmatrix} a_{2n}(E(\theta)I_1(\theta)) & b_{2n}(E(\theta)I_1(\theta)) \\ b_{2n}(E(\theta)I_1(\theta)) & -a_{2n}(E(\theta)I_1(\theta)) \end{bmatrix}. \quad (2.58b)$$

These equations reveal that the $2n$ -th harmonics of the cross section area $A_1(\theta)$ and the flexural rigidity $E(\theta)I_1(\theta)$ give rise to the deviation of M_1 and K_1 from their nominal counterparts, respectively (note that for constant A_1 and EI_1 $a_0(\cdot)$ is the only nonzero harmonic). We give two examples to demonstrate the application of Fourier analysis:

Example 2.1. Considering the continual nature of etching processes, geometric imperfections caused by them have, qualitatively speaking, low spatial frequency

content. Therefore, the harmonics $a_{2n}(\cdot)$ and $b_{2n}(\cdot)$ get smaller as n increases. Hence, higher modes exhibit better robustness in terms of *filtering out* the effects of geometric imperfections on their mass and stiffness matrices. However, we should note that this comes at the cost of lower angular gain, which is discussed in the next section.

Example 2.2. Silicon wafers with the (111) crystalline plane direction are almost in-plane isotropic [39, 40]. However, the same does not hold for (100) wafers. For the (100) plane, Young's modulus is given by

$$E(\theta) = \left(S_{11} - \left(S_{11} - S_{12} - \frac{S_{44}}{2} \right) \frac{1 - \cos(4\theta)}{4} \right)^{-1}, \quad (2.59)$$

where $S_{11} = 7.6442 \times 10^{-12} \text{ Pa}^{-1}$, $S_{12} = -2.1434 \times 10^{-12} \text{ Pa}^{-1}$, and $S_{44} = 1.2564 \times 10^{-11} \text{ Pa}^{-1}$ are the elastic compliance constants of single crystalline silicon [40] (see Figure 2.8). $E(\theta)$ is $\frac{\pi}{2}$ -periodic and admits the Fourier expansion

$$E(\theta) \approx 149.0132 - 19.3781 \cos(4\theta) + 1.2600 \cos(8\theta) - 0.0819 \cos(12\theta) \text{ (GPa)}, \quad (2.60)$$

which has the RMS approximation error of $3.8 \times 10^{-3} \text{ GPa}$ for one period. Assuming no geometric imperfections, we draw the following conclusions from (2.60):

- Since $E(\theta)$ is an even function, the off-diagonal terms in the stiffness matrix K_1 are zero. This means that the anisotropy of (100) plane contributes only to the frequency split and not to the rotation of the mode shape.
- Excluding the zeroth-order term a_0 , a_4 is the largest harmonic of $E(\theta)$. As a result, the anisotropy of (100) plane has the strongest frequency split effect on the $n = 2$ mode.

2.7 Solution of Equations of Motion

2.7.1 Ideal gyroscope

Let us consider the free vibration of an undamped ideal gyroscope without any geometric imperfections or material anisotropy. In such a case, the mass and stiffness matrices are of the form $M = mI_{2 \times 2}$ and $K = kI_{2 \times 2}$, respectively, and equations of motion (2.47) (or (2.56) because $\bar{Q} = Q$) reduce to

$$\ddot{Q} + (\omega_n^2 - \Omega_z^2) Q = 2\mu\Omega_z S_c \dot{Q}, \quad (2.61)$$

where $\omega_n = \sqrt{k/m}$ is the identical natural frequency of both modes and $\mu := m_c/m$. We assume that the gyroscope initially has a displacement of p_0 along the $\cos(n\theta)$ mode shape with zero absolute velocity. Introducing the coordinate transformation

$$X = \begin{bmatrix} \cos(\mu\Omega_z t) & -\sin(\mu\Omega_z t) \\ \sin(\mu\Omega_z t) & \cos(\mu\Omega_z t) \end{bmatrix} Q, \quad (2.62)$$

we obtain

$$\ddot{X} + (\omega_n^2 - \Omega_z^2(1 - \mu^2)) X = 0, \quad (2.63)$$

which, for $t \geq 0$, has the solution

$$X(t) = p_0 \cos(\tilde{\omega}_n t) [1, 0]^\top, \quad (2.64)$$

where $\tilde{\omega}_n := \sqrt{\omega_n^2 - \Omega_z^2(1 - \mu^2)}$. Now, transforming back to the Q -coordinates, we have

$$Q(t) = p_0 \cos(\tilde{\omega}_n t) [\cos(\mu\Omega_z t), -\sin(\mu\Omega_z t)]^\top. \quad (2.65)$$

The X -coordinates defined by (2.62) describe the gyroscope's motion in the inertial reference frame with respect to which rotation of the rate Ω_z takes place. The Coriolis acceleration appears only in the gyro-fixed frame; hence, transforming into the X -coordinates removes the Coriolis coupling [56].

2.7.1.1 Angular gain

In the absence of rotation, i.e., $\Omega_z = 0$, the gyroscope's oscillation is along the $\cos(n\theta)$ mode shape, and the corresponding trajectory in the (p_n, q_n) -plane is a straight line on the p_n -axis. In the presence of rotation, this line rotates as well, and at any time $t \geq 0$, its slope is given by

$$\frac{q_n(t)}{p_n(t)} = -\tan(\mu\Omega_z t). \quad (2.66)$$

This equation shows that for every 2π rotation of the gyroscope, its oscillation trajectory rotates by $2\pi\mu$ in the (p_n, q_n) -plane or equivalently by $2\pi\mu/n$ in the plane of the mode shapes. That is, the oscillation trajectory of the gyroscope follows the applied rotation angle $\Omega_z t$ multiplied by the gain factor

$$A_g := \frac{1}{n} \frac{m_c}{m}. \quad (2.67)$$

This parameter, known as the angular gain [19], reflects the mass distribution efficiency in Coriolis-based momentum transfer between the gyroscope's two modes. The radial displacement of the outer ring is given by

$$u_r(\theta, t) = p_0 \cos(\tilde{\omega}_n t) \cos(n(\theta + A_g \Omega_z t)), \quad (2.68)$$

which accentuates the role of A_g in the rotation of the oscillation pattern in response to the applied rotation rate, Ω_z .

2.7.2 Nonideal gyroscope

Examining nonidealities in the context of equations of motion is the key to the quantitative understanding of their effects on the gyroscope's performance. In the literature, there are studies of nonideal MEMS gyroscopes via averaging in the orbital coordinates [57] or method of decomposition [58] that attempt to separate the slow and fast dynamics associated with the natural vibration and input rate, respectively. Here, we adopt a perturbation-based method to derive the nonideal gyroscope's transfer matrix. First, we observe from (2.56) that,

in the principal coordinates, the coupling between the modes is due to angular velocity and damping asymmetry. We then rewrite (2.56) in the following form:

$$\bar{M}\ddot{\bar{Q}} + \bar{D}_0\dot{\bar{Q}} + \bar{K}\bar{Q} = \bar{F}_e + \Omega_z^2 \bar{f}_{Ce} + 2\bar{m}_c \Omega_z S_c \dot{\bar{Q}} + \Omega_z^2 \bar{M}\bar{Q} - \bar{d}_{12} S_d \dot{\bar{Q}}, \quad (2.69)$$

where

$$\bar{D}_0 = \begin{bmatrix} \bar{d}_{11} & 0 \\ 0 & \bar{d}_{22} \end{bmatrix}, \quad (2.70a)$$

$$S_d = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (2.70b)$$

We note that, in the case of a nonideal gyroscope, the transformation (2.62) cannot be used to decouple the equations of motion because the modes, in general, do not share the same modal mass, stiffness, and damping coefficient. Hence, we resort to a *multi-parameter perturbation analysis*. We note that in practical gyroscopes Ω_z is orders of magnitude smaller than the natural frequencies, and $\bar{d}_{12}$ is also a small parameter due to quality factors in the orders of 10^3 . Therefore, using the perturbation analysis is justified. In that respect, we consider the solution $\bar{Q}(t)$, $t \geq 0$, of (2.69) to be of the form

$$\bar{Q}(t) = \sum_{i,j=0}^{\infty} \bar{Q}_{i,j}(t) \frac{\Omega_z^i \bar{d}_{12}^j}{i!j!} \quad (2.71)$$

which is the Taylor expansion of $\bar{Q}(t)$ in terms of the coupling parameters Ω_z and $\bar{d}_{12}$. Plugging (2.71) into (2.69) results in

$$\bar{M}\ddot{\bar{Q}}_{0,0} + \bar{D}_0\dot{\bar{Q}}_{0,0} + \bar{K}\bar{Q}_{0,0} = \bar{F}_e, \quad (2.72a)$$

$$\bar{M}\ddot{\bar{Q}}_{1,0} + \bar{D}_0\dot{\bar{Q}}_{1,0} + \bar{K}\bar{Q}_{1,0} = 2\bar{m}_c S_c \dot{\bar{Q}}_{0,0}, \quad (2.72b)$$

$$\bar{M}\ddot{\bar{Q}}_{0,1} + \bar{D}_0\dot{\bar{Q}}_{0,1} + \bar{K}\bar{Q}_{0,1} = -S_d \dot{\bar{Q}}_{0,0}, \quad (2.72c)$$

$$\bar{M}\ddot{\bar{Q}}_{2,0} + \bar{D}_0\dot{\bar{Q}}_{2,0} + \bar{K}\bar{Q}_{2,0} = \bar{f}_{Ce} + 4\bar{m}_c S_c \dot{\bar{Q}}_{1,0} + 2\bar{M}\bar{Q}_{0,0}, \quad (2.72d)$$

$$\begin{aligned} \bar{M}\ddot{\bar{Q}}_{i,j} + \bar{D}_0\dot{\bar{Q}}_{i,j} + \bar{K}\bar{Q}_{i,j} &= 2i\bar{m}_c S_c \dot{\bar{Q}}_{i-1,j}, \\ &+ i(i-1)\bar{M}\bar{Q}_{i-2,j} - jS_d \dot{\bar{Q}}_{i,j-1}, \text{ for } i \geq 2, j \geq 1. \end{aligned} \quad (2.72e)$$

Remark 2.5. The main advantage of the expansion (2.71) is that it decouples the effects of Ω_z and $\bar{d}_{12}$. From (2.72a), we first find the response $\bar{Q}_0(t)$ to the excitation forces. Then, by using $\bar{Q}_0(t)$, we obtain the responses associated with Ω_z and $\bar{d}_{12}$ from (2.72b) and (2.72c), respectively.

Define the transfer function

$$\bar{H}(s) = \begin{bmatrix} \bar{H}_1(s) & 0 \\ 0 & \bar{H}_2(s) \end{bmatrix}, \quad (2.73)$$

where s is the Laplace variable and

$$\begin{aligned} \bar{H}_1(s) &= \frac{1}{\bar{m}_1 s^2 + \bar{d}_{11} s + \bar{k}_1}, \\ \bar{H}_2(s) &= \frac{1}{\bar{m}_2 s^2 + \bar{d}_{22} s + \bar{k}_2}. \end{aligned} \quad (2.74)$$

Taking the Laplace transform of (2.72a)-(2.72c) and assuming zero initial values, we obtain

$$\begin{aligned} \bar{Q}_{0,0}(s) &= \bar{H}(s) \bar{F}_e(s), \\ \bar{Q}_{1,0}(s) &= 2s\bar{m}_c \bar{H}(s) S_c \bar{Q}_{0,0}(s), \\ \bar{Q}_{0,1}(s) &= -s\bar{H}(s) S_d \bar{Q}_{0,0}(s), \\ \bar{Q}_{2,0}(s) &= \bar{H}(s) \bar{f}_{Ce} + 4m_c s \bar{H}(s) S_c \bar{Q}_{1,0} + 2\bar{H}(s) \bar{M} \bar{Q}_{0,0}, \end{aligned} \quad (2.75)$$

that results in

$$\begin{aligned} \bar{Q}(s) &\approx \bar{H}(s) \bar{F}_e(s) - s\bar{H}_1(s) \bar{H}_2(s) \begin{bmatrix} 0 & -(2\bar{m}_c \Omega_z - \bar{d}_{12}) \\ (2\bar{m}_c \Omega_z + \bar{d}_{12}) & 0 \end{bmatrix} \bar{F}_e(s) \\ &+ \frac{1}{2} \Omega_z^2 \bar{H}(s) \bar{f}_{Ce} \\ &+ \Omega_z^2 \begin{bmatrix} \bar{m}_1 \bar{H}_1^2(s) \left(1 - 4\bar{m}_c \left(\frac{\bar{m}_c}{\bar{m}_1}\right) s^2 \bar{H}_2(s)\right) & 0 \\ 0 & \bar{m}_2 \bar{H}_2^2(s) (1 - 4\bar{m}_c s^2 \bar{H}_1(s)) \end{bmatrix} \bar{F}_e(s). \end{aligned} \quad (2.76)$$

2.7.2.1 Force-displacement transfer matrix

Using (2.53), substituting $\bar{F} = T^\top F$, and neglecting Ω_z^2 terms, we obtain the force-displacement transfer matrix $H(s)$, i.e., $Q(s) \approx H(s)F(s)$, as

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix}, \quad (2.77)$$

$$\begin{aligned}
H_{11}(s) &= \bar{H}_1(s) \left(\frac{1 + \cos(2\psi_1)}{2} \right) + \bar{H}_2(s) \left(\frac{1 - \cos(2\psi_2)}{2} \right) \\
&\quad + 2s\bar{H}_1(s)\bar{H}_2(s)\bar{d}_{12} \cos(\psi_1) \sin(\psi_2), \\
H_{22}(s) &= \bar{H}_1(s) \left(\frac{1 - \cos(2\psi_1)}{2} \right) + \bar{H}_2(s) \left(\frac{1 + \cos(2\psi_2)}{2} \right) \\
&\quad - 2s\bar{H}_1(s)\bar{H}_2(s)\bar{d}_{12} \cos(\psi_2) \sin(\psi_1), \\
H_{12}(s) &= \frac{\bar{H}_1(s) \sin(2\psi_1) - \bar{H}_2(s) \sin(2\psi_2)}{2} - s\bar{H}_1(s)\bar{H}_2(s)\bar{d}_{12} \cos(\psi_1 + \psi_2) \\
&\quad + 2\bar{m}_c\Omega_z s\bar{H}_1(s)\bar{H}_2(s) \cos(\psi_1 - \psi_2), \\
H_{21}(s) &= \frac{\bar{H}_1(s) \sin(2\psi_1) - \bar{H}_2(s) \sin(2\psi_2)}{2} - s\bar{H}_1(s)\bar{H}_2(s)\bar{d}_{12} \cos(\psi_1 + \psi_2) \\
&\quad - 2\bar{m}_c\Omega_z s\bar{H}_1(s)\bar{H}_2(s) \cos(\psi_1 - \psi_2).
\end{aligned}$$

$H_{11}(s)$ and $H_{21}(s)$ are the transfer functions from the nominal drive to the nominal drive and sense axes, respectively; similarly, $H_{12}(s)$ and $H_{22}(s)$ are the transfer functions from the nominal sense to the nominal drive and sense, respectively.

2.7.2.2 Estimating mode shapes' rotation angle

Using the coordinate transformation (2.53) and the solution (2.76) of equations of motion, we present a new method to estimate the mode shapes rotation angles of a fabricated gyroscope. We assume that the frequency split between the modes is much larger than their bandwidths. Based on the transfer matrix (2.77), the method consists of the following steps: 1) in the zero-rate condition, drive from the nominal $\cos(n\theta)$ mode and record the frequency responses $H_{11}(j\omega)$ and $H_{21}(j\omega)$; 2) similarly, drive from the nominal $\sin(n\theta)$ mode and record $H_{12}(j\omega)$ and $H_{22}(j\omega)$; 3) using the recorded frequency responses, calculate

$$\tan(2\psi_{1(2)}) \approx \frac{\text{Im}(H_{12}(j\omega_{1(2)}) + H_{21}(j\omega_{1(2)}))}{\text{Im}(H_{11}(j\omega_{1(2)}) - H_{22}(j\omega_{1(2)}))}. \quad (2.78)$$

2.8 Quadrature and In-phase Errors

A salient outcome of our modeling is that it enables analysis of quadrature and in-phase errors associated with frequency split and rotation of the mode shapes. As explained in the previous sections, in an ideal gyroscope, the two $n\theta$ -mode shapes share the same effective mass and stiffness. In turn, the mode shapes have a matched frequency ($\omega_1 = \omega_2$) and are in alignment with the designated drive $\theta = 0$ and sense $\theta = \frac{\pi}{2n}$ axes. Fabrication-induced nonuniformities alter the ideal mass and stiffness distribution across the mechanical structure, and environmental variations such as stress and temperature perturb the stiffness. These factors manifest in the M and K matrices as mismatches between the diagonal entries and nonzero off-diagonal coupling terms. Therefore, in a nonideal gyroscope, we could have a frequency split between the mode shapes, and their orientation could deviate from the nominal drive and sense axes. Referring back to Figure 2.7, the modal space of the nonideal gyroscope comprises the rotated mode shapes $\cos(n\theta - \psi_1)$ and $\sin(n\theta - \psi_1)$. In the following, we examine the effects of such nonidealities on the gyroscope's output.

We rewrite the force-displacement transfer functions (2.74) as

$$\bar{H}_i(s) = \frac{1}{\bar{m}_i} \frac{1}{s^2 + BW_i s + \omega_i^2}, \quad (2.79)$$

where $BW_i = \omega_i/q_{f_i}$ and q_{f_i} , $i = 1, 2$, are the bandwidths and quality factors, respectively. In our gyroscopes, a phase-locked loop (PLL) uses the outputs of the drive pick electrodes located at 90° and 270° as the feedback variables and generates an AC force of the form $F(t) = F_0 \cos(\omega_1 t)$ along the $\theta = 0$ axis via the drive electrodes at 0° and 180° . The sense electrodes located at 135° and 315° pick up the response $Q_2(t)$ along the $\theta = 135^\circ$ axis which then is demodulated (see Figure 2.5 for the locations of electrodes). The rotation angles ψ_1 and ψ_2 of the mode shapes disturb this drive-sensing apparatus. The driving force actuates both modes, and the picked up signal $Q_2(t)$ is a mixture of the responses of both modes. Under the drive signal $F(t) = F_0 \cos(\omega_1 t)$, the transfer functions (2.79)

reduce to

$$\begin{aligned}\bar{H}_1(j\omega_1) &= \frac{1}{\bar{m}_1\omega_1 BW_1} \angle -90^\circ, \\ \bar{H}_2(j\omega_1) &\approx \frac{1}{2\bar{m}_2\omega_1|\Delta\omega|\sqrt{1 + \left(\frac{BW_2}{2\Delta\omega}\right)^2}} \angle -\beta_2,\end{aligned}\tag{2.80}$$

where j denotes the imaginary unit, $\Delta\omega = \omega_1 - \omega_2$ is the frequency split, and β_2 is the phase of $H_2(j\omega_1)$. The nonideal transfer function (2.77) gives us

$$Q_2(t) \approx \mathcal{A}F_0 \cos(\omega_1 t - \varphi) + \mathcal{B}F_0 \sin(\omega_1 t - \varphi),\tag{2.81}$$

where φ_{dem} is the demodulation angle and

$$\begin{aligned}\mathcal{A} &= A \sin(\varphi_{\text{dem}}) - (B - SF^{-1}(\Omega_z + \Omega_{z,d})) \cos(\beta_2 - \varphi_{\text{dem}}), \\ \mathcal{B} &= A \cos(\varphi_{\text{dem}}) - (B - SF^{-1}(\Omega_z + \Omega_{z,d})) \sin(\beta_2 - \varphi_{\text{dem}}), \\ A &= \frac{1}{2} |\bar{H}_1(j\omega_1)| \sin(2\psi_1), \\ B &= \frac{1}{2} |\bar{H}_2(j\omega_1)| \sin(2\psi_2), \\ SF &= \frac{-1}{2m_c\omega_1 |\bar{H}_1(j\omega_1)| |\bar{H}_2(j\omega_1)| \cos^2(\psi_1 - \psi_2)}, \\ \Omega_{z,d} &= \frac{\bar{d}_{12} \cos(\psi_1 + \psi_2)}{2m_c \cos^2(\psi_1 - \psi_2)}.\end{aligned}\tag{2.82}$$

We note that SF is the nominal mechanical scale factor, and $\Omega_{z,d}$ is the equivalent rotation rate of the damping coupling. The demodulation angle φ_{dem} is selected to compensate for the nonideal phase of the sense mode caused by electronics. Based on (2.81), we analyze the gyroscope's errors in two operation scenarios: mode-matched and mode-mismatched.

- *Mode-matched:* $\Delta\omega \approx 0$ and $\beta_2 - \varphi_{\text{dem}} \approx \frac{\pi}{2}$. The Coriolis response lies in the $\sin(\omega_1 t - \varphi_{\text{dem}})$ component of $Q_2(t)$. Multiplying Equation (2.81) by $\frac{SF}{F_0 \sin(\beta_2 - \varphi_{\text{dem}})}$ gives us the quadrature $\Omega_{z,q}$ and in-phase $\Omega_{z,in}$ errors as

$$\begin{aligned}
\Omega_{z,q} &= \left(\frac{\bar{m}_1}{4m_c} \right) \frac{\sin(2\psi_2)}{\cos^2(\psi_1 - \psi_2) \tan(\beta_2 - \varphi_{\text{dem}})} BW_1 \\
&\quad - \left(\frac{\bar{m}_2}{4m_c} \right) \frac{\sin(2\psi_1) \sin(\varphi_{\text{dem}})}{\cos^2(\psi_1 - \psi_2) \sin(\beta_2 - \varphi_{\text{dem}})} BW_2 + \frac{\Omega_{z,d}}{\tan(\beta_2 - \varphi_{\text{dem}})}, \\
\Omega_{z,in} &= \left(\frac{\bar{m}_1}{4m_c} \right) \frac{\sin(2\psi_2)}{\cos^2(\psi_1 - \psi_2)} BW_1 \\
&\quad - \left(\frac{\bar{m}_2}{4m_c} \right) \frac{\sin(2\psi_1) \cos(\varphi_{\text{dem}})}{\cos^2(\psi_1 - \psi_2) \sin(\beta_2 - \varphi_{\text{dem}})} BW_2 + \Omega_{z,d}.
\end{aligned} \tag{2.83}$$

- *Mode-mismatched:* $\Delta\omega \neq 0$, $\Delta\omega \gg BW_2$, and $\beta_2 - \varphi_{\text{dem}} \approx \pi$. The Coriolis response is carried by the $\cos(\omega_1 t - \varphi_{\text{dem}})$ component of $Q_2(t)$. Multiplying Equation (2.81) by $\frac{SF}{F_0 \cos(\beta_2 - \varphi_{\text{dem}})}$ gives us

$$\begin{aligned}
\Omega_{z,q} &= - \left(\frac{\bar{m}_2}{2m_c} \right) \frac{\sin(2\psi_1) \cos(\varphi_{\text{dem}})}{\cos^2(\psi_1 - \psi_2) \cos(\beta_2 - \varphi_{\text{dem}})} |\Delta\omega| \\
&\quad + \left(\frac{\bar{m}_1}{4m_c} \right) \frac{\sin(2\psi_2) \tan(\beta_2 - \varphi_{\text{dem}})}{\cos^2(\psi_1 - \psi_2)} BW_1 + \Omega_{z,d} \tan(\beta_2 - \varphi_{\text{dem}}), \\
\Omega_{z,in} &= - \left(\frac{\bar{m}_2}{2m_c} \right) \frac{\sin(2\psi_1) \sin(\varphi_{\text{dem}})}{\cos^2(\psi_1 - \psi_2) \cos(\beta_2 - \varphi_{\text{dem}})} |\Delta\omega| \\
&\quad + \left(\frac{\bar{m}_1}{4m_c} \right) \frac{\sin(2\psi_2)}{\cos^2(\psi_1 - \psi_2)} BW_1 + \Omega_{z,d}.
\end{aligned} \tag{2.84}$$

The quadrature and in-phase errors depend on mode shape rotation angles, the demodulation angle, and the sense mode's (mechanical) phase.

2.9 Simulation and Experimental Results

The flowchart of the main steps of our modeling approach along with the relevant equations is given in Figure 2.9. We implemented our model in MATLAB and also performed finite element simulations in COMSOL Multiphysics. We note that the computation of eigenvectors is very sensitive to numerical errors, which could lead to erroneous rotation angles $\psi_{1,2}$ for the mode shapes. This issue was pervasive in our FE simulations, prohibiting a correct assessment of the mode shapes' orientation. For example, we observed that even for a perfect

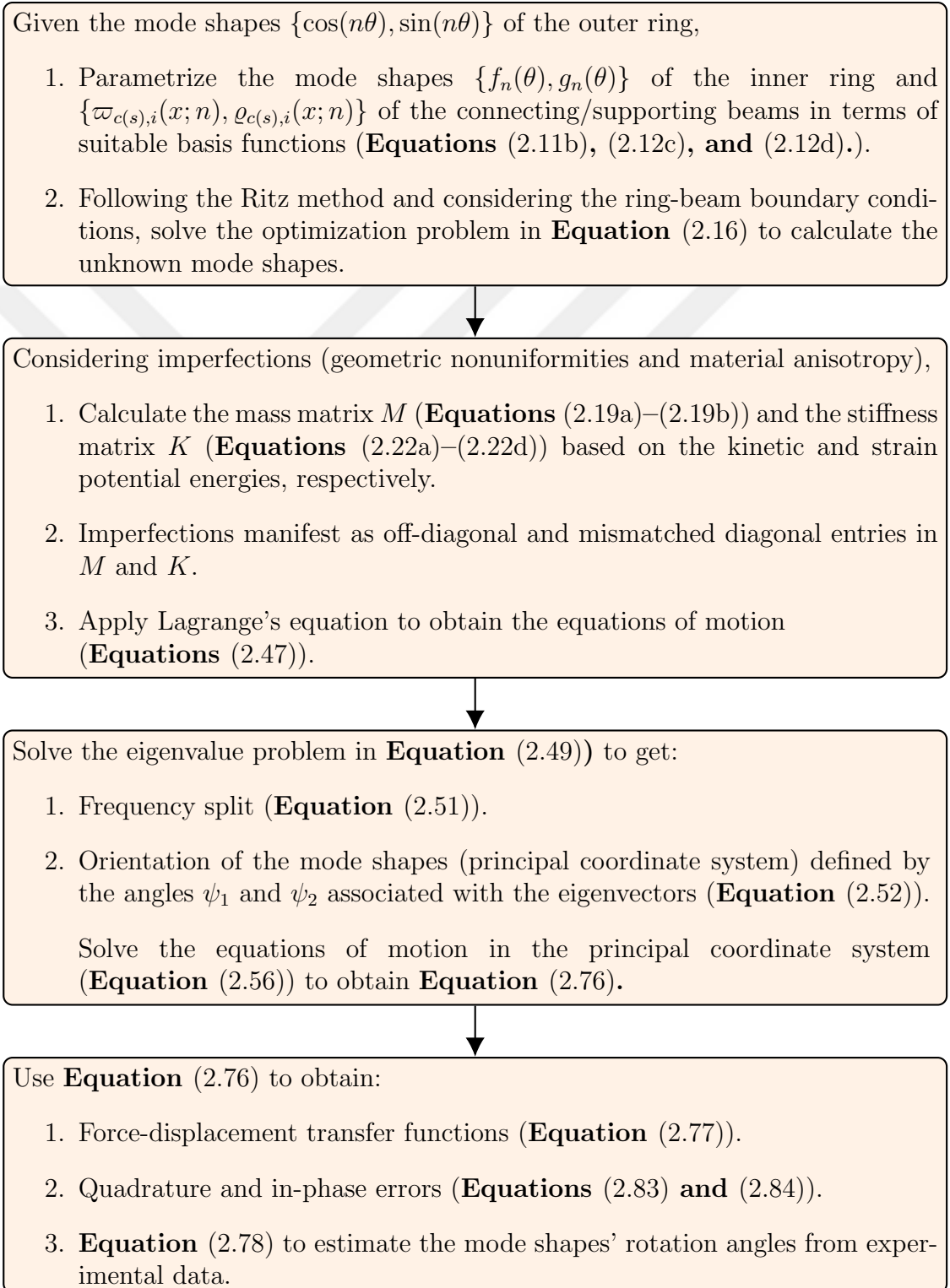


Figure 2.9: Flowchart of the modeling's main steps.

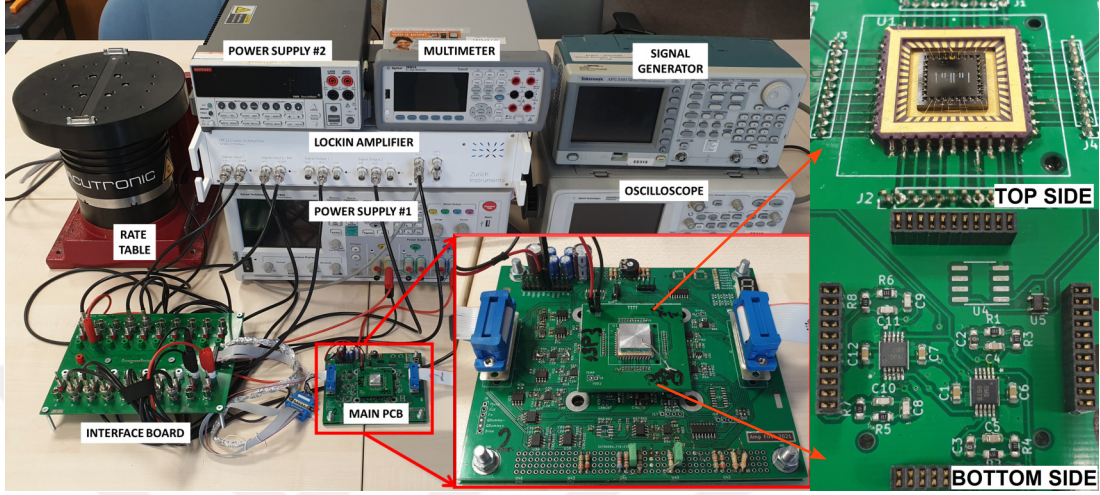


Figure 2.10: Experimental setup.

structure, the $\cos(2\theta)$ mode shape obtained from a FE simulation could exhibit a few degrees of angle lag with respect to $\theta = 0^\circ$. To address this problem, we used the four points Gauss-Legendre formula to develop an accurate numerical integration method for the Fourier-type oscillatory integrals that appear in the mass and stiffness matrices (see Appendix B.1).

2.9.1 Experimental results

Figure 2.10 shows our experimental setup with its electronics. We use Zurich Instruments HF2LI [59] to operate both drive and sense modes. Figure 2.11 depicts the measured frequency response (normalized with respect to the drive mode's maximum gain). With a 62 Hz frequency split, the resonance frequencies and quality factors of the drive ($\cos(2\theta)$) and the sense ($\sin(2\theta)$) modes are 58.7581 kHz, 12.5119×10^3 and 58.8206 kHz, 12.3581×10^3 , respectively. We measured the quadrature and in-phase error as $2200^\circ/\text{s}$ and $8^\circ/\text{s}$, respectively. Using the method of Section 2.7.2.2, we obtain the mode shapes' rotation angle as 1.9681° ($\psi_1 = 3.9362^\circ$) for $\cos(2\theta)$ and 2.2363° ($\psi_2 = 4.4726^\circ$) for $\sin(2\theta)$. Using these rotation angles and Equation (2.84), we calculate the quadrature error as $\Omega_{z,q} = 1978^\circ/\text{s}$, which is in good agreement with the observed value. Assuming a negligible cross-damping between the modes, we reconstructed the frequency

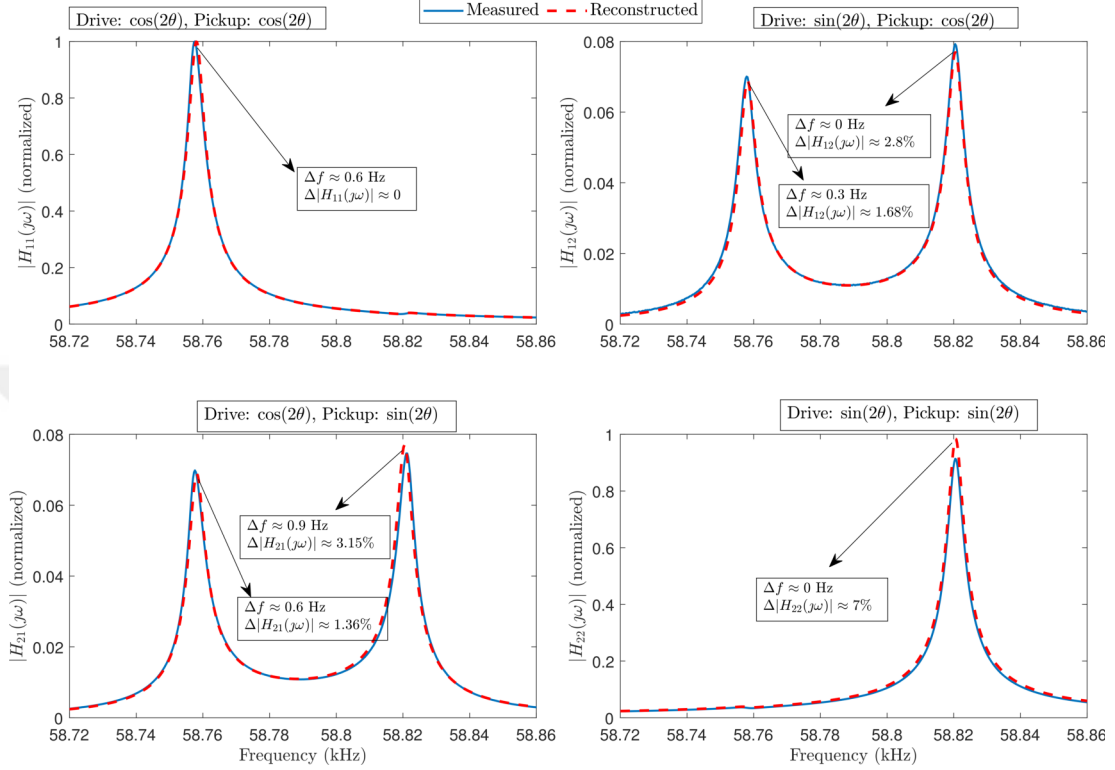


Figure 2.11: Measured and reconstructed frequency responses of the fabricated VRG.

response using the analytical expressions in (2.77) for the transfer functions. As shown in Figure 2.11, our reconstruction closely follows the experimental data, especially for the drive mode. However, we have a 7% mismatch between the peak values for the sense mode. A plausible source of this error is the gap discrepancy between the drive and sense mode electrodes that causes different actuation/sensing gains for these two modes.

2.9.2 Calculated mode shapes

We first calculate the mode shapes of the inner ring and connecting/supporting beams. Figure 2.12 shows the comparative graphs for the $n = 2$ mode; for these results, we used $N = 6$ cosine basis functions in (2.11b). Compared to the FE results, the calculated (normalized) mode shape of the inner ring has a relative error of 3% (in the root mean square (RMS) sense for one period), confirming

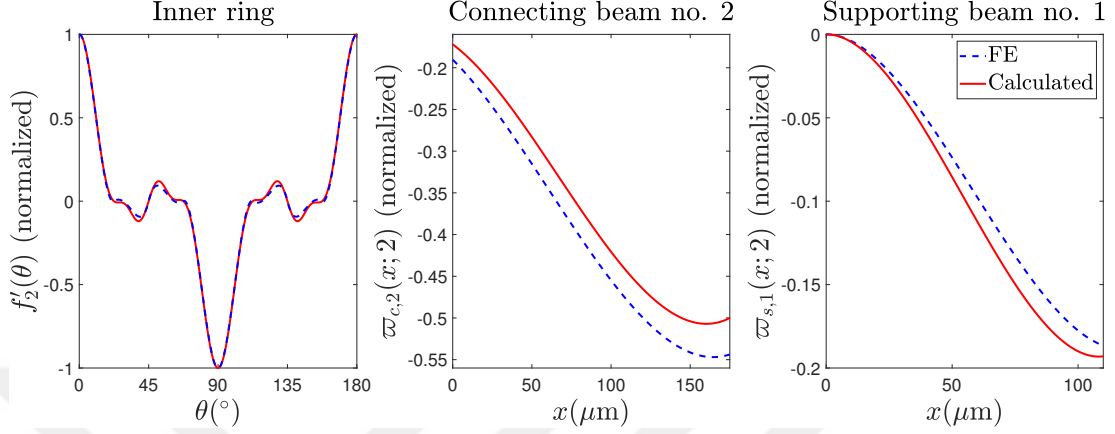


Figure 2.12: Comparison of our calculated and FE-based $n = 2$ mode shapes.

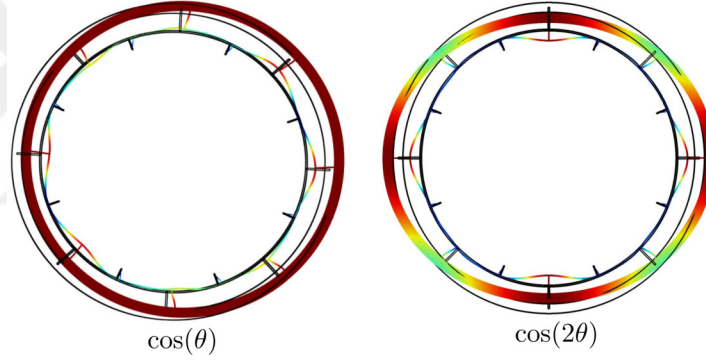


Figure 2.13: Rigid body mode $n = 1$ versus operation mode $n = 2$.

the efficacy of our method in capturing the inner ring's displacement field. The discrepancy between the calculated and the FE beam mode shapes is due to the error induced by the thin ring assumption. More explicitly, neglecting the small variations of the tangential displacement along the rings' widths gives rise to the deviations in the beams' displacement boundary conditions. The calculated natural frequencies are as follows: FE: 65.67 kHz, our method: 64.88 kHz (1.20% relative error). We point out that our model tends to give a bit larger natural frequencies because, in our design, the anchors are located at some distance from the supporting beams (see Figure 2.1), which slightly perturb their displacement and slope at $x = 0$ from the assumed fixed boundary conditions. As a confirmation, FE simulation with fixed points at the anchors gives us 62.60 kHz as the natural frequency. This error acts as a constant offset in the frequencies, but it does not affect their variation with respect to imperfections. Considering this point, along

Table 2.3: Contributions of different components to the VRG’s effective mass, Coriolis mass, and stiffness

Component	Mass	Coriolis mass	Stiffness
Outer ring	95.30%	97.80%	43.52%
Inner ring	4.42%	2.20%	18.89%
Connecting beams	0.26%	0%	20.65%
Supporting beams	0.02%	0%	17.03%

with the mismatch between our calculated and experimentally measured frequencies, we obtain the value $1\mu\text{m}$ for the scaling error caused by the combination of mask bias, lithography, and etching. As given in Table 2.3, our method provides insights into the contribution of the different parts of the VRG’s structure to the total effective mass, Coriolis mass, and stiffness. We observe that the inner suspension’s predominant effect is in the device’s stiffness, as the outer ring mainly dictates the mass properties. We note that the underlying reason for designing a stiff suspension is to safely separate the $n = 1$ rigid body mode of about 45 kHz (see Figure 2.13) from the $n = 2$ operation mode. In the rigid body mode, the effective mass is equivalent to the outer ring’s mass, while the suspension system determines the total stiffness. In the $n = 2$ operation mode, however, the outer ring and suspension both contribute to the total stiffness, and only a fraction of the outer ring’s mass corresponds to the effective mass. Accordingly, the rigid body mode, which has a lower frequency than the $n = 2$ mode, is responsible for the VRG’s acceleration sensitivity. A sufficiently stiff suspension will confine the undesired effects ensuing the acceleration sensitivity, but at the same time, it will increase the frequency of the $n = 2$ operation mode. Additionally, our results show that mounting a ring on an internal suspension decreases the Coriolis mass-to-total mass ratio from 0.8 to 0.7796. This result indicates that the constraining effect of the inner suspension reduces VRG’s angular gain.

2.9.3 Effects of imperfections

Outer and inner rings. To incorporate the etching-induced imperfections in our mathematical model, we consider the profile in Figure 2.14 for the rings’

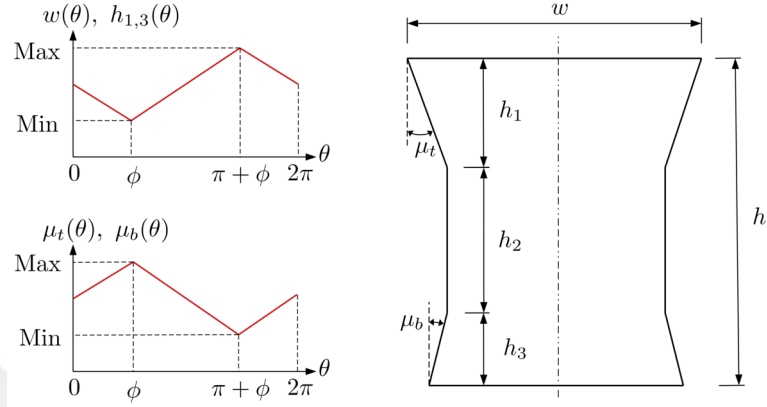


Figure 2.14: Cross-section profile of rings.

cross-sections. This profile captures both width variation and sidewall angles (denoted by μ_t and μ_b) due to undercuts. The maximal and minimal cross-sections occur at the angles $\phi \in [0, \pi]$ and $\phi + \pi$, respectively, where the difference π between these two angles reflects the circular progression of etching fronts. The cross-section parameters are piecewise linear functions of θ that vary between the respective maximum and minimum values. To model a common over-etching scenario with undercuts, for both rings, we consider a width variation of 50 nm, $h_1 \in [10, 11] \text{ } \mu\text{m}$, $h_3 \in [5, 6] \text{ } \mu\text{m}$, and the sidewall angles $\mu_t, \mu_b \in [3^\circ, 5^\circ]$. To accentuate the effect of sidewall angles, we separately examine the case of pure width variation (i.e., $\mu_t = \mu_b = 0$). The following results are in order:

- When one of the rings is the only imperfect component of the VRG, the ensuing mode shape rotation angle, $\psi/2$ depends only on the angle ϕ of the cross-section profile (see Figure 2.15). For the outer ring, ψ is a $\pi/4$ -periodic function of ϕ and $\psi = 2\phi$ for $\phi \in [-\frac{\pi}{8}, \frac{\pi}{8}]$. For the inner ring, the period is $\pi/2$ and the $\phi - \psi$ relation is nonlinear.
- Width variation alone produces negligible frequency split and, as shown in Figure 2.16, relatively small quadrature errors. Introducing sidewall angles raises the order of the frequency split from 10^{-3} Hz to 10^{-1} Hz and increases the quadrature error by a factor of 10. For the outer ring, the frequency split is a constant function of ϕ while we observe a $\pi/2$ -periodic oscillatory function for the inner ring.

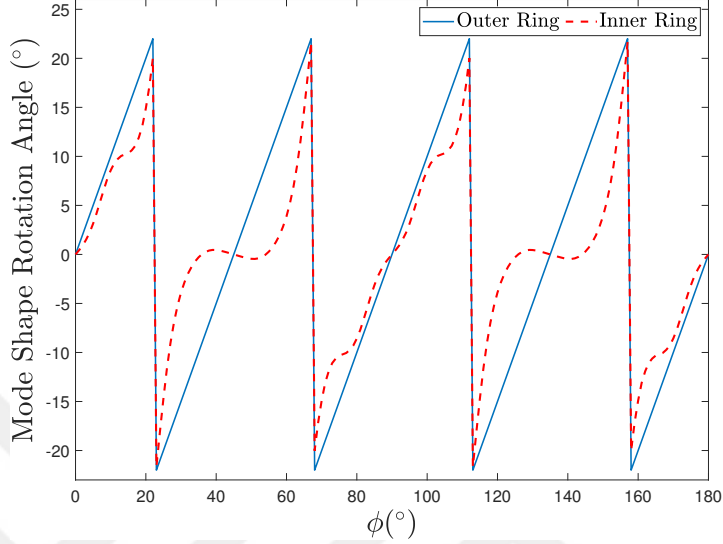


Figure 2.15: Mode shape rotation angle ($\psi/2$) versus the angle ϕ of the imperfection profiles of outer and inner rings (only outer or inner ring is imperfect at a time).

We used the experimentally observed quality factors in our quadrature calculations.

Connecting beams. According to (2.19d) and (2.22d), imperfections in these beams perturb only the diagonal elements of the mass and stiffness matrices, which translates into mass/stiffness mismatch between the modes; the connecting beams do not induce any coupling (in the form of off-diagonal elements) that could rotate the mode shapes. Therefore, the defects in these beams alone contribute to the frequency split and not the quadrature error. To explain this point, one can assume that one of the connecting beams is subjected to width variation. If this defective beam is even-numbered, it affects the frequency of $\cos(n\theta)$ mode (i.e., K_{11} entry of the stiffness matrix); otherwise, its effect is on the frequency of $\sin(n\theta)$ mode (i.e., K_{22} entry of the stiffness matrix). The absolute value of the frequency split, however, is independent of the beam's number. Figure 2.17 shows the frequency split varies linearly with respect to the width variation of the defective beam.

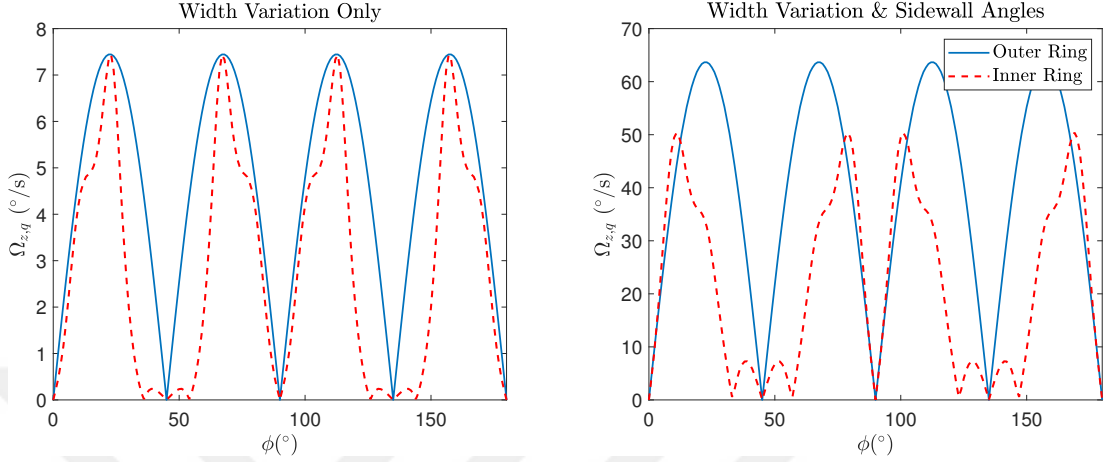


Figure 2.16: Quadrature error versus the angle ϕ of the imperfection profiles of outer and inner rings.

Supporting beams. According to (2.19d) and (2.22d), imperfections in these beams perturb both diagonal and off-diagonal entries of the mass and stiffness matrices. Hence, any defects in these beams contribute to the quadrature error as well as frequency split. Width variation in only one supporting beam results in a $\pm\pi/8$ mode shape rotation and affects the frequency of only one of the modes. Therefore, these beams are most likely to be the source of quadrature error. For example, a 50 nm (0.5%) width variation in the supporting beam no. 1 leads to a rather large quadrature error of $4700^\circ/\text{s}$. As shown in Figure 2.17, similar to the connecting beams, the induced frequency split is linear in the beam's width variation.

Material anisotropy. Our device is fabricated from silicon wafers with the (111) crystalline plane direction, which are almost in-plane isotropic [39, 40]. According to analysis presented in Section 2.6, if our VRG was made of (100) wafers, we would observe a large frequency split of 332 Hz between the mode shapes.

Combined effects. When considering the combined effects of the above imperfection, one should note that the induced mode shape rotation angles cannot be superposed. This stems from the fact that the eigenvectors of sum of matrices, in general, are not equivalent to the sum of their eigenvectors. Also, one needs to evaluate the effect of an imperfection based on the affected mass/stiffness matrix

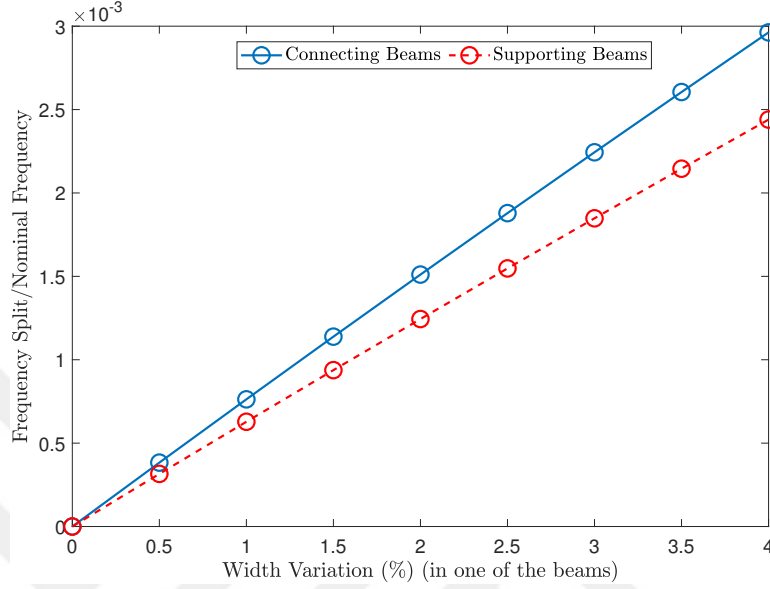


Figure 2.17: Frequency split versus width variation in connecting and supporting beams.

entries. We explain these points by the following examples:

Example 2.3. The outer ring has a variation profile with the angle $\phi = 10^\circ$, implying a rotation angle $\psi = 20^\circ$ and a negligible frequency split for the mode shapes. Connecting beam#1 has a 50 nm width variation that causes 15 Hz frequency split and no mode shape rotation. When these two imperfections are considered together, we still get the 15 Hz frequency split, but the mode shapes' rotation is nullified. This means that the beam's width variation counters the effect of the outer ring's imperfection on mode shapes' orientation.

Example 2.4. Assume that connecting beams#1 and 2 have the same width variation. The first beam affects the entry k_{22} in the VRG's stiffness matrix, and the second one has the same effect on k_{11} . Thus, no mismatch between the modes is induced.

Example 2.5. According to (2.22a) and (2.22b), the anisotropic Young's modulus of (100) Silicon decreases k_{11} and increases k_{22} . To eliminate the resultant mismatch, we can reduce the widths of the odd-numbered connecting beams as they affect k_{22} . Our model predicts that 330nm width reduction would remove the 332 Hz frequency split.

Possible imperfection scenarios for the fabricated VRG. We use our model to find possible imperfection scenarios that fit to the frequency split and quadrature error of our device. We recast this task as a nonlinear optimization problem with the frequency split and mode shape rotation angle as the objectives and the followings as variables: width variations $\Delta w_{c,e}$ and $\Delta w_{c,o}$ of even-numbered and odd-numbered connecting beams, respectively, width variations $\Delta w_{s,e}$ and $\Delta w_{s,o}$ of even-numbered and odd-numbered supporting beams, respectively, width variation Δw of both outer and inner rings, and angle ϕ of the rings' imperfection profile; the sidewall angles are fixed as $\mu_t, \mu_b \in [3^\circ, 5^\circ]$. We solved this problem in MATLAB using the genetic algorithm; two sample solutions that we obtained are

1. $\Delta w_{c,e} = 297.18$ nm, $\Delta w_{c,o} = 70.77$ nm, $\Delta w_{s,e} = 474.73$ nm, $\Delta w_{s,o} = 441.46$ nm, $\Delta w = 279.60$ nm, $\phi = 22.04^\circ$;
2. $\Delta w_{c,e} = 385.14$ nm, $\Delta w_{c,o} = 160.37$ nm, $\Delta w_{s,e} = 355$ nm, $\Delta w_{s,o} = 324.3$ nm, $\Delta w = 395.6$ nm, $\phi = 37.22^\circ$.

Despite the existence of many solutions to our optimization problem, we observed the following common motifs among them:

- The solutions give us (approximately) 61.84 Hz frequency split, 2° mode shape rotation, and $2000^\circ/\text{s}$ quadrature error that fit well to the experimental data.
- A 225 nm (1.4%) width difference between the odd-numbered and even-numbered connecting beams generates the 62 Hz frequency split, and a 30 nm (0.3%) width difference between the odd-numbered and even-numbered supporting beams causes the 2° mode shape rotation. The combination of these two factors produces the $2000^\circ/\text{s}$ quadrature error. It is reported in [30] that there might be a mask bias between 0° and 90° aligned beams. Equally spaced connecting beams start from 0° location, and equally spaced supporting beams start from 22.5° . So it is more likely that connecting beams would suffer from mask bias. Supporting beams are more evenly

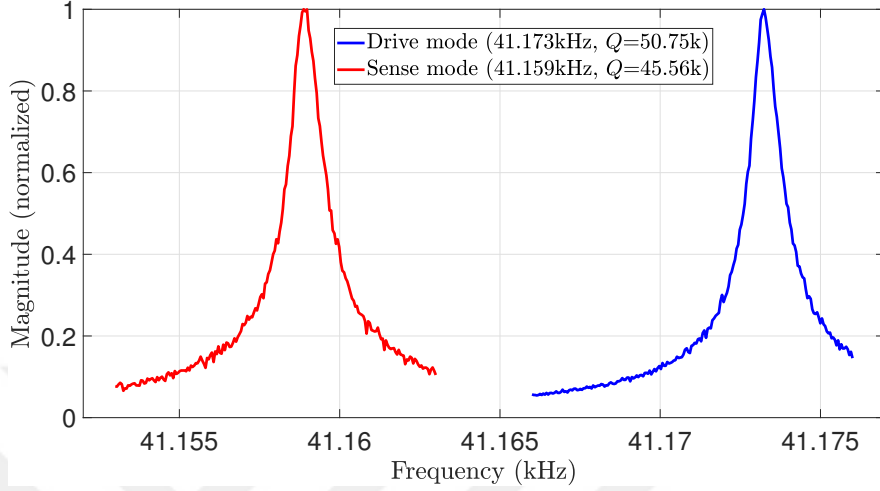


Figure 2.18: Measured frequency response of DRG.

distributed in terms of mask bias. As a result, we would expect the frequency split to be originated by the width variations in connecting beams, which is consistent with the our model's predictions.

- We got diverse values for Δw and ϕ since the stiffness matrix variation caused by the imperfections in the beams dominates that of the rings.

2.9.4 Results for DRG

According to the DRG's measured frequency response plotted in Figure 2.18, we obtained the resonance frequencies and quality factors as 41.173 kHz, 50.57k for the drive mode ($\cos(2\theta)$) and 41.159 kHz, 45.56k for the sense mode ($\sin(2\theta)$); there is a 14 Hz frequency split between the modes, which is attributable to geometric imperfections. The FEM gave us the natural frequency 45.216 kHz. With $N = 3$ basis functions, the comparative graphs of our calculated and FEM mode shapes are given in Figure 2.19. At the spoke angles, the mode shapes match very closely; however, our calculated mode shapes exhibit larger amplitudes at other angles. We obtained the modal properties as $m = 4.765 \times 10^{-8}$ kg, $m_c = 3.545 \times 10^{-8}$ kg, and $k = 3.339 \times 10^3$ N/m; these values give us the natural frequency as 42.134 kHz, which has the relative errors of 2.4% and 6.8% with

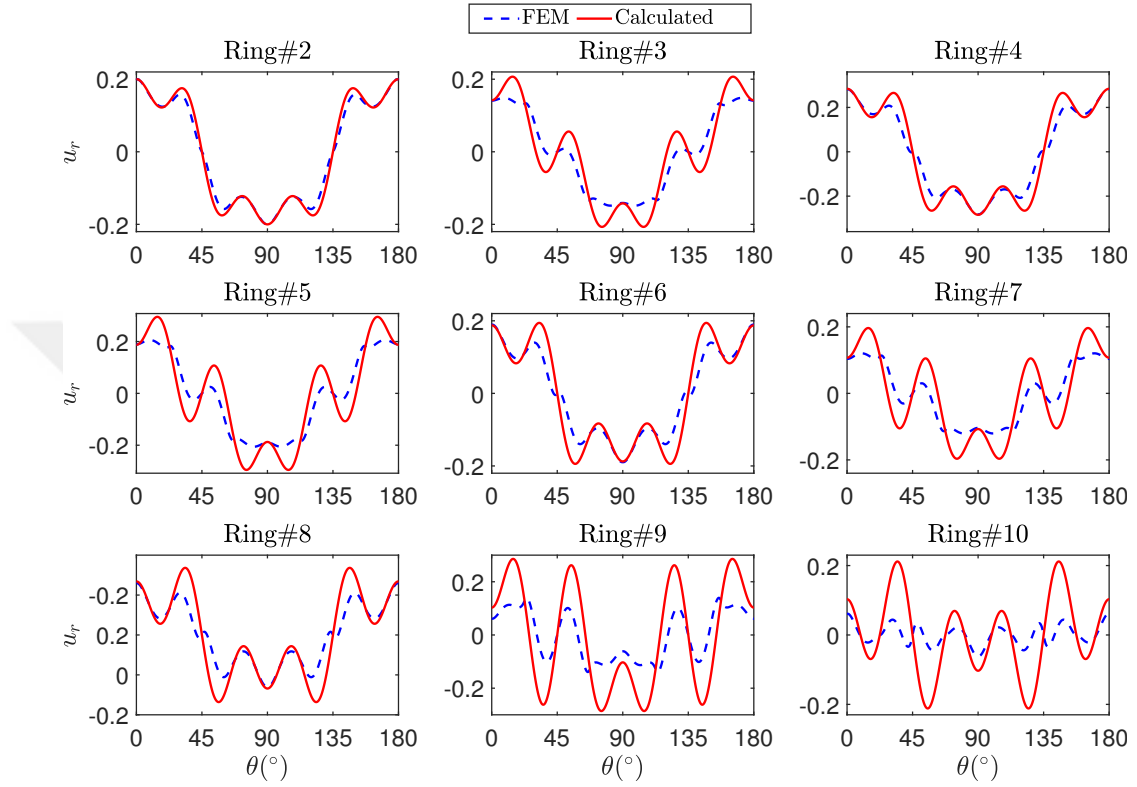


Figure 2.19: Mode shapes of DRG's inner rings; calculated versus FEM.

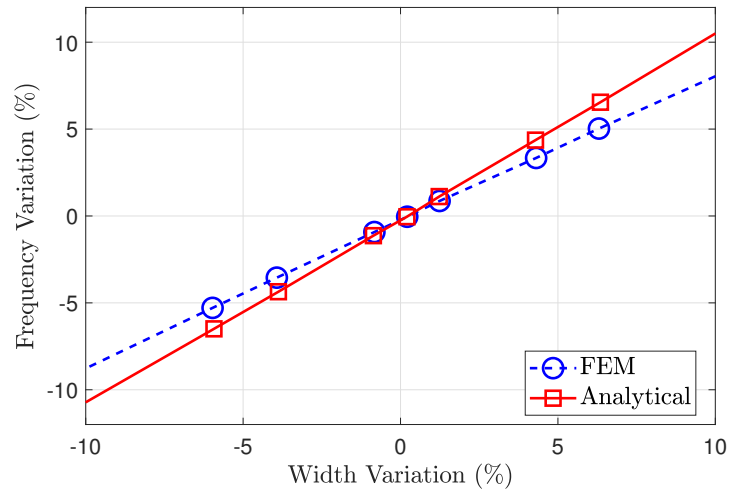


Figure 2.20: Variations of calculated natural frequencies with width variation of rings; calculated versus FEM.

respect to the experimental and FEM values, respectively. The mismatch between our results and FEM is due to neglected strain energies of the spokes and variations of the displacement fields across rings. Furthermore, based on the mismatch between calculated and measured frequencies, we estimate that a 200 nm mask bias has occurred in the lithography. We also obtain the angular gain of our gyroscope as 0.3723, which is 7% less than the theoretical value of 0.4 for a single ring. This observation (similar to the VRG) points out the limiting effect of the suspension system on the angular gain of circular gyroscopes. Figure 2.20 shows the comparative graphs of natural frequency variation against the width variation of our method and FEM; we observe a linear pattern in both methods, which guarantees the consistency of our modeling.

Chapter 3

Modeling Stress Effects in Circular MEMS Gyroscopes¹

3.1 Introduction

As pointed out in Chapter 1, bias drift is one of the long-standing obstacles that prevent MEMS gyroscopes from attaining navigation-grade performance. Based on the temperature-drift correlation, temperature calibration is the widely used drift compensation method in both industry and academic research [62, 63, 64]. It has been shown that stress is another contributing factor in the drift, and one can achieve a superior drift compensation using both stress and temperature [9]. Along the same line, the recent paper [10] demonstrated that a time-partitioned stress-based drift compensation could significantly improve the bias stability of a 57 kHz, 3.2 mm diameter vibrating ring gyroscope (VRG) from $0.7^\circ/\text{h}$ to $0.008^\circ/\text{h}$. The connection established in [9, 10] between stress and gyroscope performance is, in effect, a statistical correlation. Our goal in this chapter is to examine and understand the impact of stress from a first-principle analytical modeling viewpoint. Our aspiration is that this approach has the potential to pave the way for online stress-based drift calibration.

¹The material presented in this chapter are based on the author's publications [60] and [61].

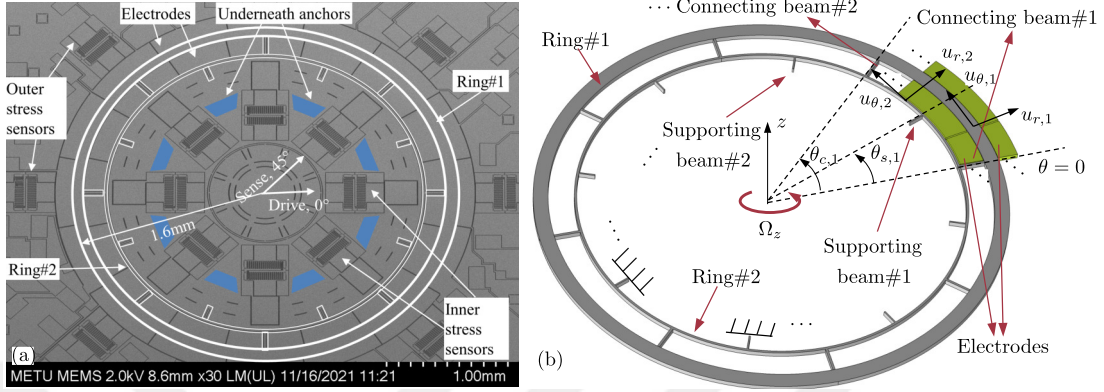


Figure 3.1: Ring gyroscope with 16 stress sensors: SEM (a) and schematics (b).

Related literature. The adverse effects of stress in MEMS devices have been well-studied in the literature. Inertial MEMS sensors used in strapdown navigation applications are often subject to shock, external vibration, and cyclic mechanical loads that impose undesired stresses on the device [65]. The die attachment quality in the packaging process is, however, a more critical factor that gives rise to stress in a MEMS device. The mismatch between the coefficients of thermal expansion of the MEMS die, the package substrate, and, if any, the gluing material could lead to residual and dynamic stresses as the device experiences abrupt temperature changes or undergoes thermal cycles [66]. One way to deal with this issue is to use modified, adhesive-free bonding techniques to minimize the packaging stress [67]. Another way is to apply a compensation algorithm using the data provided by deployed stress sensors [9]. The recent paper [68] showed that stress-based correction is remarkably effective in reducing humidity-related scale factor drifts of plastic packaged gyroscopes. To the best of our knowledge, there has been very limited attempts on analytical modeling of stress effects. Apart from our results, the paper [69] reported a modeling procedure for the scale factor changes under one-dimensional PCB bending stresses, focusing on electrostatics as the main affected component.

Contribution. Figure 3.1 shows the SEM and schematics of our 59 kHz, $n = 2$ wineglass modes VRG, which is equipped with 16 capacitive stress sensors. We lay out an analytical framework to model the effects of stress on frequencies, mode shape orientations, and quadrature/in-phase errors of this VRG. The key

findings and novelty aspects of this framework are as follows.

- We present a solid mechanical formulation of rings and beams, the components of the gyroscope’s mechanical structure, using Green-Lagrange strain to take geometric nonlinearity into account. Moreover, following Chapter 2, our formulation accommodates fabrication-induced (e.g., nonuniform etching, mask bias) imperfections and elastic anisotropy.
- The fundamental effect of stress on the gyroscope’s structure is the displacement of anchors from their designated locations. In terms of modeling, these anchors displacements translate into
 1. Nonhomogeneous (i.e., nonzero) boundary conditions at the interface between the gyroscope’s suspension and the internal anchored structure. In our gyroscope, the anchor displacements within the internal structure push/pull the eight supporting beams and, in turn, impose additional bending and extensional stresses on the inner ring. We show that the extensional component of these imposed stresses is the major contributor to the stiffness, and the inner ring’s extensibility is necessary for its correct modeling.
 2. Variations of the electrostatic gap due to the displacements of anchored electrodes, which then alters the electrostatic softening.
- The stress sensors in our gyroscope provide us with the local radial strains at the substrate level. We use these measurements to interpolate the strain field across the substrate and then obtain the anchor displacements that determine the nonhomogeneous boundary conditions and the electrostatic gap variations.
- Our formulation shows that stress could affect both diagonal and off-diagonal entries of the gyroscope’s stiffness matrix. We study the subsequent effects on the natural frequencies and quadrature/in-phase errors.

We performed extensive tests by applying bending loads on the gyroscope’s die and recorded the stress sensors’ outputs, frequency variations, and quadrature/in-phase errors. We also implemented our analytical model in a simulation setup

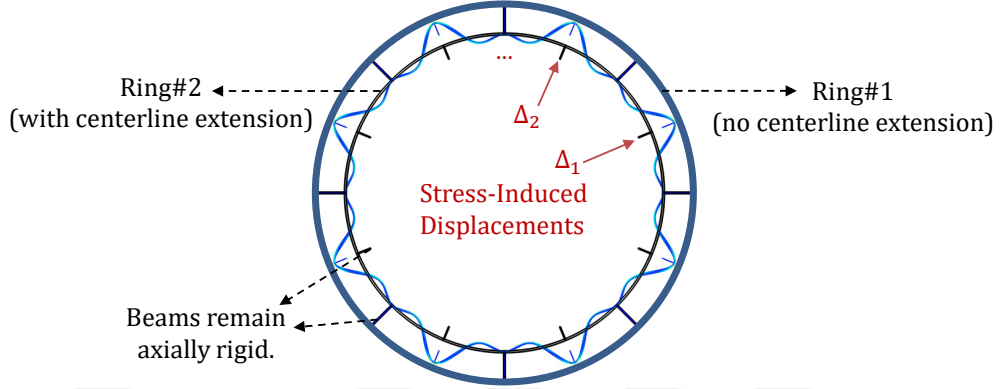


Figure 3.2: Illustration of rings' extensibility under stress effects; the inner ring experiences extensibility while the outer rings remains unaffected.

that accepts the stress sensors' data as the input. Comparison between the experimental and simulation results for frequency shifts and quadrature/in-phase errors demonstrates the validity of our modeling.

Notation. We will follow the same mathematical notation as Chapter 2.

3.2 Main Idea of Our Approach

Our goal, in a nutshell, is to mathematically model the link between stress at the substrate level and the subsequent changes in the stiffness matrix. As shown in Figure 3.1-(b), the VRG's mechanical structure comprises the following components: Ring#1 (outer ring), Ring#2 (inner ring), $4n$ connecting beams between the two rings, and $4n$ supporting beams that connect Ring#2 to the anchored internal structure. Here, $n = 2, 3, \dots$ is the mode number. As illustrated by Figure 3.2, Ring#2 takes up the load resulting from anchor displacements while Ring#1 remains unaffected. As a result, Ring#2 is subject to additional extensional stress besides the preexisting bending generated by the gyroscope's vibration. This key observation implies that we need to consider centerline extensibility only for Ring#2. The beams are all axially rigid, so they experience pure bending. In the structures that are under both axial and bending loads, second-order

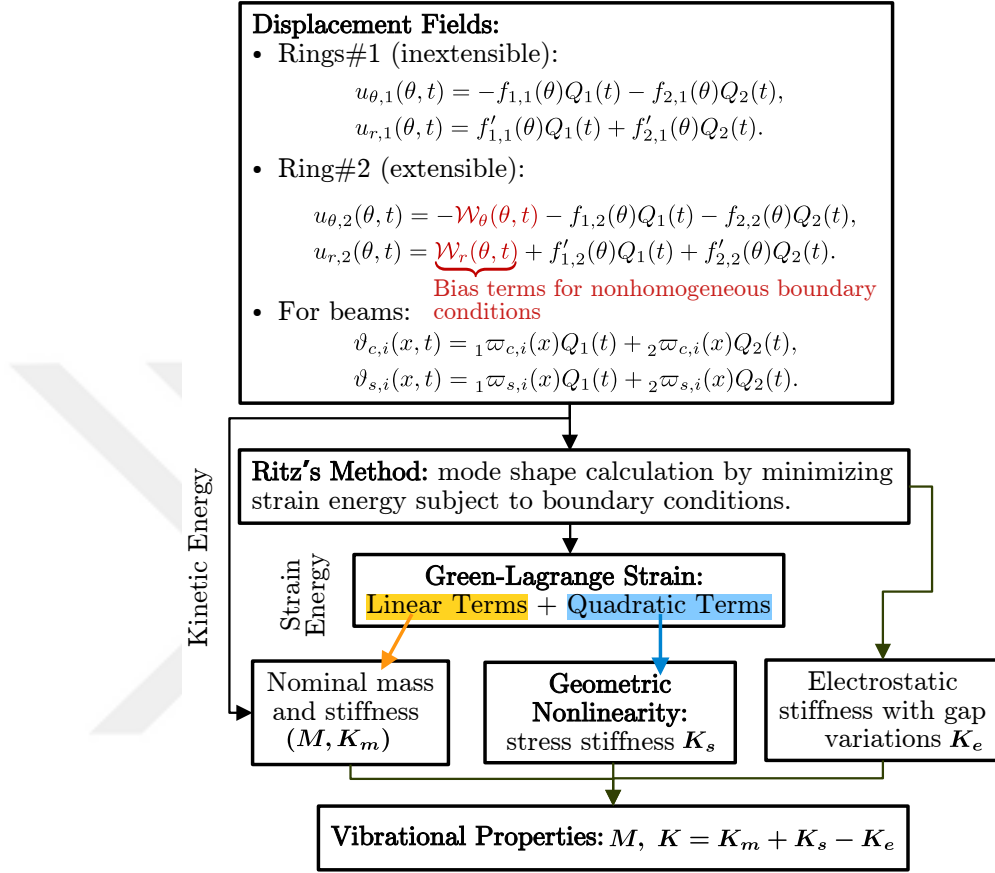


Figure 3.3: Flowchart of main steps of analytical modeling.

displacement-strain relations should be considered to fully characterize the stiffness [70]. We, therefore, adopt the Green-Lagrange strain to analyze Ring#2 and proceed with the usual linear strain for the rest of the structure. The flowchart in Figure 3.3 explains the big picture of our modeling. The essence of our method is to calculate the unknown mode shapes to have a complete description of the displacement fields, form the kinetic and strain energies, and finally obtain the mass and stiffness matrices.

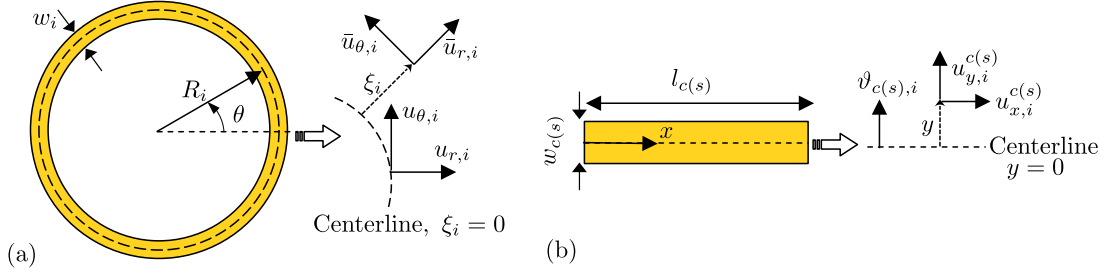


Figure 3.4: Displacement fields of Ring# i (a) and connecting (supporting) beam# i (b).

3.3 Modeling of Stress Effects

3.3.1 Solid mechanical formulation with geometric nonlinearity

We follow the basic assumptions and formulation setting of Chapter 2 with a slight change in notation.

Rings. We use the cylindrical coordinate system $\{r, \theta, z\}$ as the gyroscope-fixed reference frame. Ring# i , $i = 1, 2$, has a fixed mean radius of R_i , but we allow a varying width, $w_i(\theta)$ to accommodate for possible geometric imperfections. $A_i(\theta)$ and $I_i(\theta)$ are the area and the second moment of area the Ring# i 's cross-section, respectively. We introduce the coordinate ξ_i as the distance from the Ring# i 's centerline and denote the displacement field at $\xi_i = 0$ by $(u_{r,i}(\theta, t), u_{\theta,i}(\theta, t))$ with $t \geq 0$ being time (see Figure 3.4-(a)). Following the Euler-Bernoulli beam theory, we obtain the displacement field $(\bar{u}_{r,i}(\xi_i, \theta, t), \bar{u}_{\theta,i}(\xi_i, \theta, t))$ across Ring# i as

$$\begin{aligned}\bar{u}_{r,i}(\xi_i, \theta, t) &= u_{r,i}(\theta, t), \\ \bar{u}_{\theta,i}(\xi_i, \theta, t) &= u_{\theta,i}(\theta, t) + \xi_i \gamma_i(\theta, t),\end{aligned}\tag{3.1}$$

where

$$\gamma_i(\theta, t) = \frac{1}{R_i} \left(u_{\theta,i}(\theta, t) - \frac{\partial u_{r,i}}{\partial \theta}(\theta, t) \right),\tag{3.2}$$

is the rotation caused by tangential motion and bending of the cross-section located at the angle θ . Based on Equation (3.1), the extensional Green-Lagrange

strain in Ring# i is

$$\mathcal{E}_{\theta\theta,i} = \varepsilon_{\theta\theta,i} + \eta_{\theta\theta,i}, \quad (3.3)$$

where $\varepsilon_{\theta\theta,i}$ and $\eta_{\theta\theta,i}$ —the linear and nonlinear strain components, respectively—are given by

$$\begin{aligned} \varepsilon_{\theta\theta,i} &= \varepsilon_{0,i} + \xi_i \kappa_i, \\ \eta_{\theta\theta,i} &= \frac{1}{2} \left((\varepsilon_{0,i} + \xi_i \kappa_i)^2 + \left(1 + \frac{\xi_i}{R_i} \right)^2 \gamma_i^2 \right). \end{aligned} \quad (3.4)$$

Here, $\varepsilon_{0,i}$ and κ_i are the centerline's extensional strain and curvature change, respectively:

$$\begin{aligned} \varepsilon_{0,i}(\theta, t) &= \frac{1}{R_i} \left(u_{r,i}(\theta, t) + \frac{\partial u_{\theta,i}}{\partial \theta}(\theta, t) \right), \\ \kappa_i(\theta, t) &= \frac{1}{R_i} \frac{\partial \gamma_i}{\partial \theta}(\theta, t). \end{aligned} \quad (3.5)$$

Based on Equation (3.3), the total strain energy in Ring# i is given by

$$\mathcal{U}_i = \mathcal{U}_i^L + \mathcal{U}_i^{NL}, \quad (3.6)$$

$$\begin{aligned} \mathcal{U}_i^L &= \frac{1}{2} R_i h \int_0^{2\pi} E(\theta) \int_{-w_i(\theta)/2}^{w_i(\theta)/2} \varepsilon_{\theta\theta,i}^2(\xi_i, \theta, t) d\xi_i d\theta, \\ \mathcal{U}_i^{NL} &= \frac{1}{2} R_i h \int_0^{2\pi} E(\theta) \int_{-w_i(\theta)/2}^{w_i(\theta)/2} \eta_{\theta\theta,i}^2(\xi_i, \theta, t) + 2\eta_{\theta\theta,i}(\xi_i, \theta, t) \varepsilon_{\theta\theta,i}(\xi_i, \theta, t) d\xi_i d\theta, \end{aligned}$$

where $\mathcal{U}_i^L$ and $\mathcal{U}_i^{NL}$ are the energy components associated with the linear and nonlinear strain terms, respectively.

Beams. Connecting beam# i and supporting beams# i are located at the angles $\theta_{c,i} = (i-1)\frac{\pi}{2n}$ and $\theta_{s,i} = (2i-1)\frac{\pi}{4n}$, respectively, where $i = 1:4n$. We denote by $\vartheta_{c,i}$ and $\vartheta_{s,i}$ the deflection curves of connecting beam# i and supporting beam# i , respectively. As the coordinates, we use the abscissa, x along the beam's centerline and the ordinate, y orthogonal to the centerline (see Figure 3.4-(b)). The connecting (supporting) beams all have a fixed length of $l_{c(s)}$, but we consider the variable width $w_{c(s),i}(x)$ for connecting (supporting) beam# i to account for imperfections, resulting in variable area, $A_{c(s),i}(x)$ and second moment

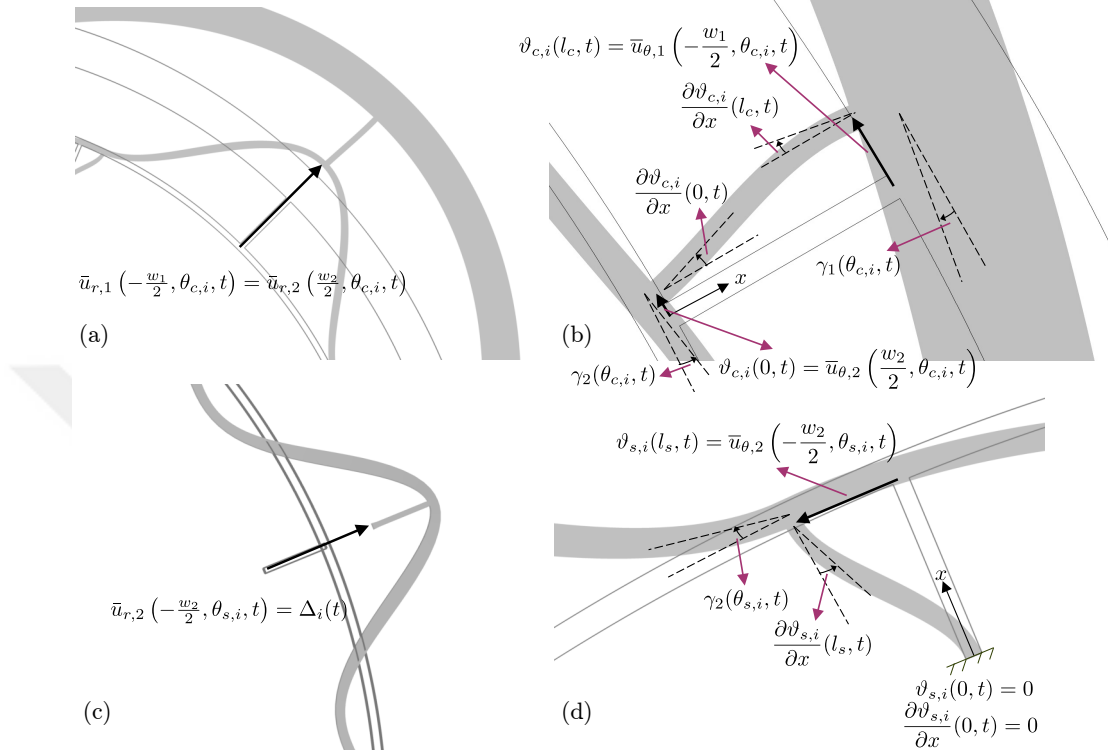


Figure 3.5: Illustration of the boundary conditions of connecting beam# i ((a), (b)) and supporting beam# i ((c), (d)).

of area, $I_{c(s),i}(x)$ of the beam's cross-section. The displacement field of connecting (supporting) beam# i is given by

$$\begin{aligned} u_{x,i}^{c(s)}(x, y, t) &= -y \frac{\partial \vartheta_{c(s),i}}{\partial x}(x, t), \\ u_{y,i}^{c(s)}(x, y, t) &= \vartheta_{c(s),i}(x, t), \end{aligned} \quad (3.7)$$

and the resulting strain is

$$\varepsilon_{xx,i}^{c(s)} = -y \frac{\partial^2 \vartheta_{c(s),i}}{\partial x^2}(x, t). \quad (3.8)$$

We obtain the connecting (supporting) beam# i 's strain energy as

$$\mathcal{U}_{c(s),i}^L = \frac{1}{2} h E(\theta_{c(s),i}) \int_0^{l_{c(s)}} \int_{-\frac{w_{c(s),i}(x)}{2}}^{\frac{w_{c(s),i}(x)}{2}} \varepsilon_{xx,i}^{c(s)2}(x, y, t) dy dx. \quad (3.9)$$

Based on Equations (3.6) and (3.9), we can decompose the total strain energy

into linear and nonlinear parts:

$$\begin{aligned}
\mathcal{U} &= \mathcal{U}^L + \mathcal{U}^{NL}, \\
\mathcal{U}^L &= \mathcal{U}_1^L + \mathcal{U}_2^L + \sum_{i=1}^{4n} \mathcal{U}_{c,i}^L + \mathcal{U}_{s,i}^L, \\
\mathcal{U}^{NL} &= \mathcal{U}_2^{NL}.
\end{aligned} \tag{3.10}$$

3.3.2 Boundary conditions

The displacement fields are subject to the following kinematic constraints that arise from ring-beam connections:

1. *Connecting beam#i*: we have

$$\bar{u}_{r,1} \left(-\frac{w_1}{2}, \theta_{c,i}, t \right) = \bar{u}_{r,2} \left(\frac{w_2}{2}, \theta_{c,i}, t \right), \tag{3.11}$$

which states that the rings both have the same radial displacement at $\theta = \theta_{c,i}$ (Figure 3.5-(a)). Furthermore, the following deflection/slope continuity conditions hold (Figure 3.5-(b)):

$$\begin{aligned}
\vartheta_{c,i}(0, t) &= \bar{u}_{\theta,2} \left(\frac{w_2}{2}, \theta_{c,i}, t \right), \\
\frac{\partial \vartheta_{c,i}}{\partial x}(0, t) &= \gamma_2(\theta_{c,i}, t), \\
\vartheta_{c,i}(l_c, t) &= \bar{u}_{\theta,1} \left(-\frac{w_1}{2}, \theta_{c,i}, t \right), \\
\frac{\partial \vartheta_{c,i}}{\partial x}(l_c, t) &= \gamma_1(\theta_{c,i}, t), \quad i = 1 : 4n.
\end{aligned} \tag{3.12}$$

2. *Supporting beam#i*: we have

$$\bar{u}_{r,2} \left(-\frac{w_2}{2}, \theta_{s,i}, t \right) = \Delta_i(t), \quad i = 1 : 4n. \tag{3.13}$$

Here, $\Delta_i(\cdot)$ is the stress-induced displacement that the supporting beam transmits to Ring#2 (Figure 3.5-(c)). We also have the following deflection/slope continuity conditions (Figure 3.5-(d)):

$$\vartheta_{s,i}(0, t) = 0,$$

$$\begin{aligned}
\frac{\partial \vartheta_{s,i}}{\partial x}(0, t) &= 0, \\
\vartheta_{s,i}(l_s, t) &= \bar{u}_{\theta,2} \left(-\frac{w_2}{2}, \theta_{s,i}, t \right), \\
\frac{\partial \vartheta_{s,i}}{\partial x}(l_s, t) &= \gamma_2(\theta_{s,i}, t), \quad i = 1: 4n.
\end{aligned} \tag{3.14}$$

3.3.3 Mode shapes

The inextensible vibration of Ring#1 corresponds to a displacement field of the form

$$\begin{aligned}
u_{\theta,1}(\theta, t) &= -f_{1,1}(\theta)Q_1(t) - f_{2,1}(\theta)Q_2(t), \\
u_{r,1}(\theta, t) &= f'_{1,1}(\theta)Q_1(t) + f'_{2,1}(\theta)Q_2(t),
\end{aligned} \tag{3.15}$$

where

$$f_{1,1}(\theta) = \frac{\sin(n\theta)}{n}, \quad f_{2,1}(\theta) = -\frac{\cos(n\theta)}{n}, \tag{3.16}$$

are the $n\theta$ -mode shapes and Q_1 , Q_2 are the generalized coordinates. Taking the extensibility of the Ring#2's centerline into account, we parametrize its displacement field as

$$\begin{aligned}
u_{\theta,2}(\theta, t) &= -\mathcal{W}_\theta(\theta, t) - f_{1,2}(\theta)Q_1(t) - f_{2,2}(\theta)Q_2(t), \\
u_{r,2}(\theta, t) &= \mathcal{W}_r(\theta, t) + f'_{1,2}(\theta)Q_1(t) + f'_{2,2}(\theta)Q_2(t),
\end{aligned} \tag{3.17}$$

where $f_{1,2}(\theta)$ and $f_{2,2}(\theta)$ are the two orthogonal mode shapes of Ring#2 corresponding to the generalized coordinates Q_1 and Q_2 , respectively. The functions $\mathcal{W}_r(\theta, t)$ and $\mathcal{W}_\theta(\theta, t)$ are bias-like terms that account for the nonhomogeneous boundary conditions caused by the displacements $\Delta_i(t)$. For the beams, we have

$$\begin{aligned}
\vartheta_{c,i}(x, t) &= {}_1\varpi_{c,i}(x)Q_1(t) + {}_2\varpi_{c,i}(x)Q_2(t), \\
\vartheta_{s,i}(x, t) &= {}_1\varpi_{s,i}(x)Q_1(t) + {}_2\varpi_{s,i}(x)Q_2(t),
\end{aligned} \tag{3.18}$$

where ${}_1\varpi_{c(s),i}(x)$ and ${}_2\varpi_{c(s),i}(x)$, $i = 1: 4n$, are the beam mode shapes corresponding to Q_1 and Q_2 , respectively. According to Equations (3.17) and (3.18), a complete description of the gyroscope's displacement fields amounts to calculation of the unknown mode shapes $\mathcal{W}_\theta$, $\mathcal{W}_r$, $f_{1,2}$, $f_{2,2}$, ${}_1\varpi_{c(s),i}$ and ${}_2\varpi_{c(s),i}$ subject to

the boundary conditions (3.11)-(3.14). We elaborated the calculation procedure of $f_{1,2}$, $f_{2,2}$, ${}_1\varpi_{c(s),i}$, and ${}_2\varpi_{c(s),i}$ in Chapter 2, and hence, we now focus on $\mathcal{W}_r$ and $\mathcal{W}_\theta$. According to Equations (3.13) and (3.14), $\mathcal{W}_r(\theta, t)$ and $\mathcal{W}_\theta(\theta, t)$ satisfy the following boundary conditions:

1. *Connecting beam#i*: no radial displacement and no cross-sectional rotation:

$$\begin{aligned}\mathcal{W}_r(\theta_{c,i}, t) &= 0, \quad \frac{\partial \mathcal{W}_r}{\partial \theta}(\theta_{c,i}, t) = 0, \\ \mathcal{W}_\theta(\theta_{c,i}, t) &= 0, \quad i = 1: 4n.\end{aligned}\tag{3.19}$$

2. *Supporting beam#i*: stress-induced radial displacement and no cross-sectional rotation:

$$\begin{aligned}\mathcal{W}_r(\theta_{s,i}, t) &= \Delta_i(t), \quad \frac{\partial \mathcal{W}_r}{\partial \theta}(\theta_{s,i}, t) = 0, \\ \mathcal{W}_\theta(\theta_{s,i}, t) &= 0, \quad i = 1: 4n.\end{aligned}\tag{3.20}$$

Mathematically, any pair of functions $(\mathcal{W}_\theta, \mathcal{W}_r)$ that satisfy (3.19) and (3.20) constitutes an admissible choice to be used in Equation (3.17). We, however, impose additional conditions to ensure that the calculated functions do not deviate drastically from the true, physical displacement field. For this purpose, at any given time instant $t \geq 0$, we require $\mathcal{W}_r(\theta, t)$ and $\mathcal{W}_\theta(\theta, t)$ to be the radial and tangential components, respectively, of the equilibrium displacement field that the applied displacements $\Delta_i(t)$, $i = 1: 4n$, generate in Ring#2. Taking advantage of the symmetry of the beams' arrangement, we calculate $\mathcal{W}_r$ and $\mathcal{W}_\theta$ by the following steps:

1. Consider the portion $\theta \in [0, \frac{\pi}{2n}]$ of Ring#2, which contains connecting beams#1, 2 and supporting beam#1; assume a given radial displacement Δ_0 that supporting beam#1 exerts on Ring#2.
2. Calculate the functions $\overline{\mathcal{W}}_r(\theta)$ and $\overline{\mathcal{W}}_\theta(\theta)$ for $\theta \in [0, \frac{\pi}{2n}]$ that correspond to the static equilibrium displacement field in step 1; this displacement field is illustrated in Figure 3.6.

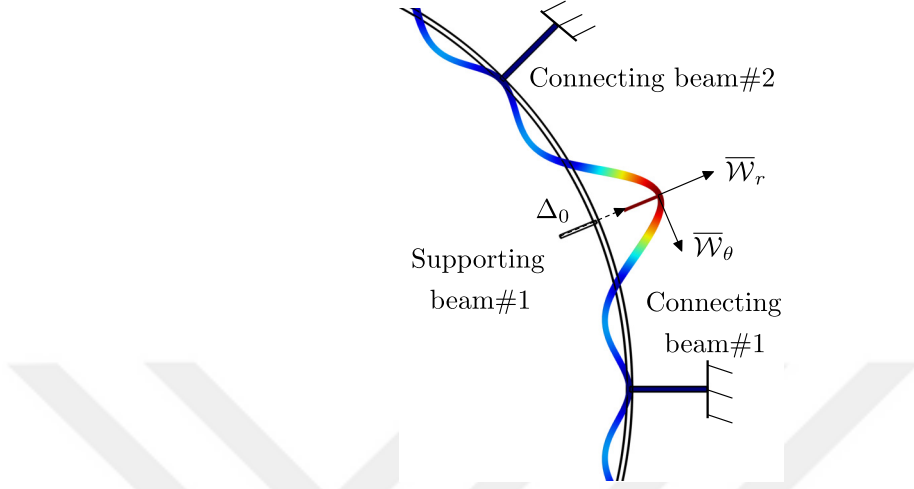


Figure 3.6: Displacement field of Ring#2 generated by the radial displacement at supporting beam#1.

3. For any portion $\theta \in [\frac{\pi}{2n}(i-1), \frac{\pi}{2n}i]$, $i = 1 : 4n$, of Ring#2 with the given displacement $\Delta_i(t)$, we have

$$\begin{aligned}\mathcal{W}_\theta(\theta, t) &= \frac{\Delta_i(t)}{\Delta_0} \overline{\mathcal{W}}_\theta \left(\theta - \frac{\pi}{2n}(i-1) \right), \\ \mathcal{W}_r(\theta, t) &= \frac{\Delta_i(t)}{\Delta_0} \overline{\mathcal{W}}_r \left(\theta - \frac{\pi}{2n}(i-1) \right).\end{aligned}\tag{3.21}$$

Calculating $\overline{\mathcal{W}}_r$ and $\overline{\mathcal{W}}_\theta$. The extensional strain and cross-sectional rotation angle corresponding to the displacement field $(\overline{\mathcal{W}}_r, \overline{\mathcal{W}}_\theta)$ are given by

$$\begin{aligned}\bar{\varepsilon}_{0,2}(\theta) &= \frac{1}{R_2} \left(\overline{\mathcal{W}}_r(\theta) - \overline{\mathcal{W}}_\theta'(\theta) \right), \\ \bar{\gamma}_{0,2}(\theta) &= \frac{-1}{R_2} \left(\overline{\mathcal{W}}_\theta(\theta) + \overline{\mathcal{W}}_r'(\theta) \right),\end{aligned}\tag{3.22}$$

respectively. We now minimize the strain potential energy stored in Ring#2 over $\theta \in [0, \frac{\pi}{2n}]$,

$$\overline{\mathcal{U}} = \frac{1}{2} E A_2 R_2 \int_0^{\frac{\pi}{2n}} \bar{\varepsilon}_{0,2}^2(\theta) d\theta + \frac{E I_2}{2 R_2} \int_0^{\frac{\pi}{2n}} \bar{\gamma}_{0,2}^2(\theta) d\theta,\tag{3.23}$$

considering the kinematic constraints that follow from (3.19) and (3.20):

$$\begin{aligned}\overline{\mathcal{W}}_r(0) &= 0, \overline{\mathcal{W}}_r'(0) = 0, \overline{\mathcal{W}}_\theta(0) = 0, \\ \overline{\mathcal{W}}_r\left(\frac{\pi}{4n}\right) &= \Delta_0, \overline{\mathcal{W}}_r'\left(\frac{\pi}{4n}\right) = 0, \overline{\mathcal{W}}_\theta\left(\frac{\pi}{4n}\right) = 0, \\ \overline{\mathcal{W}}_r\left(\frac{\pi}{2n}\right) &= 0, \overline{\mathcal{W}}_r'\left(\frac{\pi}{2n}\right) = 0, \overline{\mathcal{W}}_\theta\left(\frac{\pi}{2n}\right) = 0.\end{aligned}\tag{3.24}$$

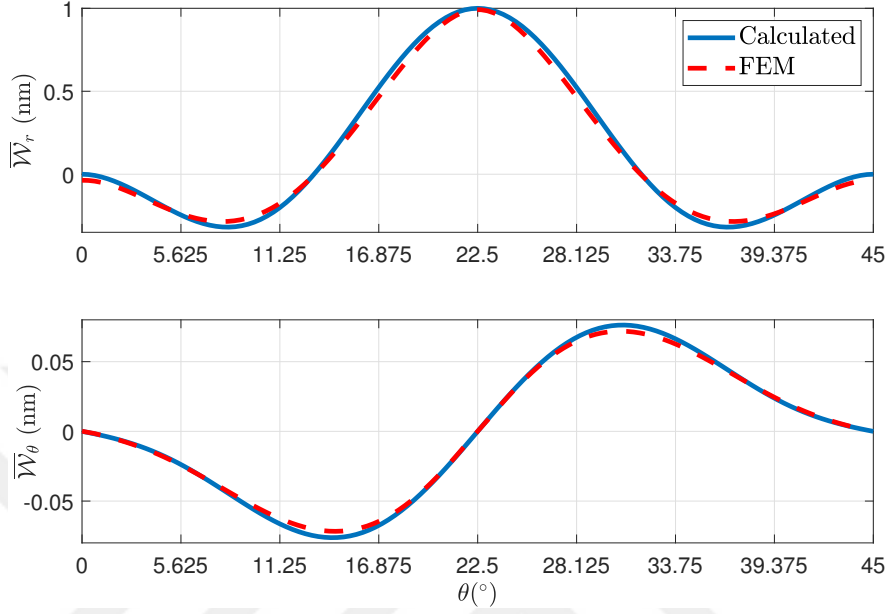


Figure 3.7: Displacement field $(\bar{W}_r, \bar{W}_\theta)$ of Ring#2 under the imposed displacement $\Delta_0 = 1$ nm by supporting beam#1 at 22.5° ; calculated versus FEM.

The above formulation is a variational problem; following the Ritz method [31], we convert it into numerical optimization using the parametrizations

$$\begin{aligned}\bar{W}_\theta(\theta) &\approx \chi_\theta^\top(\theta)\alpha_\theta, \\ \bar{W}_r(\theta) &\approx \chi_r^\top(\theta)\alpha_r,\end{aligned}\tag{3.25}$$

where $\chi_\theta(\cdot), \chi_r(\cdot) \in \mathbb{R}^N$ (with $N \geq 1$) are the vectors of suitable basis functions, and $\alpha_\theta, \alpha_r \in \mathbb{R}^N$ are the corresponding coefficients. By substituting (3.25) into (3.23) and (3.24), we obtain the following QP problem in the unknowns α_θ, α_r :

$$\begin{bmatrix} \alpha_\theta^* \\ \alpha_r^* \end{bmatrix} = \arg \min_{\alpha_\theta, \alpha_r \in \mathbb{R}^N} \frac{1}{2} \begin{bmatrix} \alpha_\theta \\ \alpha_r \end{bmatrix}^\top \begin{bmatrix} P_{\theta\theta} & P_{\theta r} \\ P_{\theta r}^\top & P_{rr} \end{bmatrix} \begin{bmatrix} \alpha_\theta \\ \alpha_r \end{bmatrix}\tag{3.26a}$$

subject to

$$\begin{cases} A_\theta \alpha_\theta = 0_{3 \times 1}, \\ A_r \alpha_r = \Delta_0 b_r, \end{cases}\tag{3.26b}$$

where

$$\begin{aligned}
P_{\theta\theta} &= \left(\frac{EA_2}{R_2} + \frac{EI_2}{R_2^3} \right) \int_0^{\frac{\pi}{2n}} \chi'_\theta(\theta) \chi'^\top_\theta(\theta) d\theta, \\
P_{rr} &= \frac{EA_2}{R_2} \int_0^{\frac{\pi}{2n}} \chi_r(\theta) \chi_r^\top(\theta) d\theta + \frac{EI_2}{R_2^3} \int_0^{\frac{\pi}{2n}} \chi''_r(\theta) \chi''^\top_r(\theta) d\theta, \\
P_{\theta r} &= -\frac{EA_2}{R_2} \int_0^{\frac{\pi}{2n}} \chi_r(\theta) \chi'^\top_\theta(\theta) d\theta + \frac{EI_2}{R_2^3} \int_0^{\frac{\pi}{2n}} \chi''_r(\theta) \chi'^\top_\theta(\theta) d\theta, \\
A_\theta &= [\chi_\theta(0), \chi_\theta(\frac{\pi}{4n}), \chi_\theta(\frac{\pi}{2n})]^\top, \\
A_r &= [\chi_r(0), \chi'_r(0), \chi_r(\frac{\pi}{4n}), \chi'_r(\frac{\pi}{4n}), \chi_r(\frac{\pi}{2n}), \chi'_r(\frac{\pi}{2n})]^\top, \\
b_r &= [0, 0, 1, 0, 0, 0]^\top.
\end{aligned} \tag{3.26c}$$

Basis functions $\chi_r(\theta)$, $\chi_\theta(\theta)$. Since $\overline{\mathcal{W}}_r$ has an even symmetry about $\theta = \frac{\pi}{4n}$ and $\overline{\mathcal{W}}_r - \overline{\mathcal{W}}'_\theta \approx 0$, we use the following polynomial basis functions:

$$\begin{aligned}
\chi_r(\theta) &= \left[1, \left(\theta - \frac{\pi}{4n} \right)^2, \dots, \left(\theta - \frac{\pi}{4n} \right)^{2N-2} \right]^\top, \\
\chi_\theta(\theta) &= \left[\left(\theta - \frac{\pi}{4n} \right), \dots, \frac{1}{2N-1} \left(\theta - \frac{\pi}{4n} \right)^{2N-1} \right]^\top.
\end{aligned} \tag{3.27}$$

For a unit displacement of $\Delta_0 = 1$ nm and $N = 6$ basis functions, Figure 3.7 shows the comparative graphs of our calculated functions $\overline{\mathcal{W}}_r(\theta)$ and $\overline{\mathcal{W}}_\theta(\theta)$ and those obtained from FEM (done in COMSOL Multiphysics). Taking FEM results as the reference, the small RMS errors 3.5×10^{-2} nm and 2.6×10^{-3} nm of the calculated $\overline{\mathcal{W}}_r(\theta)$ and $\overline{\mathcal{W}}_\theta(\theta)$, respectively, confirm our method's accuracy.

3.3.4 Electrostatic softening

We detailed the modeling of electrostatic softening in Chapter 2, Section 2.2.6.3. Here, we further consider (θ, t) -dependent outer $g_\mathcal{O}(\theta, t)$ and inner $g_\mathcal{I}(\theta, t)$ gaps to account for stress-induced gap variations.

3.3.5 Stiffness matrix

The total stiffness matrix $K \in \mathbb{R}^{2 \times 2}$ is given by

$$K = K_m + K_s - K_e, \quad (3.28)$$

where

- $K_m = [k_{ij}^m] \in \mathbb{R}^{2 \times 2}$ is the nominal mechanical stiffness matrix that one would observe in the absence of external stress effects, and it follows from the linear strains:

$$k_{ij}^m = \frac{\partial^2 \mathcal{U}^L}{\partial Q_i \partial Q_j} \Big|_{Q_1, Q_2=0} = k_{ij,1} + k_{ij,2} + k_{ij,c} + k_{ij,s}, \quad (3.29)$$

where

- Ring# ℓ , $\ell = 1, 2$:

$$k_{ij,\ell} = \frac{1}{R_\ell^3} \int_0^{2\pi} F'_{i,\ell}(\theta) F'_{j,\ell}(\theta) E(\theta) I_\ell(\theta) d\theta, \quad (3.30)$$

- Connecting (supporting) beams:

$$k_{ij,c(s)} = \sum_{\ell=1}^{4n} E(\theta_{c(s),\ell}) \int_0^{l_{c(s)}} {}_i\varpi''_{c(s),\ell}(x) {}_j\varpi''_{c(s),\ell}(x) I_{c(s),\ell}(x) dx. \quad (3.31)$$

- $K_s = [k_{ij}^s] \in \mathbb{R}^{2 \times 2}$ is the *stress stiffness* matrix; the stress-induced displacements Δ_i alter the strain energy stored in Ring#2, and the subsequent effect on the stiffness is associated with the geometric nonlinearity, i.e., the nonlinear strain terms:

$$k_{ij}^s = \frac{\partial^2 \mathcal{U}^{NL}}{\partial Q_i \partial Q_j} \Big|_{Q_1, Q_2=0}. \quad (3.32)$$

Neglecting second-order and product strain terms, we obtain the linearized expression

$$k_{ij}^s \approx k_{ij}^{\text{ext}} + k_{ij}^{\text{ben}}, \quad (3.33)$$

where

$$\begin{aligned}
k_{ij}^{\text{ext}}(t) &= \frac{1}{R_2} \int_0^{2\pi} E(\theta) A_2(\theta) \varepsilon_{0,2}(\theta, t) F_{i,2}(\theta) F_{j,2}(\theta) d\theta, \\
k_{ij}^{\text{ben}}(t) &= \frac{1}{R_2^3} \int_0^{2\pi} E(\theta) I_2(\theta) \varepsilon_{0,2}(\theta, t) \{ F_{i,2}(\theta) F_{j,2}(\theta) + 3F'_{i,2}(\theta) F'_{j,2}(\theta) \} d\theta \quad (3.34) \\
&\quad + \frac{2}{R_2^3} \int_0^{2\pi} E(\theta) I_2(\theta) \frac{\partial}{\partial \theta} \{ \gamma_{0,2}(\theta, t) F_{i,2}(\theta) F_{j,2}(\theta) \} d\theta.
\end{aligned}$$

The first term denotes an extensional stiffness, $k_{ij}^{\text{ext}} \propto EA_2/R_2$ while the second one stands for bending stiffness, $k_{ij}^{\text{ben}} \propto EI_2/R_2^3$. Since $k_{ij}^{\text{ben}}/k_{ij}^{\text{ext}} \propto (w_2/R_2)^2$, for a narrow ring, the extensional part has a much larger contribution to the stress stiffness than the bending part. It is worth mentioning that the extensional stiffness k_{ij}^{ext} does not appear in the k_{ij}^s formula if one proceeds with the inextensibility assumption prevalent in the literature.

- $K_e = [k_{ij}^e] \in \mathbb{R}^{2 \times 2}$ is the electrostatic softening matrix:

$$\begin{aligned}
k_{ij}^e &= \frac{\partial^2 \mathcal{U}_e}{\partial Q_i \partial Q_j} \Big|_{Q_1, Q_2=0} = \epsilon_0 h \sum_{\ell=1}^{16} \left\{ V_{1,\ell}^2 \int_{\theta_{e,\ell}-\frac{\Delta\theta_{1,\ell}}{2}}^{\theta_{e,\ell}+\frac{\Delta\theta_{1,\ell}}{2}} \frac{R_1 + \frac{w(\theta)}{2}}{g_{\mathcal{O}}^3(\theta, t)} f'_{i,1}(\theta) f'_{j,1}(\theta) d\theta \right. \\
&\quad \left. + V_{2,\ell}^2 \int_{\theta_{e,\ell}-\frac{\Delta\theta_{2,\ell}}{2}}^{\theta_{e,\ell}+\frac{\Delta\theta_{2,\ell}}{2}} \frac{R_1 - \frac{w(\theta)}{2}}{g_{\mathcal{I}}^3(\theta, t)} f'_{i,1}(\theta) f'_{j,1}(\theta) d\theta \right\}. \quad (3.35)
\end{aligned}$$

3.3.6 Further remarks on stress stiffness

Remark 3.1. The stress stiffness varies with time as it depends on the displacements $\Delta_i(t)$. Based on the procedure detailed in Section 3.3.3 and Equation (3.21), we simplify the online calculation of $k_{ij}^s(t)$ as

$$k_{ij}^s(t) \approx \sum_{\ell=1}^{4n} \frac{\Delta_\ell(t)}{\Delta_0} \bar{k}_{ij,\ell}, \quad (3.36)$$

where

$$\begin{aligned}
\bar{k}_{ij,\ell} = & \frac{1}{R_2} \int_{\frac{\pi(\ell-1)}{2n}}^{\frac{\pi\ell}{2n}} E(\theta) A_2(\theta) \bar{\varepsilon}_{0,2}(\theta - \frac{\pi(\ell-1)}{2n}) F_{i,2}(\theta) F_{j,2}(\theta) d\theta \\
& + \frac{1}{R_2^3} \int_{\frac{\pi(\ell-1)}{2n}}^{\frac{\pi\ell}{2n}} E(\theta) I_2(\theta) \bar{\varepsilon}_{0,2}(\theta - \frac{\pi(\ell-1)}{2n}) \{F_{i,2}(\theta) F_{j,2}(\theta) 3F'_{i,2}(\theta) F'_{j,2}(\theta)\} d\theta \quad (3.37) \\
& + \frac{2}{R_2^3} \int_{\frac{\pi(\ell-1)}{2n}}^{\frac{\pi\ell}{2n}} E(\theta) I_2(\theta) \frac{d}{d\theta} \{\bar{\gamma}_{0,2}(\theta - \frac{\pi(\ell-1)}{2n}) F_{i,2}(\theta) F_{j,2}(\theta)\} d\theta.
\end{aligned}$$

The interpretation of Equation (3.36) is as follows. We first calculate the displacement field $(\bar{W}_r(\theta), \bar{W}_\theta(\theta))$ resulting from a unit displacement Δ_0 at supporting beam#1. This calculated displacement field then enables us to compute $\bar{k}_{ij,\ell}$, the unit stiffness matrix across the portion $\theta \in [\frac{\pi}{2n}(\ell-1), \frac{\pi}{2n}\ell]$, $\ell = 1 : 4n$, of Ring#2. At any given time t with the displacements $\Delta_\ell(t)$ at the supporting beams, we obtain the stress stiffness $k_{ij}^s(t)$ as a linear combination of the unit stiffness matrices with the weights $\Delta_\ell(t)/\Delta_0$. We note that, in the absence of imperfections and elastic anisotropy, all the unit stiffness matrices are equivalent, i.e., $\bar{k}_{ij,\ell} = \bar{k}_{ij,1}$ for $\ell = 2 : 4n$, and the calculation of stress stiffness is further simplified.

Remark 3.2. The displacement field $(\bar{W}_r, \bar{W}_\theta)$ in Figure 3.6 is symmetric with respect to both $n\theta$ -mode shapes of the inner ring, and, as a result, $k_{11}^s = k_{22}^s$. However, this displacement field spoils the orthogonality of the $n\theta$ -mode shapes, resulting in a stress-induced stiffness coupling. For example, with our VRG's parameters, we have $k_{12}^s/k_{11}^s \approx 1.8\%$ under $\Delta_0 = 1$ nm displacement.

3.4 Stress Sensors

As shown in Figure 3.8, we have distributed 16 stress sensors with 45° symmetry across the gyroscope, eight in the internal structure and eight outside Ring#1. Figure 3.9 explains the working principle of the stress sensors as capacitive strain gauges that measure the local displacement gradient of the substrate via their anchors [10, 71]. Following the notation in the figure, consider a stress sensor

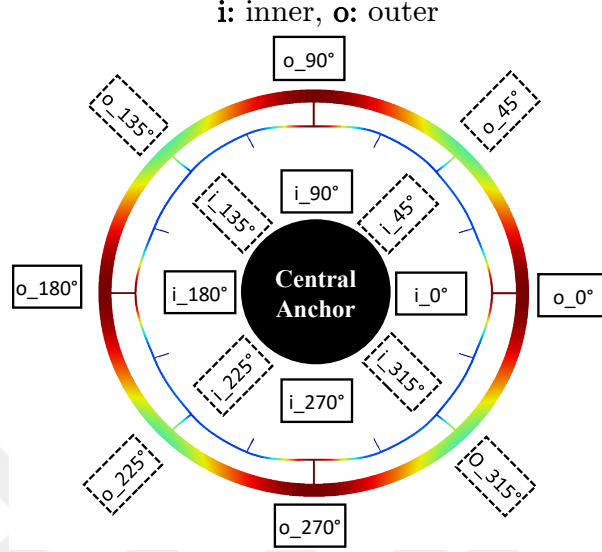


Figure 3.8: Stress sensors' arrangement across the gyroscope.

located at the radial position $\tilde{R}$ and the polar angle $\tilde{\theta}$. Let $\tilde{u}_r(r, \theta, t)$ be the radial component of the substrate's displacement field, and $\tilde{\varepsilon}_{rr}(r, \theta, t) = \partial \tilde{u}_r(r, \theta, t) / \partial r$ be the corresponding radial strain. Since

$$\tilde{u}_r(\tilde{R} + d_s, \tilde{\theta}, t) - \tilde{u}_r(\tilde{R}, \tilde{\theta}, t) \approx \tilde{\varepsilon}_{rr}(\tilde{R}, \tilde{\theta}, t) d_s, \quad (3.38)$$

the stress sensor's output is proportional to the local radial strain, and, in turn, to the radial stress, $\tilde{\sigma}_{rr} = E \tilde{\varepsilon}_{rr}$. On this basis, we use the stress sensors outputs to interpolate the radial strain field across the substrate. The interpolation amounts to setting forth an approximation of the form

$$\tilde{\varepsilon}_{rr}(r, \theta, t) \approx \sum_{i=1}^{\tilde{N}} \phi_i(r, \theta) \alpha_i(t) = \phi^\top(r, \theta) \alpha(t), \quad (3.39)$$

where $\phi(r, \theta) := [\phi_1(r, \theta), \dots, \phi_{\tilde{N}}(r, \theta)] \in \mathbb{R}^{\tilde{N}}$ is a vector of suitable basis functions, $\alpha(t) = [\alpha_1(t), \dots, \alpha_{\tilde{N}}(t)]^\top \in \mathbb{R}^{\tilde{N}}$ is the vector of basis functions' coefficients, and $\tilde{N} \geq 1$ is the number of basis functions. At time t , we obtain the optimal vector of coefficients, $\alpha^*(t)$ by solving the following least-squares problem:

$$\begin{aligned} \alpha^*(t) &:= \arg \min_{\alpha \in \mathbb{R}^{\tilde{N}}} J(\alpha, t), \\ J(\alpha, t) &= \frac{1}{2} \sum_{i,j} \left(\tilde{\varepsilon}_{ij}(t) - \phi^\top(\tilde{R}_i, \tilde{\theta}_j) \alpha \right)^2, \end{aligned} \quad (3.40)$$

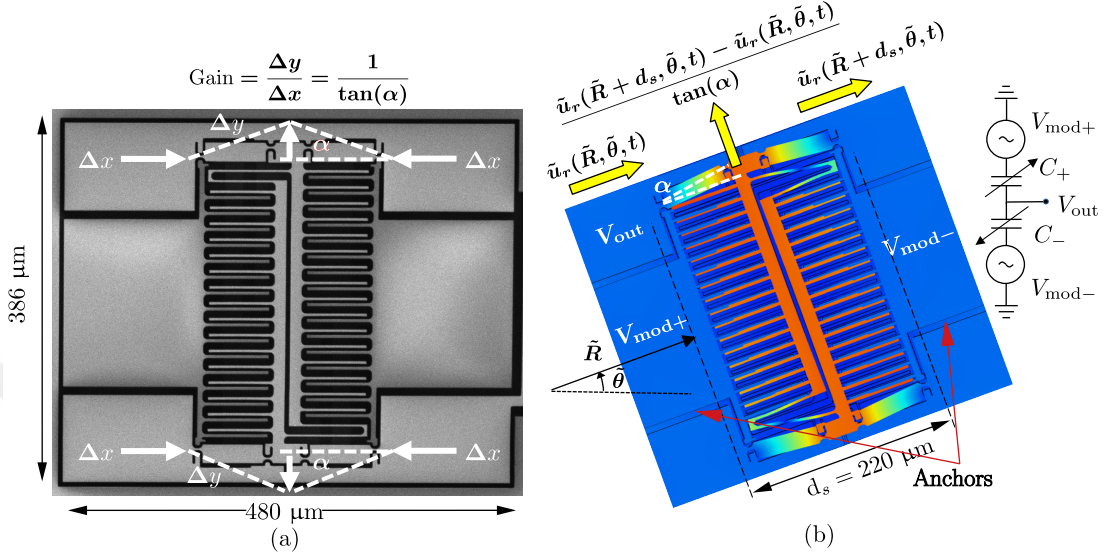


Figure 3.9: Stress sensors' SEM (a), and working principle (b).

where $\tilde{\varepsilon}_{ij}$ is the output of the stress sensor located at $(\tilde{R}_i, \tilde{\theta}_j)$. By applying the optimality condition $\partial J / \partial \alpha = 0$, we obtain the normal equation of the least squares problem as

$$A\alpha^*(t) = b(t), \quad (3.41)$$

where

$$\begin{aligned} A &= \sum_{i,j} \phi(R_i, \theta_j) \phi^\top(R_i, \theta_j), \\ b(t) &= \sum_{i,j} \phi(R_i, \theta_j) \tilde{\varepsilon}_{ij}(t). \end{aligned} \quad (3.42)$$

Upon solving (3.41) and interpolating the strain field $\tilde{\varepsilon}_{rr}(r, \theta, t)$, we obtain the substrate's radial displacement field as

$$\tilde{u}_r(r, \theta, t) \approx \int_0^r \phi^\top(\varrho, \theta) \alpha^*(t) d\varrho, \quad (3.43)$$

which provides us with the displacements $\Delta_i(t)$ at the supporting beams and the electrostatic gaps $g_{\mathcal{I}}(\theta, t)$, $g_{\mathcal{O}}(\theta, t)$ that are required for calculating the stress stiffness and softening matrices, respectively. Figure 3.11 pictorially demonstrates the interpolation process.

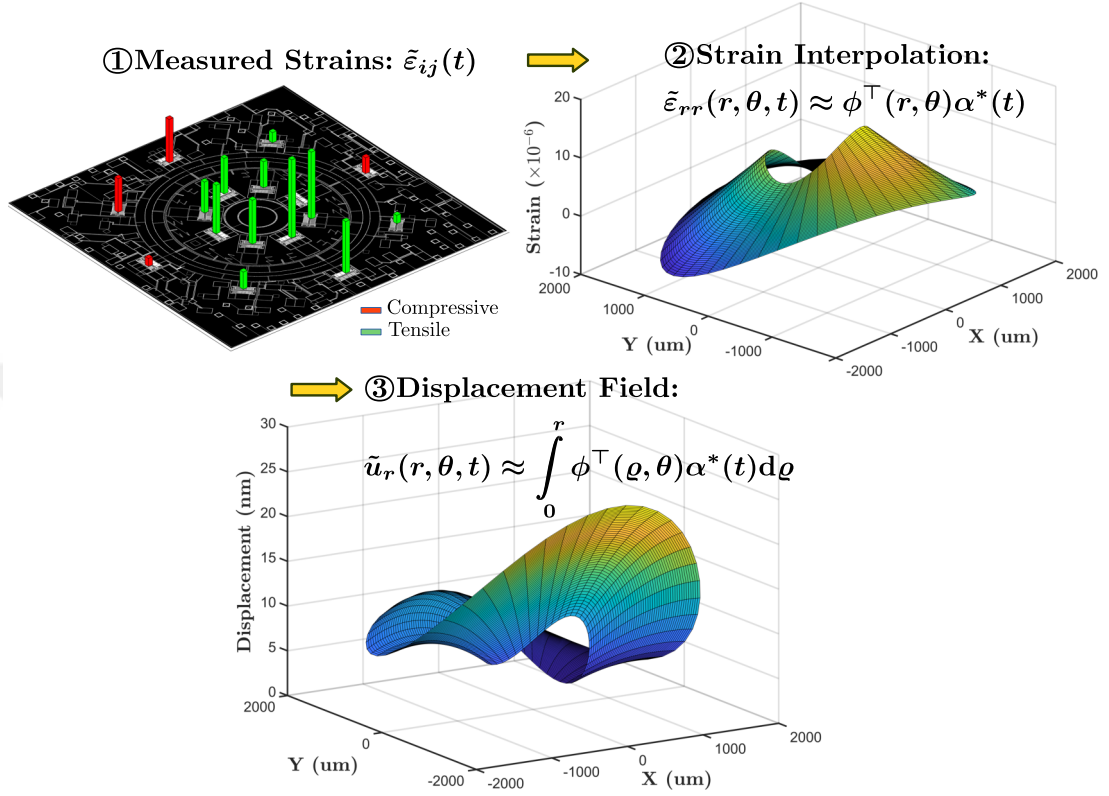


Figure 3.10: Interpolation of substrate's strain and displacement fields at time t .

Basis functions $\phi(r, \theta)$. The strain field $\tilde{\epsilon}_{rr}(r, \theta, t)$ is continuous, 2π -periodic in θ , and, therefore, admits a convergent Fourier series. Hence, assuming a linear variation in the r -direction, we use the tensor product of the linear polynomial basis and the usual Fourier sine-cosine pairs as the (r, θ) -basis functions:

$$\phi(r, \theta) := [1, r]^\top \otimes [1, \cos(\theta), \sin(\theta), \dots, \cos(N_h \theta), \sin(N_h \theta)]^\top. \quad (3.44)$$

In selecting the number of harmonics N_h , we should be careful about the rank deficiency of the matrix A in (3.42).

3.4.1 Comments on accuracy and performance of stress sensors

Accuracy. Ideally, the stress sensors do not respond to the tangential strain up to the first order [71]. Tangential strains rotate the entire sensor, resulting

in a negligible response due to the differential sensing mechanism. However, considering fabrication tolerances, tangential strains could affect practical sensors and contaminate the radial strain measurements.

Performance. The stress sensors exhibit a mechanical resonance frequency of around 300 kHz, while they operate at 10 kHz. Hence, electronics noise dominates the performance of the stress sensors. We calculated the resolution of the stress sensor as 0.9 kPa/ $\sqrt{\text{Hz}}$, corresponding to 5.3 nano-Strain. Given the substantial levels of stress that we apply during the tests, this resolution proves adequate for our analysis. In practical stress calibration scenarios, we can take advantage of the slow variations of stress and further improve the resolution by heavily averaging the stress sensor outputs.

3.5 Analysis of Stress Effects

In Chapter 2, we showed that imperfections can affect the mass M and stiffness K matrices of the gyroscope, resulting in mismatched diagonal entries and nonzero off-diagonal coupling terms. Moreover, as we observed in this chapter, stress perturbs the gyroscope's stiffness distribution and contributes to the effects of imperfections on M and K . Such nonidealities manifest as frequency mismatch between the modes ($\omega_1 \neq \omega_2$) and rotated mode shapes $\cos(n\theta - \psi_1)$ and $\sin(n\theta - \psi_2)$ (see Section 2.5 and Figure 2.7). We recall that in the principal coordinate system characterized by the transformation matrix

$$T = \begin{bmatrix} \cos(\psi_1) & -\sin(\psi_2) \\ \sin(\psi_1) & \cos(\psi_2) \end{bmatrix}, \quad (3.45)$$

the mass and stiffness matrices take the diagonal forms

$$\begin{aligned} \bar{M} &= T^\top M T = \begin{bmatrix} \bar{m}_1 & 0 \\ 0 & \bar{m}_2 \end{bmatrix}, \\ \bar{K} &= T^\top K T = \begin{bmatrix} \bar{k}_1 & 0 \\ 0 & \bar{k}_2 \end{bmatrix}, \end{aligned} \quad (3.46)$$

and the natural frequencies are give by $\omega_i = \sqrt{\frac{\bar{k}_i}{\bar{m}_i}}$, $i = 1, 2$.

3.5.1 Variations of vibrational properties

In the absence of stress, the total stiffness matrix is

$$K_0 = K_m - K_{e,0}, \quad (3.47)$$

where $K_{e,0} \in \mathbb{R}^{2 \times 2}$ is the initial electrostatic stiffness matrix. The pair (M, K_0) is characterized by the frequencies $\omega_{1,0}, \omega_{2,0}$ and mode shape rotation angles $\psi_{1,0}, \psi_{2,0}$. In the presence of stress, the stiffness variation $\delta K = K_s - (K_e - K_{e,0})$ acts as a small perturbation of the initial stiffness matrix, i.e., $K_0 \mapsto K_0 + \delta K$. We can therefore perform an eigenvalue perturbation analysis (see Appendix B.2) to approximate the variations of the vibrational properties:

1. Eigenvalue variations, $\omega_{i,0} \mapsto \omega_{i,0} + \delta\omega_i$:

$$\delta\omega_i \approx \frac{\delta\bar{k}_{ii}}{2\bar{m}_i\omega_i}, \quad i = 1, 2; \quad (3.48)$$

we note that the initial frequency split between the modes, $\Delta\omega_0 := \omega_{1,0} - \omega_{2,0}$ is perturbed as well:

$$\begin{aligned} \Delta\omega_0 &\mapsto \Delta\omega_0 + \delta\omega_1 - \delta\omega_2, \\ \frac{\delta\omega_1 - \delta\omega_2}{\omega_{1,0}} &\approx \frac{\delta\bar{k}_{11} - \delta\bar{k}_{22}}{2\bar{m}_1\omega_{1,0}^2} = \frac{\delta k_{11} - \delta k_{22}}{4\bar{k}_{11}} (\cos(2\psi_{1,0}) + \cos(2\psi_{2,0})) \\ &\quad + \frac{\delta k_{12}}{2\bar{k}_{11}} (\sin(2\psi_{1,0}) + \sin(2\psi_{2,0})), \end{aligned} \quad (3.49)$$

where $\bar{k}_1 = \bar{m}_1\omega_{1,0}^2$.

2. Eigenvector variations, $\psi_{i,0} \mapsto \psi_{i,0} + \delta\psi_i$:

$$\delta\psi_i \approx \frac{\delta\bar{k}_{12}}{\bar{m}_i(\omega_{1,0}^2 - \omega_{2,0}^2)}, \quad i = 1, 2. \quad (3.50)$$

In Equations (3.48) and (3.50), $\delta\bar{k}_{ij}$ are the components of δK in the principal coordinate system:

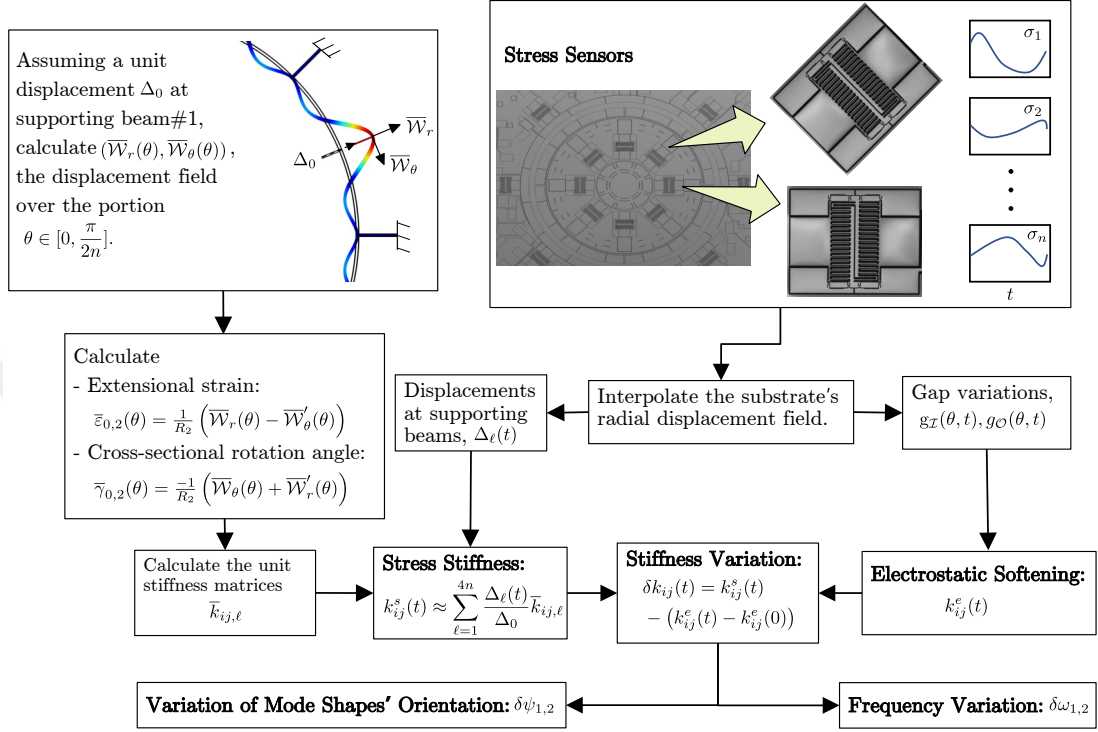


Figure 3.11: Flowchart of calculating stress stiffness, electrostatic softening, and their effects on vibrational properties using the stress sensors' outputs.

$$\begin{aligned}
\delta \bar{k}_{11} &= \left(\frac{\delta k_{11} + \delta k_{22}}{2} \right) + \left(\frac{\delta k_{11} - \delta k_{22}}{2} \right) \cos(2\psi_{1,0}) + \delta k_{12} \sin(2\psi_{1,0}), \\
\delta \bar{k}_{22} &= \left(\frac{\delta k_{11} + \delta k_{22}}{2} \right) - \left(\frac{\delta k_{11} - \delta k_{22}}{2} \right) \cos(2\psi_{2,0}) - \delta k_{12} \sin(2\psi_{2,0}), \\
\delta \bar{k}_{12} &= \left(\frac{\delta k_{11} + \delta k_{22}}{2} \right) \sin(\psi_{1,0} - \psi_{2,0}) - \left(\frac{\delta k_{11} - \delta k_{22}}{2} \right) \sin(\psi_{1,0} + \psi_{2,0}) \\
&\quad + \delta k_{12} \cos(\psi_{1,0} + \psi_{2,0}).
\end{aligned} \tag{3.51}$$

3.5.2 Quadrature and in-phase errors

The quadrature and in-phase errors depend on mode shape rotation angles, the demodulation angle, and the sense mode's (mechanical) phase (see Section 2.8). As shown by Equations (3.48) and (3.50), stress can affect all these quantities and induce additional errors.

3.6 Results and Discussion

3.6.1 Gyroscope specifications

The geometric parameters of the VRG are given in Table 2.1. We measured the frequencies and quality factors of the drive ($\cos(2\theta)$) and sense ($\sin(2\theta)$) modes as 58.919 kHz, 32.42×10^3 and 58.858 kHz, 31.54×10^3 , respectively. The gyroscope has an uncompensated quadrature error of $-2125^\circ/\text{s}$. Using the method proposed in Section 2.7.2.2, we estimated the mode shape rotation angle to be -2.4° .

3.6.2 Simulation setup

We built a simulation model in MATLAB that uses the recorded outputs of the stress sensors as the input to interpolate the substrate's strain and displacement fields. The simulation model follows the flowchart in Figure 3.11, which explains the key steps of our modeling, and implements Equations (3.48), (3.50), and (2.84). We replaced the missing data of the nonfunctional stress sensor at 0° by a cubic interpolation of the functioning stress sensors along the $\theta = 0$ axis. We used the first three harmonics $N_h = 3$ in Equation (3.44) as the basis functions for strain interpolation.

3.6.3 Comparison against FEM

We first assess the accuracy of our model in predicting frequency variations against FEM. For this, we performed a prestressed modal analysis in COMSOL Multiphysics (free tetrahedral elements with quadratic serendipity shape functions), where all supporting beams undergo the same displacement as visualized in Figure 3.13. Figure 3.13 shows the comparative graphs; the analytical results have a (least squares) slope of $\approx 0.7147 \text{ Hz/nm}$, larger than $\approx 0.5682 \text{ Hz/nm}$ of the FEM. This discrepancy is attributable to the analytical model's idealized

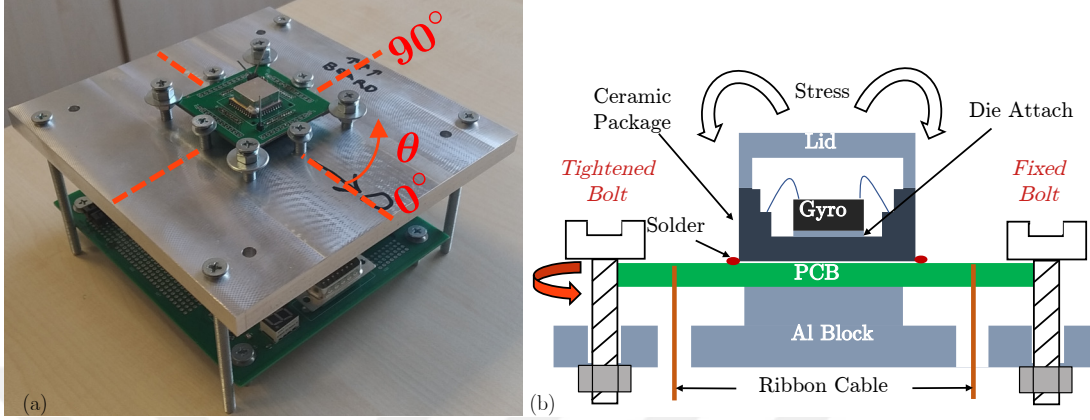


Figure 3.12: Stress test setup (a), and concept (b).

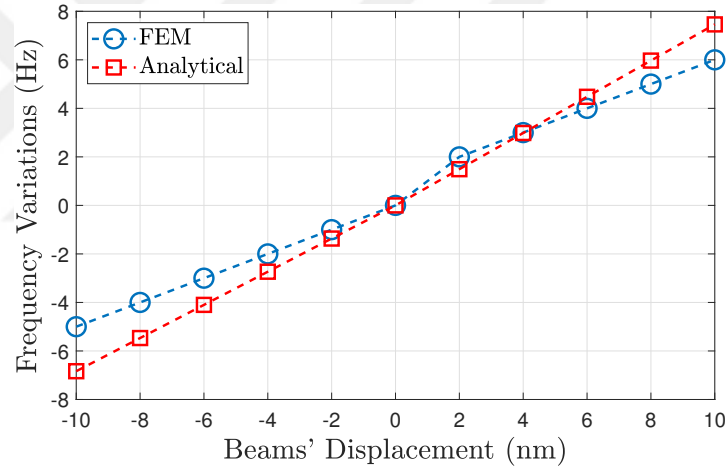


Figure 3.13: Frequency variations for different displacements of supporting beams; analytical versus FEM.

boundary conditions that lead to higher stiffness estimates. Moreover, in our model, the beams' displacements coincide with the vibration of the gyroscope, whereas in the prestressed analysis, these displacements precede the vibration. We note that, unlike our model, these FEM simulations do not provide the quadrature and in-phase errors.

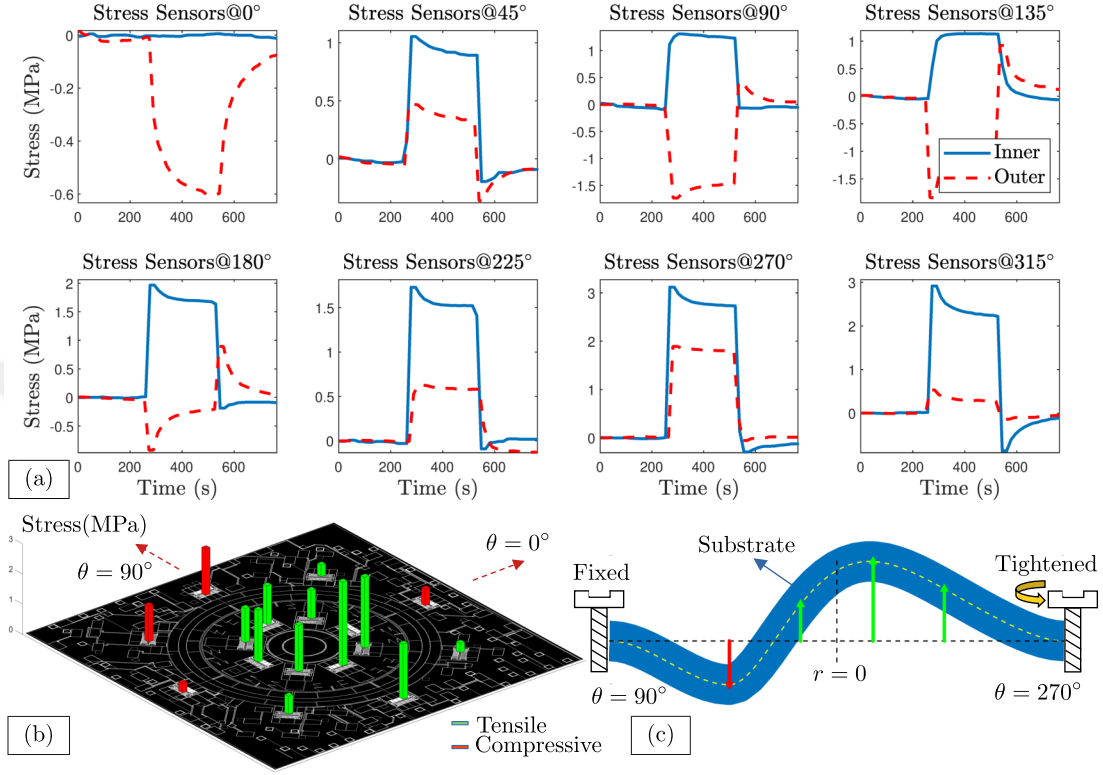


Figure 3.14: Stress sensors' outputs during test#1 along 90° (a); maximal stress state's top (b) and (exaggerated) side views (c); inner stress sensor at 0° was nonfunctional.

3.6.4 Experimental setup

Figure 3.12 shows the setup for the stress tests and explains their underlying concept. The gyro PCB is sitting on top of an aluminum block and is held by a set of bolts. By tightening suitable bolts, we can apply bending loads in selected directions. To achieve this, we keep one of the bolts fixed and fasten the bolt on the opposite side. Ribbon cables connect the gyro PCB to a main PCB that acts as the central signal transmission hub for actuation and readouts. We use two 8×1 multiplexers as the readout apparatus for the 16 stress sensors such that each stress sensor's output is sampled at every 16 s. The high voltage across the gyroscope's structure was set to 40 V during the tests. Further technical details about the experimental setup can be found in [10].

We performed four tests: test#1 along 90° , test#2 along 135° , test#3 along

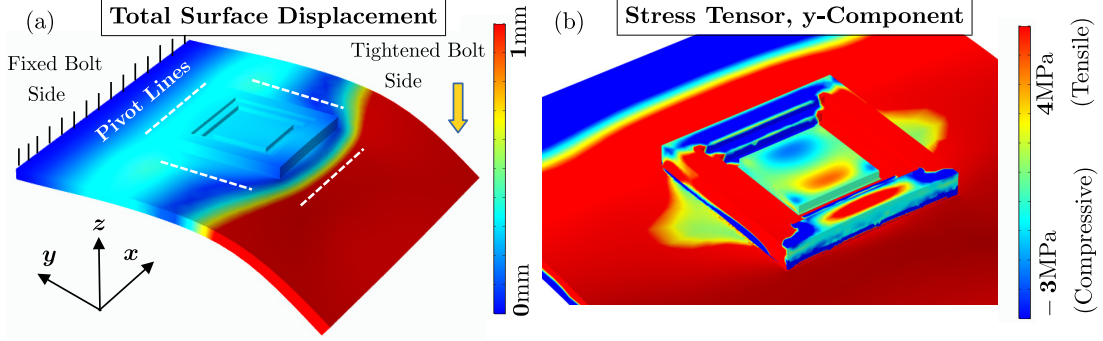


Figure 3.15: FEM model of bending along 90° when the bolt and PCB are not centered, generating both tensile and compressive stresses on the MEMS die; displacement profile (a) and stress distribution (b).

0° direction, and test#4 along 90° . During tests#1-3, the device was in the gyro mode, and we recorded the drive mode's frequency, quadrature, and in-phase errors. During test#4, however, the gyroscope was actuated through the sense mode, and we recorded only the frequency. Figure 3.14 shows the outputs of the stress sensors during test#1, following the usual convention of positive for tensile and negative for compressive stresses. As shown in Figure 3.14-(b) and -(c), the stress is tensile on the tightening side and gradually transitions to compressive on the fixed side. Because we tighten only one of the bolts, the stress profile is not perfectly symmetrical. To further clarify this point, we created a FEM model of our system where the MEMS chip is replaced with a silicon block. The bending was applied only in one direction to reflect the practical scenario. There is a hexagonal bolt under the PCB to generate stresses efficiently; however, the daughter board is susceptible to being misplaced and not perfectly centered with the bolt. Figure 3.15 presents the y -stress when the PCB and bolt are shifted by 5 mm, which leads to both compressive and tensile stresses on the chip. We remark that the symmetry of the bending profile is not critical for our result, and nonuniform stress distribution helps us better evaluate our model's capabilities. We also note that the stress sensor located at 0° was nonfunctional (likely short-circuited due to metal routing issues). The maximum stress we observed was around 3 MPa during test#1.

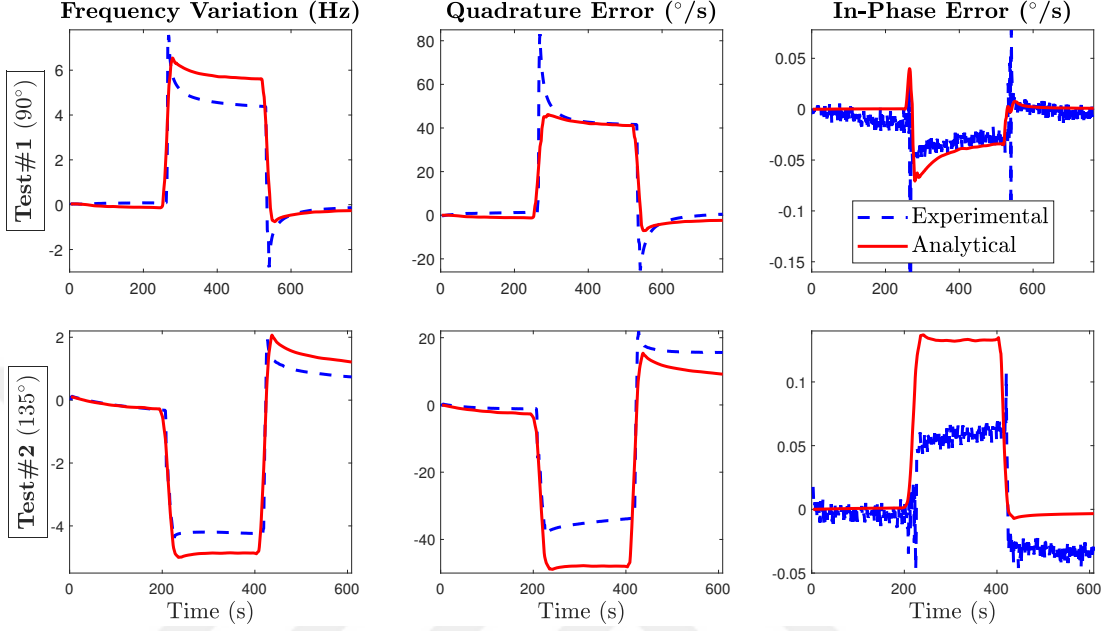


Figure 3.16: Variations of drive mode's frequency, quadrature, and in-phase errors during tests#1 and 2; analytical versus experimental.

3.6.5 Results

Figure 3.16 shows the comparative graphs for the drive mode's frequency variations and quadrature/in-phase errors during tests#1 and 2. The analytical predictions successfully track the corresponding experimental references. We can explain the model's consistency with both increase (test#1) and decrease (test#2) in the frequency by Equation (3.36). When there is a net tensile stress distribution, the supporting beams are mostly pushed outward ($\sum_{\ell=1}^{4n} \Delta_{\ell}(t) > 0$), and we have a hardening effect. On the other hand, a net compressive stress pulls most of the beams inward (e.g., stress relaxation, $\sum_{\ell=1}^{4n} \Delta_{\ell}(t) < 0$), resulting in a softening effect.

Effect of nonfunctional stress sensors. Accurate interpolation of the substrate's strain is critical for the correct evaluation of stress stiffness. To examine the effect of missing stress sensor data at 0° , we did test#3 along 0° direction so that the maximal stress occurs in the direction of the nonfunctional stress sensor. The results in Figure 3.17 suggest that the model still successfully predicts the

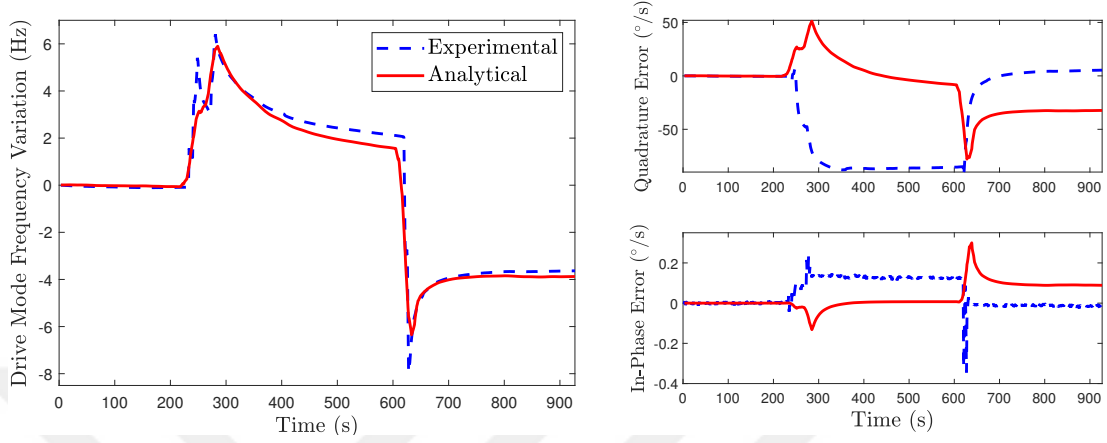


Figure 3.17: Analytical versus experimental results for test#3 (0° direction); maximum stress data are missing because of the nonfunctional stress sensor at 0° .

frequency variation. However, the same does not hold for the quadrature and in-phase errors, implying that the interpolation accuracy is not sufficient to capture the off-diagonal stiffness, δk_{12} .

Stress stiffness versus electrostatic softening. Figure 3.18 compares the stress stiffness and variations of the electrostatic softening, as predicted by our model, during test#1. We observe that the stress stiffness dominates the electrostatic softening by several orders of magnitude in both diagonal and off-diagonal entries. We can explain this by noting that, due to the differential arrangement of the electrodes, the anchors on both inner and outer sides have similar displacements. These results accord with the experimental observation in [10] where reducing the high structural voltage from 50 V to 20 V had a negligible effect on the frequency shift.

Sense mode's frequency. According to Remark 2 and Equation (3.49), we expect the sense mode's frequency to have a similar level of variation to the drive mode's since $\delta \bar{K}_{11} \approx \delta \bar{K}_{22}$ for small mode shape rotation angles. To validate this point, we conducted test#4 along the 90° direction with the PLL around the sense mode. The stress values during this test were comparable to those of test#1 (maximum stress of 2.5 MPa similar to 3 MPa of test#1). Figure 3.19

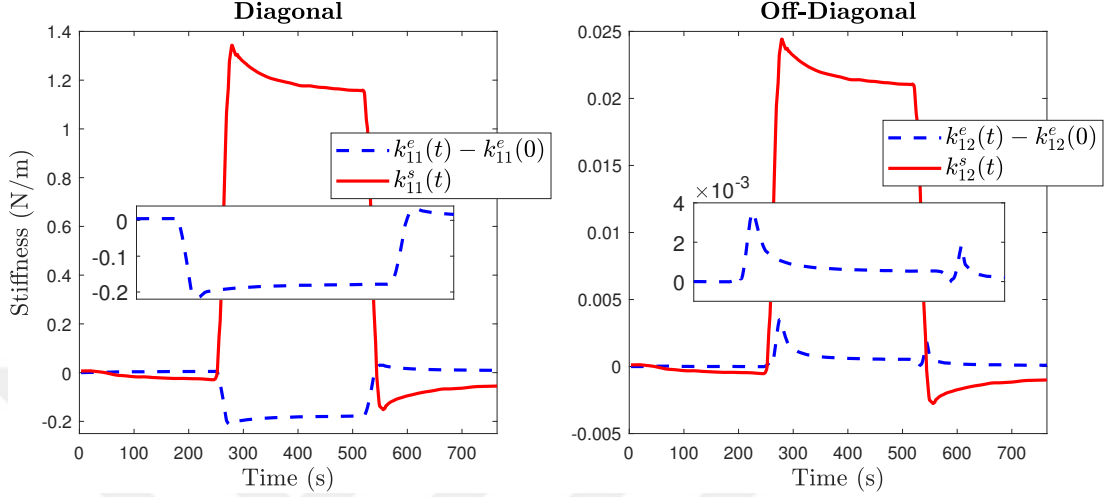


Figure 3.18: Stress stiffness versus variations of electrostatic softening during test#1.

shows that the sense mode's frequency shifts by about 5 Hz, similar to the drive mode's frequency in test#1.

Frequency split. Based on Equation (3.49), Figure 3.20 shows that the initial frequency split of 61 Hz is perturbed by ≈ 0.1 -0.2 Hz during the stress tests. Variations of frequency split contribute to the scale factor drift, which, according to our analysis, could be calibrated with stress.

Error sources. Based on the presented results, we can categorize the error sources of our modeling as follows.

1. *Errors at the modeling level:* Apart from the approximations inherent in Euler-Bernoulli beam theory, the idealized boundary conditions we assume in deriving our mode shapes and displacement fields tend to produce a *stiffer* model than the actual structure.
2. *Stress measurement errors:* Missing or faulty stress data directly affect the interpolation accuracy, calculated radial displacements of the supporting beams, and the resulting stress stiffness matrix. Our main fabrication-related challenge is the low yield due to process nonidealities, affecting the stress sensors' balance and sensitivity. Overall, stress measurement is the main deciding factor for our model's accuracy.

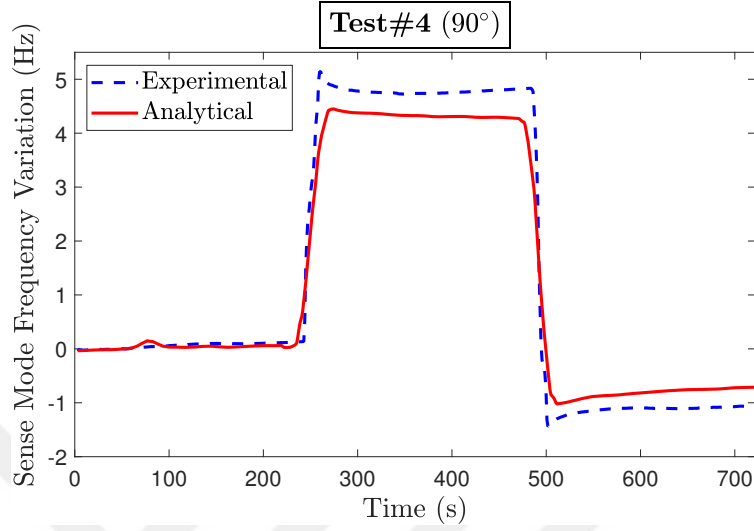


Figure 3.19: Sense mode's frequency variation during test#4; analytical versus experimental.

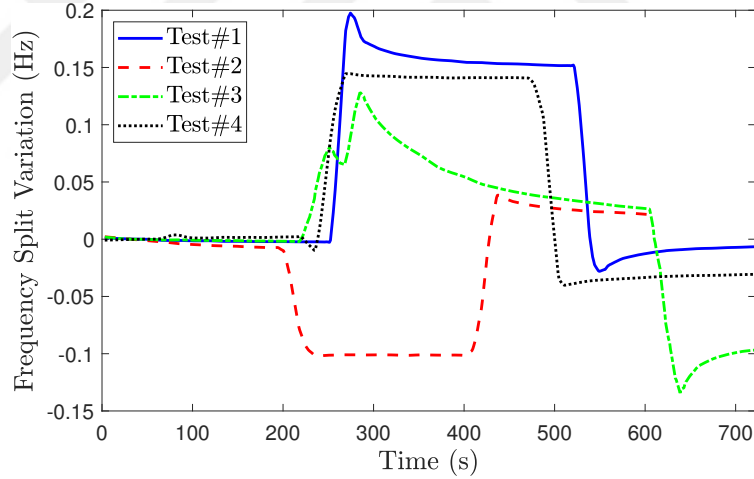


Figure 3.20: Variations of frequency split, $\omega_1 - \omega_2$ during stress tests.

3.6.6 Supplementary data

To provide additional supporting data for our modeling, we present partial results from tests done with another 57 kHz, 3.2 mm diameter gyroscope. As shown in Figure 3.21, the level of agreement between the experimental and analytical results parallels the first device. We note that the second gyroscope had four nonfunctional stress sensors at 90° (both inner and outer), 180° (outer), and 225° (inner).

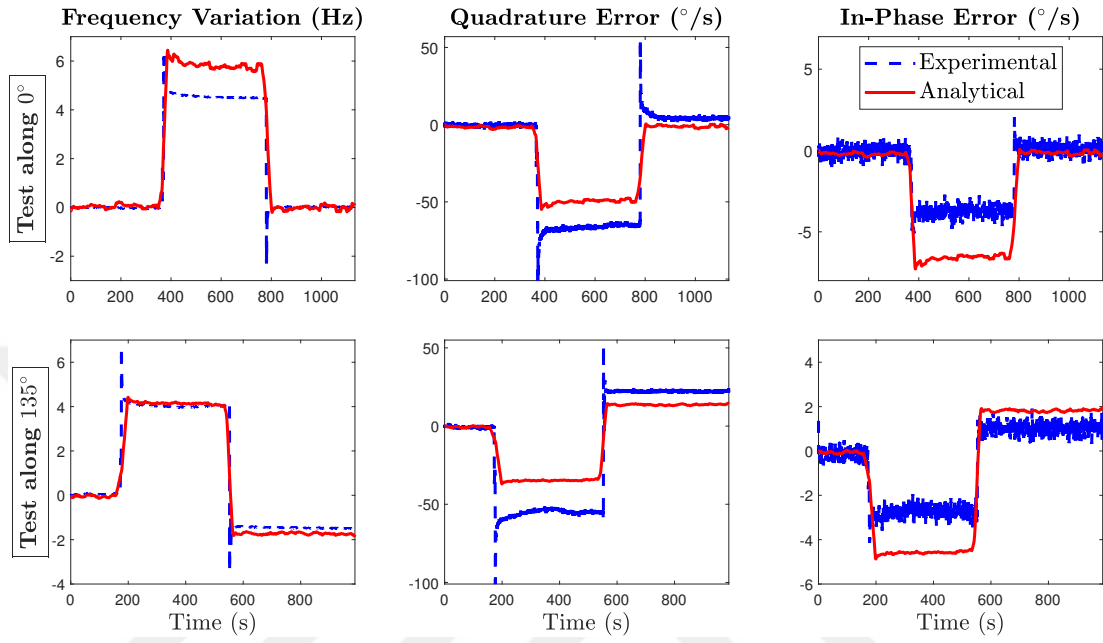


Figure 3.21: Analytical versus experimental results for the second gyroscope.

Chapter 4

Modeling Temperature Effects in Circular MEMS Gyroscopes

4.1 Introduction

MEMS devices are more prone to temperature in comparison with their macro counterparts. Temperature has a prominent role in the performance of MEMS gyroscopes due to its correlation with scale factor and bias drift. Therefore, understanding the effects of temperature is critical for the performance improvement of these sensors. Following this motivation, we performed several tests on our ring gyroscopes by exposing them to controlled temperature cycles. These experiments led to two key observations:

1. According to the well-known ~ 60 ppm/ $^{\circ}\text{C}$ temperature dependency of silicon's Young's modulus [72], we expected to see a temperature coefficient of frequency (TCF) of ~ 30 ppm/ $^{\circ}\text{C}$; however, we measured very different values such as -10 ppm/ $^{\circ}\text{C}$ and -14 ppm/ $^{\circ}\text{C}$.
2. Over temperature cycles, hysteresis loops formed in the frequency-temperature plots, which point to some irreversible residual effects.

These above observations reveal thermal stresses as a major mechanism of temperature effects. In the previous chapter, we highlighted the potential of stress calibration for drift compensation and developed a model for the effects of stress. In practice, the source of stresses in MEMS gyroscopes is usually thermal rather than mechanical. Accordingly, our goal in this chapter is to extend the analytical modeling’s ambit to encompass the temperature effects and explain the experimental observations. This attempt will serve as a basis for further exploration of the temperature dependency of long-term bias and scale factor drift.

Related literature. In [62], a drift compensation was developed for a high quality factor quadruple mass MEMS gyroscope, which obviated the need for a temperature sensor by taking advantage of the drive mode’s linear temperature dependence. For a NEMS-based gyroscope, it was shown in [63] that the drive and sense modes’ TCF mismatch is responsible for the scale factor drift. The temperature dependency of drift for a tuning fork gyroscope was investigated in [73] based on temperature chamber tests, and material properties and electronics parameters variations were introduced as the main factors. For a JPL/Boeing Gyroscope, it was shown in [74] that drive/sense frequencies and quality factors have an exponential response—with a considerable time lag—to slow step-like temperature changes. A pole-zero cancellation controller was developed in [75] to guarantee above 90 Hz bandwidth for a dual-mass MEMS gyroscope against temperature-dependent frequency variations. The bias drift behavior of a group of consumer-grade MEMS gyroscopes was studied in [76] over -40°C to 85°C , taking into account the temperature-dependency of quadrature and sense mode phase errors. Mechanical stresses were cited as the source of discrepancy between the experimental and predicted quadrature errors.

Analytical models for temperature effects in MEMS gyroscopes, based on the underlying solid mechanics, are absent in the literature. Moreover, the emphasis is often on the temperature dependency of the material properties, and the role and significance of thermal stresses are not meticulously explored.

Contribution. We set out an analytical model for the effects of temperature on the stiffness distribution of our double-ring VRG described in Chapter 3. The

modeling's departure point is that we can delineate two mechanisms for such effects:

1. Change in material properties, especially the ~ 30 ppm/ $^{\circ}\text{C}$ dependency of Young's modulus;
2. Thermal stresses that arise from 1) conflict between thermal deformations and structural constraints, 2) CTE mismatches at the silicon-glass and die attachment interfaces.

Our modeling adheres to the linear theory of thermoelasticity where strains are the linear sum of a thermal and a mechanical part [77]. The innovative features of our approach are as follows.

- We present an Euler-Bernoulli-based geometrically nonlinear formulation of the gyroscope's structure. In the same line as the stress effect, Green-Lagrange strains should be considered to reveal the stiffness induced by extensional thermal stresses. We proceed by decomposing the total displacement field into a thermal component that is superposed on the nominal vibration of the gyroscope. We provide a potential function that accounts for initial stresses and allows us to calculate the stiffness matrix associated with this thermal displacement.
- We further prove that the thermal displacement field is the superposition of two components:
 1. The displacement field resulting from the thermal load under homogeneous boundary conditions (i.e., fixed supporting beams). We refer to the stiffness matrix induced by this component as *thermal stiffness*.
 2. The displacement field that is generated by the nonhomogeneous boundary conditions (i.e., displacements of supporting beams) under zero thermal strain. We refer to the stiffness matrix corresponding to this component as *stress stiffness*.

We obtain these displacement fields by solving suitable QP problems obtained via the Ritz method. The thermal and stress stiffness matrices appear as additive perturbations of the gyroscope's nominal stiffness.

- The on-chip stress sensors measure the mechanical strains that, in combination with thermal strains, enable us to interpolate the substrate's strain field. We use this interpolation to calculate the displacements of the supporting beams and changes in the electrostatic gap. Furthermore, we show the mechanical strains contain residual effects over temperature cycles that account for frequency-temperature hysteresis.

We establish the efficacy of our model by comparing its estimated frequencies with the experimental data. The model addresses the experimental observations by successfully predicting the frequency variations, formation of frequency-temperature hysteresis loops, and TCF. Additionally, the model provides insight into the behavior of different types of stiffness (nominal, thermal, stress, and electrostatic) in terms of their contribution to the gyroscope's frequency changes.

Notation. We will use the same mathematical notation as Chapters 2 and 3.

4.2 Modeling Basics

4.2.1 Solid-mechanical formulation

We follow the same modeling setting and nomenclature for the components of the gyroscope's structure as Chapter 3. We briefly review the displacement-strain formulation of beams and rings.

Rings. Ring# i 's displacement field in the polar (r, θ) -coordinates is given by

$$\begin{aligned}\bar{u}_{r,i}(\xi_i, \theta, t) &= u_{r,i}(\theta, t), \\ \bar{u}_{\theta,i}(\xi_i, \theta, t) &= u_{\theta,i}(\theta, t) + \xi_i \gamma_i(\theta, t),\end{aligned}\tag{4.1}$$

where $\xi_i = r - R_i$ is the radial coordinate across the ring, $(u_{r,i}(\theta, t), u_{\theta,i}(\theta, t))$ is the displacement field at the ring's centerline $\xi_i = 0$, and $\gamma_i(\theta, t) = \frac{1}{R_i} \left(u_{\theta,i}(\theta, t) - \frac{\partial u_{r,i}}{\partial \theta}(\theta, t) \right)$ is the cross-section rotation angle (see Figure 3.4-(a)). The nonlinear extensional strain in Ring# i is

$$\mathcal{E}_{\theta\theta,i} = \varepsilon_{\theta\theta,i} + \eta_{\theta\theta,i},\tag{4.2}$$

$$\begin{aligned}
\varepsilon_{\theta\theta,i} &= \varepsilon_{0,i} + \xi_i \kappa_i, \\
\eta_{\theta\theta,i} &= \frac{1}{2} \left((\varepsilon_{0,i} + \xi_i \kappa_i)^2 + \left(1 + \frac{\xi_i}{R_i} \right)^2 \gamma_i^2 \right), \\
\varepsilon_{0,i}(\theta, t) &= \frac{1}{R_i} \left(u_{r,i}(\theta, t) + \frac{\partial u_{\theta,i}}{\partial \theta}(\theta, t) \right), \\
\kappa_i(\theta, t) &= \frac{1}{R_i} \frac{\partial \gamma_i}{\partial \theta}(\theta, t),
\end{aligned}$$

where $\varepsilon_{0,i}$ and κ_i are the extensional strain and the curvature change at the ring's centerline, respectively.

Beams. Connecting (supporting) beam# i 's displacement field in the Cartesian (x, y) -coordinates is given by

$$\begin{aligned}
u_{x,i}^{c(s)}(x, y, t) &= u_{0,i}^{c(s)}(x, t) - y \frac{\partial \vartheta_{c(s),i}}{\partial x}(x, t), \\
u_{y,i}^{c(s)}(x, y, t) &= \vartheta_{c(s),i}(x, t),
\end{aligned} \tag{4.3}$$

where $\left(u_{0,i}^{c(s)}(x, t), \vartheta_{c(s),i}(x, t) \right)$ is the displacement field at the beam's centerline $y = 0$ (Figure 3.4-(b)). The nonlinear extensional strain in the beam is

$$\begin{aligned}
\mathcal{E}_{xx,i}^{c(s)} &= \varepsilon_{xx,i}^{c(s)} + \eta_{xx,i}^{c(s)}, \\
\varepsilon_{xx,i}^{c(s)} &= \varepsilon_{0,i}^{c(s)}(x, t) - y \kappa_{c(s),i}(x, t), \\
\eta_{xx,i}^{c(s)} &= \frac{1}{2} \left(\varepsilon_{0,i}^{c(s)}(x, t) - y \kappa_{c(s),i}(x, t) \right)^2, \\
\varepsilon_{0,i}^{c(s)}(x, t) &= \frac{\partial u_{0,i}^{c(s)}}{\partial x}(x, t), \\
\kappa_{c(s),i}(x, t) &= \frac{\partial^2 \vartheta_{c(s),i}}{\partial x^2}(x, t),
\end{aligned} \tag{4.4}$$

where $\varepsilon_{0,i}^{c(s)}$ and $\kappa_{c(s),i}$ are the extensional strain and the curvature at the beam's centerline, respectively.

4.2.2 Main idea

Similar to stress effects in Chapter 3, our goal is to derive the change in the stiffness matrix under the temperature variations. Toward this end, we decompose

the general displacement field $\mathbf{u} \in \mathbb{R}^2$ of the gyroscope's structure as

$$\mathbf{u} \approx \mathbf{u}_0 + \mathbf{u}_n, \quad (4.5)$$

where

- $\mathbf{u}_0 \in \mathbb{R}^2$ is the displacement field due to the temperature variations. This component of $\mathbf{u}$ evolves in a slow time scale determined by the temperature cycles (e.g., in the order of minutes or hours).
- $\mathbf{u}_n \in \mathbb{R}^2$ is the displacement field under the nominal vibration of the gyroscope, taking place in the fast time scale defined by the operation frequency (in the order of μs).

The above slow-fast time scale separation indicates that $\mathbf{u}_0$ can be conceived as the equilibrium displacement field of a quasistatic thermal loading. We then obtain the resulting stress field $\boldsymbol{\sigma}$, which we can treat as a *prestress* in conjunction with the gyroscope's nominal vibration. In a variational framework, we consider the modified potential function [78]

$$\frac{d\mathcal{U}}{d\mathcal{V}} = \frac{1}{2}E\mathcal{E}_n^2 + \boldsymbol{\sigma}\mathcal{E}_n \quad (4.6)$$

($d\mathcal{V}$ is the differential volume element) to describe the interaction between $\mathbf{u}_0$ and $\mathbf{u}_n$ and calculate the induced stiffness matrix k_{ij}^σ :

$$\frac{dk_{ij}^\sigma}{d\mathcal{V}} = \frac{\partial^2}{\partial Q_i \partial Q_j} \left(\frac{d\mathcal{U}}{d\mathcal{V}} - \frac{d\mathcal{U}}{d\mathcal{V}} \Big|_{\sigma=0} \right) \Big|_{Q_1, Q_2=0}, \quad i, j \in \{1, 2\}. \quad (4.7)$$

In Equation (4.6), it is essential to use the Green-Lagrange strain $\mathcal{E}_n$, associated with $\mathbf{u}_n$, to capture all the second-order terms that produce stiffness (refer to the discussion of geometric nonlinearity in Chapter 3). In the following, we provide expanded, simplified versions of (4.7) for rings and beams.

- **Ring.** Consider a narrow ring with the mean radius R , width w , cross-section area A and second moment of area I whose inextensible vibration in the polar (r, θ) -coordinates is governed by

$$\mathbf{u}_n = \begin{bmatrix} f_1'(\theta) \\ -f_1(\theta) \end{bmatrix} Q_1(t) + \begin{bmatrix} f_2'(\theta) \\ -f_2(\theta) \end{bmatrix} Q_2(t), \quad (4.8)$$

where f_i and Q_i , $i = 1, 2$, are mode shapes and generalized coordinates, respectively. An initial stress of the form

$$\sigma = E (\epsilon_0 + \xi \kappa_0), \quad \xi \in [-\frac{w}{2}, \frac{w}{2}], \quad (4.9)$$

induces the following stiffness matrix:

$$\begin{aligned} k_{ij}^\sigma = & \frac{EA}{R} \int_0^{2\pi} \epsilon_0 F_i F_j d\theta + \frac{EI}{R^3} \int_0^{2\pi} \epsilon_0 (F_i F_j + F_i' F_j') d\theta \\ & + \frac{2EI}{R^2} \int_0^{2\pi} \kappa_0 F_i F_j d\theta, \quad i, j \in \{1, 2\}, \end{aligned} \quad (4.10)$$

where $F_i = f_i + f_i''$.

- **Beam.** Consider a narrow beam with the length l , width w , cross-section area A , and second moment of area I whose (in tandem with the ring) vibration in the Cartesian (x, y) -coordinates is given by

$$\mathbf{u}_n = \begin{bmatrix} -y\varpi_1'(x) \\ \varpi_1(x) \end{bmatrix} Q_1(t) + \begin{bmatrix} -y\varpi_2'(x) \\ \varpi_2(x) \end{bmatrix} Q_2(t), \quad (4.11)$$

where ϖ_i are the mode shapes corresponding to the generalized coordinates Q_i , $i = 1, 2$. Similar to (4.9), the initial stress

$$\sigma = E (\epsilon_0 - y\kappa_0), \quad y \in [-\frac{w}{2}, \frac{w}{2}], \quad (4.12)$$

results in the following stiffness matrix:

$$k_{ij}^\sigma = EI \int_0^l \epsilon_0 \varpi_i'' \varpi_j'' dx, \quad i, j \in \{1, 2\}. \quad (4.13)$$

We note that ϵ_0 and κ_0 are the initial stress's extensional and curvature (bending) components, respectively. In Chapter 2, we calculated the mode shapes that parametrize a ring gyroscope's nominal displacement field $\mathbf{u}_n$. Therefore, the following section focuses on $\mathbf{u}_0$.

4.2.3 Temperature-dependent material properties

Based on the widely used approximation $\frac{dE}{dT} \approx -60 \text{ ppm}/^\circ\text{C}$ [72], we consider the following model for the Young's modulus:

$$E(T) \approx 170 (1 - 60 \times 10^{-6} (T - 20)) \text{ [GPa]}, \quad (4.14)$$

where T is in $^{\circ}\text{C}$. According to the experimental results in [79], we use the following polynomial approximation for the silicon's CTE valid for $T \in [20^{\circ}\text{C}, 100^{\circ}\text{C}]$:

$$\alpha(T) \approx 2.3332 + 9.2618 \times 10^{-3}T - 1.8901 \times 10^{-5}T^2 \text{ [ppm}/^{\circ}\text{C}]. \quad (4.15)$$

We consider the glass substrate's CTE, which is slightly larger than silicon's, to be 3.25 ppm/ $^{\circ}\text{C}$ [80].

4.3 Thermal Displacement Field

Let T_0 denote the reference, stress-free temperature. We aim to calculate the displacement fields of the rings and beams under the thermal load $\Delta T = T - T_0$, which we assume to be uniform across the structure. In this case, the beams will have pure axial displacements, and hence, the y -components of their displacement fields vanish, $\vartheta_{c(s),i} \equiv 0$. The ring-beam boundary conditions are summarized as follows.

1. Connecting Beam# i , $i = 1 : 4n$:

$$\begin{aligned} @x = 0 : & \ u_{0,i}^c(0) = u_{r,2}(\theta_{c,i}), \ u_{\theta,2}(\theta_{c,i}) = 0, \ \gamma_2(\theta_{c,i}) = 0; \\ @x = l_c : & \ u_{0,i}^c(l_c) = u_{r,1}(\theta_{c,i}), \ u_{\theta,1}(\theta_{c,i}) = 0, \ \gamma_1(\theta_{c,i}) = 0. \end{aligned} \quad (4.16)$$

2. Supporting Beam# i , $i = 1 : 4n$:

$$\begin{aligned} @x = 0 : & \ u_{0,i}^s(0) = \Delta_i; \\ @x = l_s : & \ u_{0,i}^s(l_s) = u_{r,2}(\theta_{s,i}), \ u_{\theta,2}(\theta_{s,i}) = 0, \ \gamma_2(\theta_{s,i}) = 0. \end{aligned} \quad (4.17)$$

Here, Δ_i is the displacement at the base of the supporting beam# i caused by

- silicon layer's thermal expansion,
- mechanical strains applied to the silicon by the glass substrate due to the CTE difference.

We will calculate Δ_i using the stress sensors' outputs.

The thermoelastic potential function of Ring# i is given by [81, Chapter 6]

$$\frac{d\mathcal{U}_i}{d\mathcal{V}_i} = \frac{1}{2}E(T)\varepsilon_{\theta\theta,i}^2 - E(T)\varepsilon_T\varepsilon_{\theta\theta,i}, \quad (4.18)$$

where $\varepsilon_T = \alpha(T)\Delta T$ is the thermal strain, and $d\mathcal{V} = hR_i d\theta d\xi_i$ is the differential volume element with h being the structural thickness. After carrying out the volume integration and using (4.2), we obtain

$$\mathcal{U}_i = \frac{1}{2}E(T)A_iR_i \int_0^{2\pi} (\varepsilon_{0,i}^2 - 2\varepsilon_T\varepsilon_{0,i}) d\theta + \frac{1}{2}E(T)I_iR_i \int_0^{2\pi} \kappa_i^2 d\theta. \quad (4.19)$$

The constitutive equation for the extensional stress $\sigma_{\theta\theta,i}$ associated with the strain $\varepsilon_{\theta\theta,i}$ is

$$\sigma_{\theta\theta,i} = \frac{\partial}{\partial \varepsilon_{\theta\theta,i}} \left(\frac{d\mathcal{U}_i}{d\mathcal{V}_i} \right) = E(T) (\varepsilon_{\theta\theta,i} - \varepsilon_T). \quad (4.20)$$

Similarly, connecting (supporting) beam# i 's potential function is given by

$$\frac{d\mathcal{U}_{c(s),i}}{d\mathcal{V}_{c(s),i}} = \frac{1}{2}E(T)\varepsilon_{xx,i}^{c(s)2} - E(T)\varepsilon_T\varepsilon_{xx,i}^{c(s)}, \quad (4.21)$$

where $d\mathcal{V}_{c(s),i} = hdx dy$ is the differential volume element. By using (4.4), we have

$$\begin{aligned} \mathcal{U}_{c(s),i} = & \frac{1}{2}E(T)A_{c(s)} \int_0^{l_{c(s)}} \left(\varepsilon_{0,i}^{c(s)2} - 2\varepsilon_T\varepsilon_{0,i}^{c(s)} \right) dx \\ & + \frac{1}{2}E(T)I_{c(s),i} \int_0^{l_{c(s)}} \kappa_{c(s),i}^2 dx, \end{aligned} \quad (4.22)$$

where the last term vanishes in the absence of bending. The constitutive equation for the axial stress $\sigma_{xx,i}^{c(s)}$ associated with the strain $\varepsilon_{xx,i}^{c(s)}$ in connecting (supporting) beam# i is

$$\sigma_{xx,i}^{c(s)} = \frac{\partial}{\partial \varepsilon_{xx,i}^{c(s)}} \left(\frac{d\mathcal{U}_{c(s),i}}{d\mathcal{V}_{c(s),i}} \right) = E(T) (\varepsilon_{xx,i}^{c(s)} - \varepsilon_T). \quad (4.23)$$

It is noteworthy that in Equations (4.20) and (4.23), only the mechanical part of the strain (total minus the thermal) produces stress. Based on (4.19) and (4.22), the total potential functions of the entire structure is

$$\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2 + \sum_{i=1}^{4n} (\mathcal{U}_{c,i} + \mathcal{U}_{s,i}) \quad (4.24)$$

According to the principle of minimum potential energy, the actual displacement fields minimize (4.24) subject to the constraints (4.16) and (4.17). We break down this problem into two subproblems that allow us to examine the effects of thermal load and nonhomogeneous boundary conditions separately.

Proposition 4.1. *The displacement field minimizing (4.24) subject to (4.16) and (4.17) is the superposition of two components:*

1. *The displacement field that minimizes (4.24) subject to (4.16) and (4.17) under $\Delta_i = 0$; that is, when we have the thermal load with homogeneous boundary conditions at fixed supporting beams.*
2. *The displacement field that minimizes (4.24) under $\varepsilon_T = 0$ subject to (4.16) and (4.17); that is, when the thermal load is absent but the supporting beams are subject to nonhomogeneous boundary conditions.*

Proof. See Appendix B.3. □

We will take advantage of the gyroscope's structural symmetry to further simplify the displacement field calculation. For this purpose, we introduce *portion#i*, for $i = 1 : 4n$, that include the segments of Ring# i , $i = 1, 2$, over the angular span $\theta \in [\theta_{c,i}, \theta_{c,i+1}]$, connecting beams# i , $i + 1 \pmod{4n}$, and supporting beam# i . For example, portion#1 for the $n = 2$ design includes the $[0, 45^\circ]$ section of the structure.

4.3.1 Thermal load, homogeneous boundary conditions

Considering the symmetry of the gyroscope's structure, for a uniform temperature change ΔT , the displacement fields of Rings#1, 2 will be $\frac{\pi}{2n}$ -periodic, and connecting/supporting beams# i , $i = 2 : 4n$, will all have the same displacement field as connecting/supporting beams#1. This periodicity reduces the number of required boundary conditions.

Ritz's Method. We first parametrize the displacement fields by suitable basis functions. In selecting the basis functions, we take into account the $\frac{\pi}{2n}$ -periodicity of the rings' radial displacements and its even-symmetry about $\theta = \theta_{s,1}$.

1. Ring# i , $i = 1, 2$:

$$\begin{aligned} u_{\theta,i}(\theta) &\approx \chi_{\theta,i}^\top(\theta) a_{\theta,i}, \\ u_{r,i}(\theta) &\approx \chi_{r,i}^\top(\theta) a_{r,i}, \\ \chi_{\theta,i}(\theta) &= [\sin(4nj\theta)]_{j=1}^{N_i}, \\ \chi_{r,i}(\theta) &= [\cos(4nj\theta)]_{j=0}^{N_i} \text{ for } \theta \in [\theta_{c,1}, \theta_{c,2}]. \end{aligned} \quad (4.25a)$$

2. Connecting beam#1:

$$\begin{aligned} u_{0,1}^c(x) &\approx \chi_c^\top(x) a_c, \\ \chi_c(x) &= [1, x]^\top \text{ for } x \in [0, l_c]. \end{aligned} \quad (4.25b)$$

3. Supporting beam#1:

$$\begin{aligned} u_{0,1}^s(x) &\approx \chi_s^\top(x) a_s, \\ \chi_s(x) &= [1, x]^\top \text{ for } x \in [0, l_s]. \end{aligned} \quad (4.25c)$$

Let $N = 2N_1 + 2N_2 + 6$ be the total number of basis functions, and define the augmented coefficients vector as

$$\underline{a} := [a_{\theta,1}^\top, a_{r,1}^\top, a_{\theta,2}^\top, a_{r,2}^\top, a_c^\top, a_s^\top]^\top \in \mathbb{R}^N. \quad (4.26)$$

By substituting the parametrizations (4.25a)-(4.25c) into (4.24), we obtain the equivalent QP of the principle of minimum potential energy:

$$\underline{a}^* := \arg \min_{\underline{a} \in \mathbb{R}^N} \frac{1}{2} \underline{a}^\top P \underline{a} + \varepsilon_T q^\top \underline{a} \left(\approx \frac{1}{4n} \mathcal{U} \right) \quad \text{subject to } A \underline{a} = 0_N, \quad (4.27a)$$

where

$$\begin{aligned} P &= \text{diag}(P_1, P_2, P_c, P_s), \\ q &= [q_1^\top, q_2^\top, q_c^\top, q_s^\top]^\top, \end{aligned}$$

$$\begin{aligned}
P_i &= \frac{E(T)A_i}{R_i} \int_{\theta_{c,1}}^{\theta_{c,2}} \begin{bmatrix} \chi'_{\theta,i} \chi_{\theta,i}^\top & \chi'_{\theta,i} \chi_{r,i}^\top \\ \star & \chi_{r,i} \chi_{r,i}^\top \end{bmatrix} d\theta + \frac{E(T)I_i}{R_i^3} \int_{\theta_{c,1}}^{\theta_{c,2}} \begin{bmatrix} \chi'_{\theta,i} \chi_{\theta,i}^\top & -\chi'_{\theta,i} \chi_{r,i}^{\prime\prime\top} \\ \star & \chi_{r,i}^{\prime\prime} \chi_{r,i}^\top \end{bmatrix} d\theta, \\
q_i &= -E(T)A_i \int_{\theta_{c,1}}^{\theta_{c,2}} \begin{bmatrix} \chi'_{\theta,i} \\ \chi_{r,i} \end{bmatrix} d\theta, \quad i = 1, 2, \\
P_{c(s)} &= E(T)A_{c(s)} \int_0^{l_{c(s)}} \chi'_{c(s)} \chi_{c(s)}^{\prime\top} dx, \quad q_{c(s)} = -E(T)A_{c(s)} \int_0^{l_{c(s)}} \chi'_{c(s)} dx, \\
A &= \begin{bmatrix} \chi_{r,2}^\top(\theta_{c,1}) \frac{\partial \underline{a}}{\partial a_{r,2}} - \chi_c^\top(0) \frac{\partial \underline{a}}{\partial a_c} \\ \chi_{r,1}^\top(\theta_{c,1}) \frac{\partial \underline{a}}{\partial a_{r,1}} - \chi_c^\top(l_c) \frac{\partial \underline{a}}{\partial a_c} \\ \chi_s^\top(0) \frac{\partial \underline{a}}{\partial a_s} \\ \chi_{r,2}^\top(\theta_{s,1}) \frac{\partial \underline{a}}{\partial a_{r,2}} - \chi_s^\top(l_s) \frac{\partial \underline{a}}{\partial a_s} \end{bmatrix}. \tag{4.27b}
\end{aligned}$$

Here, $\frac{\partial \underline{a}}{\partial a_{r,1}}$ is the Jacobian matrix of $\underline{a}$ with respect to $a_{r,1}$ and so on. We point out that the boundary conditions related to $u_{\theta,i}$ and $u'_{r,i}$ are automatically satisfied by the particular choice (4.25a) of basis functions.

Proposition 4.2. *Let $\underline{a}^*(E(T), \varepsilon_T)$ denote the solution of the QP problem (4.27a) for Young's modulus $E(T)$ and thermal strain ε_T ; then*

$$\underline{a}^*(E(T), \varepsilon_T) = \frac{\varepsilon_T}{\hat{\varepsilon}_T} \underline{a}^*(E(\hat{T}), \hat{\varepsilon}_T), \tag{4.28}$$

where $\hat{T}$ is a preselected temperature.

Proof. First, we note that QP (4.27a) is homogeneous in Young's modulus since $E(T)$ multiplies the entire cost function. Then, the proof follows as a special case of Lemma B.1 with $\zeta_1 = \varepsilon_T$ and $\zeta_2 = 0$. $\square$

The significant implication of Proposition 4.2 is that we do not need to solve different QP problems as the temperature changes. Instead, we solve the QP problem (4.27a) once for a particular temperature $\hat{T}$ with the thermal strain $\hat{\varepsilon}_T$ and then use (4.28) to obtain the solution at any desired temperature by simply scaling its ε_T .

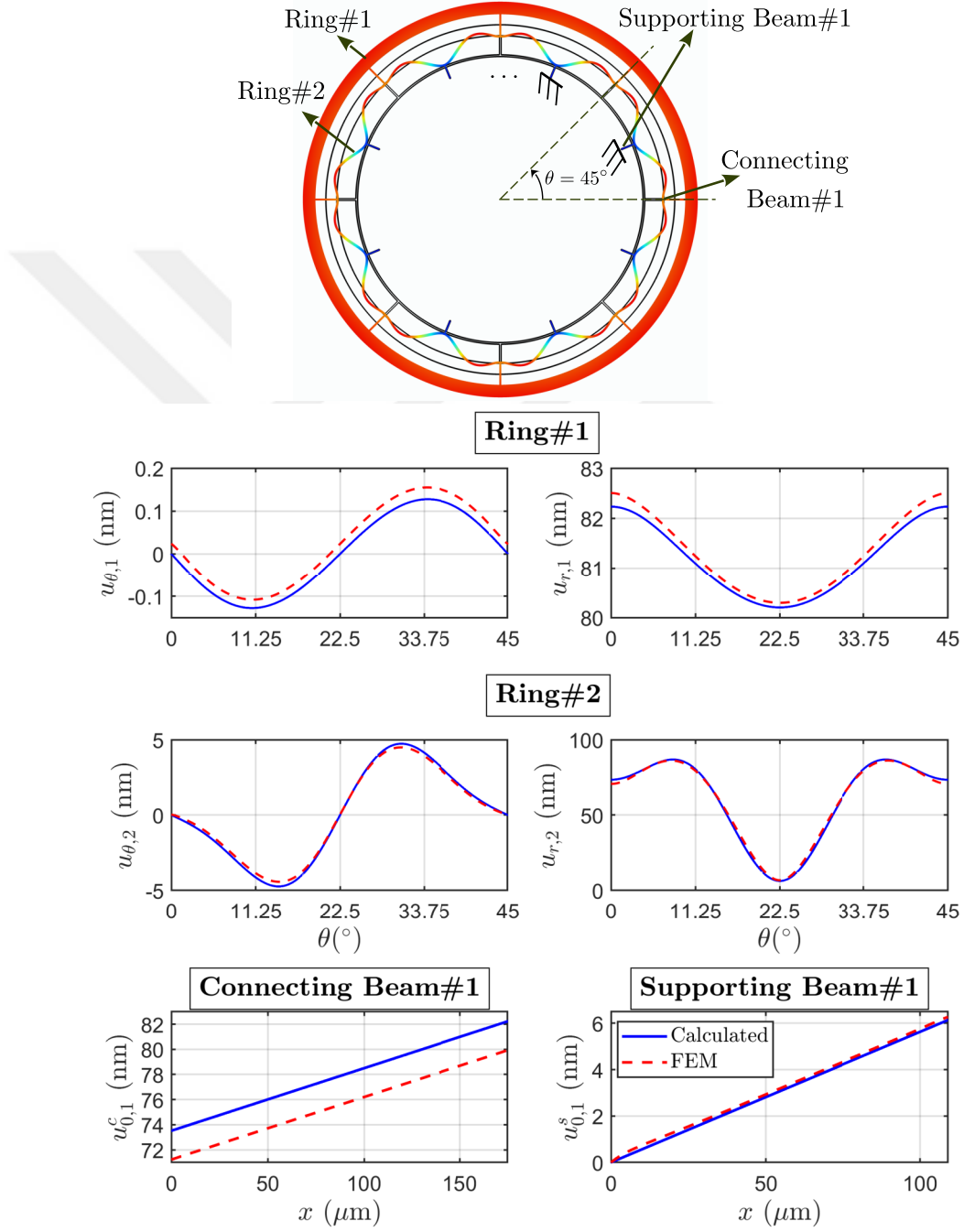


Figure 4.1: Displacement fields under the thermal load $\Delta T = 20^\circ\text{C}$ and homogeneous boundary conditions; calculated versus FEM.

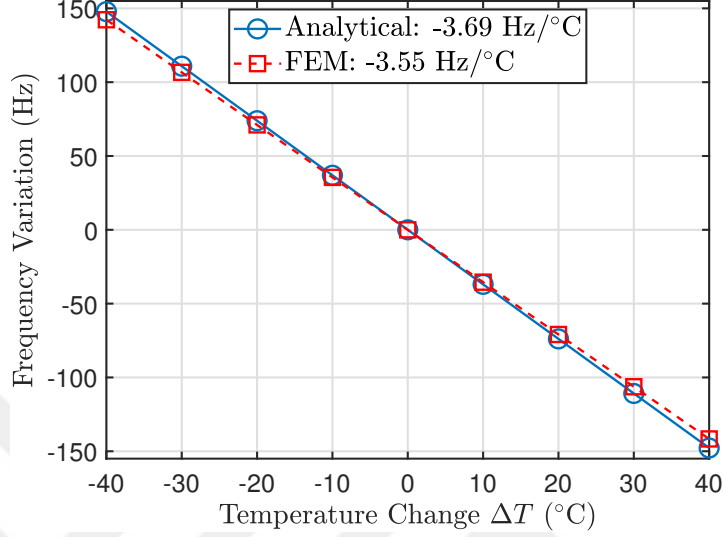


Figure 4.2: Frequency variation against temperature change with fixed supporting beams; analytical versus FEM.

Thermal stiffness. After computing the displacement field, we obtain the corresponding stresses by the constitutive equations (4.20) and (4.23). The induced stiffness, which we refer to as *thermal stiffness*, $K_T = [k_{ij}^T] \in \mathbb{R}^{2 \times 2}$, is calculated in the following:

- **Ring# i , $i = 1, 2$:** Based on the periodicity of the ring's displacement field and Proposition 4.2, we obtain the extensional stress over portion# j as

$$\sigma_{\theta\theta,i}^T(\xi_i, \theta)|_{\text{portion}\#j} = \frac{\varepsilon_T}{\hat{\varepsilon}_T} E(T) (\hat{\varepsilon}_{\theta\theta,i}(\theta - \theta_{c,j}) - \hat{\varepsilon}_T), \quad j = 1 : 4n, \quad (4.29)$$

where $\hat{\varepsilon}_{\theta\theta,i}$ is the strain calculated over portion#1 under the unit thermal load:

$$\begin{aligned} \hat{\varepsilon}_{\theta\theta,i}(\theta) &= \hat{\varepsilon}_{0,i}(\theta) + \xi_i \hat{\kappa}_{0,i}(\theta), \\ \hat{\varepsilon}_{0,i}(\theta) &= \frac{1}{R_i} (a_{r,i}^{*\top} \chi_{r,i}(\theta) + a_{\theta,i}^{*\top} \chi'_{\theta,i}(\theta)), \\ \hat{\kappa}_{0,i}(\theta) &= -\frac{1}{R_i^2} (a_{r,i}^{*\top} \chi''_{r,i}(\theta) - a_{\theta,i}^{*\top} \chi'_{\theta,i}(\theta)). \end{aligned} \quad (4.30)$$

Substituting (4.29) into (4.10) gives us the Ring# i 's thermal stiffness:

$$k_{pq,i}^T = \frac{\varepsilon_T}{\hat{\varepsilon}_T} \sum_{j=1}^{4n} \frac{E(T) A_i}{R_i} \int_{\theta_{c,j}}^{\theta_{c,j+1}} \{ \hat{\varepsilon}_{0,i}(\theta - \theta_{c,j}) - \hat{\varepsilon}_T \} F_p F_q d\theta$$

$$\begin{aligned}
& + \frac{E(T)I_i}{R_i^3} \int_{\theta_{c,j}}^{\theta_{c,j+1}} \{ \hat{\epsilon}_{0,i}(\theta - \theta_{c,j}) - \hat{\epsilon}_T \} (F_p F_q + F'_p F'_q) d\theta \\
& + \frac{2E(T)I_i}{R_i^2} \int_{\theta_{c,j}}^{\theta_{c,j+1}} \hat{\kappa}_{0,i}(\theta - \theta_{c,j}) F_p F_q d\theta, \quad p, q \in \{1, 2\}, \quad i = 1, 2. \quad (4.31)
\end{aligned}$$

- **Connecting(supporting) beam# i , $i = 1 : 4n$:** According to Proposition 4.2, we obtain the axial stress as

$$\sigma_{xx,i}^{c(s)T}(x) = \frac{\varepsilon_T}{\hat{\epsilon}_T} E(T) \left(\hat{\epsilon}_{xx,i}^{c(s)}(x) - \hat{\epsilon}_T \right), \quad (4.32)$$

where $\hat{\epsilon}_{xx,i}^{c(s)}$ is the strain under the unit thermal load:

$$\hat{\epsilon}_{xx,i}^{c(s)}(x) = \hat{\epsilon}_{0,c(s)}(x) = a_{c(s)}^{*\top} \chi'_{c(s)}(x). \quad (4.33)$$

By plugging (4.32) into (4.13), we have

$$k_{pq,c(s)}^T = \frac{\varepsilon_T}{\hat{\epsilon}_T} \sum_{i=1}^{4n} E(T) I_{c(s)} \int_0^{l_{c(s)}} \{ \hat{\epsilon}_{0,c(s)}(x) - \hat{\epsilon}_T \} {}_p\varpi''_{c(s),i}(x) {}_q\varpi''_{c(s),i}(x) dx, \quad (4.34)$$

$$p, q \in \{1, 2\},$$

where ${}_p\varpi_{c(s),i}$ is the p -th mode shape of connecting(supporting) beam# i .

The total thermal stiffness is the sum of the rings' and beams' stiffness:

$$k_{pq}^T = k_{pq,1}^T + k_{pq,2}^T + k_{pq,c}^T + k_{pq,s}^T, \quad p, q \in \{1, 2\}. \quad (4.35)$$

Comparison against FEM. Figure 4.1 provides comparative graphs of the calculated displacement fields against FEM for $\Delta T = 20^\circ\text{C}$. Here, using $N_1 = N_2 = 3$ basis functions for Rings#1 and 2, we first solved the QP problem (4.27a) for the reference temperature change $\hat{\Delta T} = 10^\circ\text{C}$ and then scaled the solution using (4.28). The FEM simulation was done in COMSOL Multiphysics' solid mechanics module, meshed by free tetrahedral elements with quadratic serendipity shape functions. Judging by the results, the error associated with our idealized boundary conditions instigates from the supporting beam and then permeates across the outer displacement fields.

To test the accuracy of the calculated thermal stiffness, we compare our predicted frequency changes for different temperature variations with those of FEM. The results in Figure 4.2 indicate that our model gives slightly larger frequencies. The least-squares slopes are $-3.69 \text{ Hz}/^\circ\text{C}$ ($-56.09 \text{ ppm}/^\circ\text{C}$) for the analytical model and $-3.55 \text{ Hz}/^\circ\text{C}$ ($-55.98 \text{ ppm}/^\circ\text{C}$) for the FEM. This observation hints that the ideal boundary conditions and Euler-Bernoulli simplification tend to overestimate the extensional stresses in the structure.

4.3.2 Zero thermal strain, nonhomogeneous boundary conditions

To simplify the displacement field calculation using the $\frac{\pi}{2n}$ -symmetry of the gyroscope's structure, we assume the connecting beams to be rigid. In such a case, the total potential is the sum of the strain energies of portions#1–4n, with each portion having its independent displacement fields and boundary conditions. By suitable choices of basis functions, we will ensure the continuity of the displacement fields of rings. Accordingly, our objective is to minimize the portion#1's potential energy

$$\begin{aligned} \mathcal{U}_{\text{portion}\#1} = & \sum_{i=1}^2 \left\{ \frac{1}{2} E(T) A_i R_i \int_{\theta_{c,1}}^{\theta_{c,2}} \varepsilon_{0,i}^2 d\theta + \frac{1}{2} E I_i R_i \int_{\theta_{c,1}}^{\theta_{c,2}} \kappa_i^2 d\theta \right\} \\ & + \frac{1}{2} E(T) A_s \int_0^{l_s} \varepsilon_{0,1}^2 dx \end{aligned} \quad (4.36)$$

subject to

$$\begin{aligned} u_{r,1}(\theta_{c,i}) &= u_{r,2}(\theta_{c,i}); \\ u_{\theta,1}(\theta_{c,i}) &= 0, \quad \gamma_1(\theta_{c,i}) = 0; \\ u_{\theta,2}(\theta_{c,i}) &= 0, \quad \gamma_2(\theta_{c,i}) = 0, \quad i = 1, 2; \\ u_{0,1}^s(0) &= \Delta_1, \quad u_{0,1}^s(l_s) = u_{r,2}(\theta_{s,1}). \end{aligned} \quad (4.37)$$

Here, Δ_1 is a unit displacement imposed on the base of supporting beam#1.

Ritz's Method. We use the same basis function parametrizations as (4.25a) and (4.25c) for Rings#1,2 and supporting beam#1, respectively. The particular choice of $\frac{\pi}{2n}$ -periodic trigonometric basis functions automatically satisfies the

boundary conditions related to $u_{\theta,i}$, γ_i , $i = 1, 2$, at $\theta_{c,1}$, $\theta_{c,2}$, and, in addition, ensures the continuity of the displacement fields across adjacent portions. Substituting these parametrizations into (4.36) results in the following QP problem:

$$\underline{a}^* := \arg \min_{\underline{a} \in \mathbb{R}^N} \frac{1}{2} \underline{a}^\top P \underline{a} \quad \text{subject to } A \underline{a} = \Delta_1 b, \quad (4.38a)$$

where

$$\underline{a} := [a_{\theta,1}^\top, a_{r,1}^\top, a_{\theta,2}^\top, a_{r,2}^\top, a_s^\top]^\top \in \mathbb{R}^N, \quad N = 2N_1 + 2N_2 + 4, \quad (4.38b)$$

is the augmented vector of coefficients and

$$\begin{aligned} P &= \text{diag}(P_1, P_2, P_s), \\ P_i &= \frac{E(T)A_i}{R_i} \int_{\theta_{c,1}}^{\theta_{c,2}} \begin{bmatrix} \chi'_{\theta,i} \chi_{\theta,i}^\top & \chi'_{\theta,i} \chi_{r,i}^\top \\ \star & \chi_{r,i} \chi_{r,i}^\top \end{bmatrix} d\theta + \frac{E(T)I_i}{R_i^3} \int_{\theta_{c,1}}^{\theta_{c,2}} \begin{bmatrix} \chi'_{\theta,i} \chi_{\theta,i}^\top & -\chi'_{\theta,i} \chi_{r,i}^{\prime\prime\top} \\ \star & \chi_{r,i}^{\prime\prime} \chi_{r,i}^\top \end{bmatrix} d\theta, \\ P_s &= E(T)A_s \int_0^{l_s} \chi'_s \chi_s^\top dx, \\ A &= \begin{bmatrix} \chi_{r,2}^\top(\theta_{c,1}) \frac{\partial \underline{a}}{\partial a_{r,2}} - \chi_{r,1}^\top(\theta_{c,1}) \frac{\partial \underline{a}}{\partial a_{r,1}} \\ \chi_s^\top(0) \frac{\partial \underline{a}}{\partial a_s} \\ \chi_{r,2}^\top(\theta_{s,1}) \frac{\partial \underline{a}}{\partial a_{r,2}} - \chi_s^\top(l_s) \frac{\partial \underline{a}}{\partial a_s} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \end{aligned} \quad (4.38c)$$

Proposition 4.3. *Let $\underline{a}^*(E(\hat{T}), \Delta_0)$ denote the solution of the QP problem (4.38a) for portion#1 with reference Young's modulus $E(\hat{T})$ at a preselected temperature $\hat{T}$ and supporting beam#1's displacement Δ_0 . Then, the solution of the same QP problem for portion#j, $j = 1 : 4n$, with Young's modulus $E(T)$ and supporting beam#j's displacement Δ_j is given by*

$$\underline{a}_j^*(E(T), \Delta_j) = \frac{\Delta_j}{\Delta_0} \underline{a}^*(E(\hat{T}), \Delta_0). \quad (4.39)$$

Proof. First, we note that

1. QP (4.38a) is homogeneous in Young's modulus since $E(T)$ is a scalar multiplying the cost function;
2. the matrices P , A , and b are the same for all portions.

Then, the proof follows as a special case of Lemma B.1 with $\zeta_1 = 0$ and $\zeta_2 = \Delta_0$. $\square$

Proposition 4.3 precludes the need for solving separate QPs for each portion, significantly simplifying the displacement field calculation task. We only solve the QP problem (4.38a) once and then use (4.39) to obtain the displacement field of the entire structure.

Stress stiffness. We refer to the stiffness induced by the displacements of the supporting beams as *stress stiffness*, $K_S = [k_{ij}^S] \in \mathbb{R}^{2 \times 2}$, and provide its calculation details in the following:

- **Ring# i , $i = 1, 2$:** Based on Proposition 4.3, the extensional stress over portion# j is given by

$$\begin{aligned} \sigma_{\theta\theta,i}^S(\xi_i, \theta)|_{\text{portion}\#j} &= \frac{\Delta_j}{\Delta_0} E(T) \hat{\varepsilon}_{\theta\theta,i}(\theta - \theta_{c,j}), \quad \theta \in [\theta_{c,j}, \theta_{c,j+1}], \\ j &= 1 : 4n, \end{aligned} \quad (4.40)$$

where $\hat{\varepsilon}_{\theta\theta,i}$ is the strain calculated over portion#1 under the unit displacement Δ_0 :

$$\begin{aligned} \hat{\varepsilon}_{\theta\theta,i}(\theta) &= \hat{\varepsilon}_{0,i}(\theta) + \xi_i \hat{\kappa}_{0,i}(\theta), \\ \hat{\varepsilon}_{0,i}(\theta) &= \frac{1}{R_i} (a_{r,i}^{*\top} \chi_{r,i}(\theta) + a_{\theta,i}^{*\top} \chi'_{\theta,i}(\theta)), \\ \hat{\kappa}_{0,i}(\theta) &= -\frac{1}{R_i^2} (a_{r,i}^{*\top} \chi''_{r,i}(\theta) - a_{\theta,i}^{*\top} \chi'_{\theta,i}(\theta)). \end{aligned} \quad (4.41)$$

By substituting (4.40) into (4.10), we obtain the Ring# i 's stress stiffness:

$$\begin{aligned} k_{pq,i}^S &= \sum_{j=1}^{4n} \frac{\Delta_j}{\Delta_0} \left\{ \frac{E(T)A_i}{R_i} \int_{\theta_{c,j}}^{\theta_{c,j+1}} \hat{\varepsilon}_{0,i}(\theta - \theta_{c,j}) F_p F_q d\theta \right. \\ &+ \frac{E(T)I_i}{R_i^3} \int_{\theta_{c,j}}^{\theta_{c,j+1}} \hat{\varepsilon}_{0,i}(\theta - \theta_{c,j}) (F_p F_q + F'_p F'_q) d\theta \\ &\left. + \frac{2E(T)I_i}{R_i^3} \int_{\theta_{c,j}}^{\theta_{c,j+1}} \hat{\kappa}_{0,i}(\theta - \theta_{c,j}) F_p F_q d\theta \right\}, \quad p, q \in \{1, 2\}, \quad i = 1, 2. \end{aligned} \quad (4.42)$$

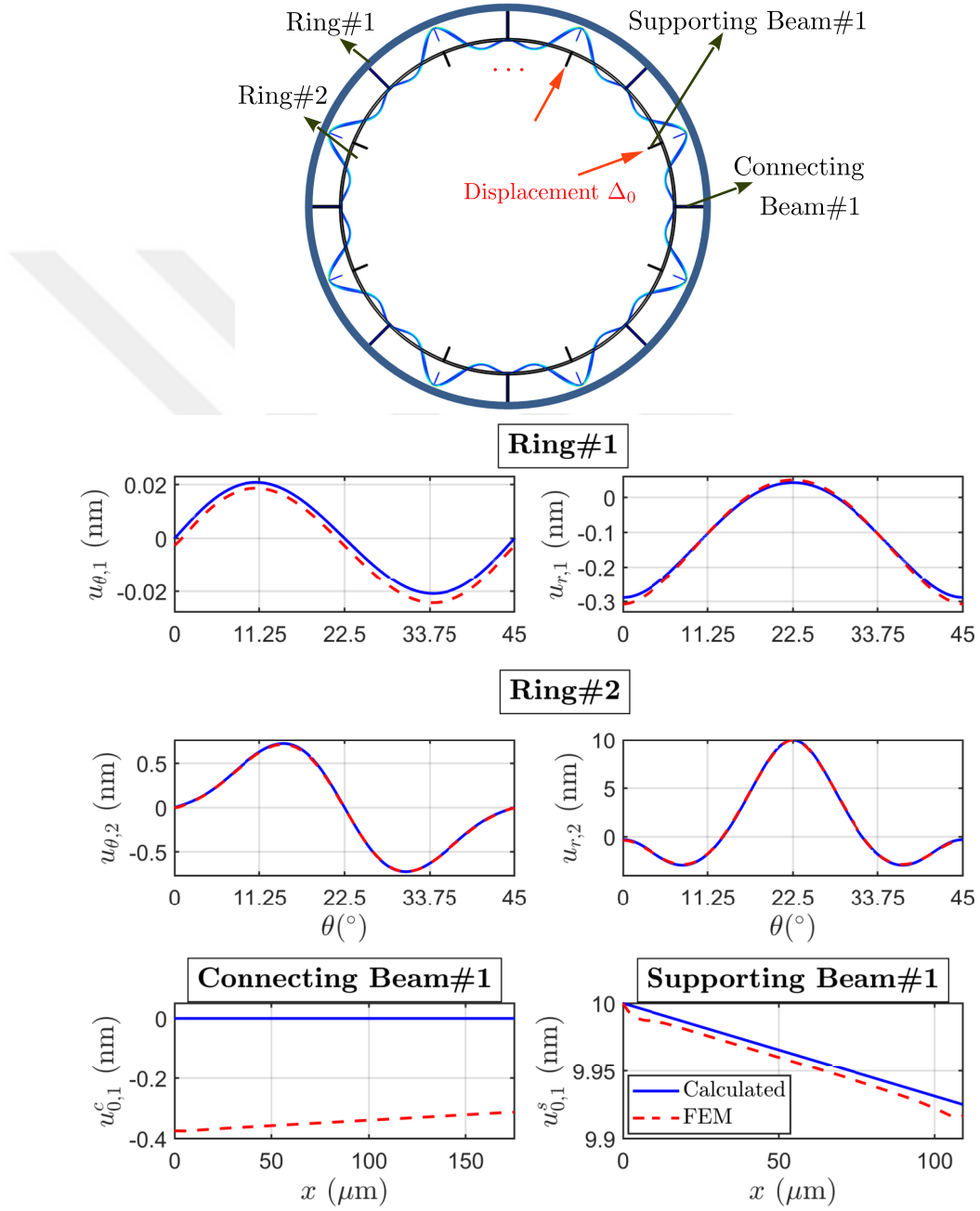


Figure 4.3: Displacement fields under the supporting beam displacement $\Delta_1 = 10$ nm and zero thermal strain; calculated versus FEM. In our model, the connecting beams are assumed to be rigid.

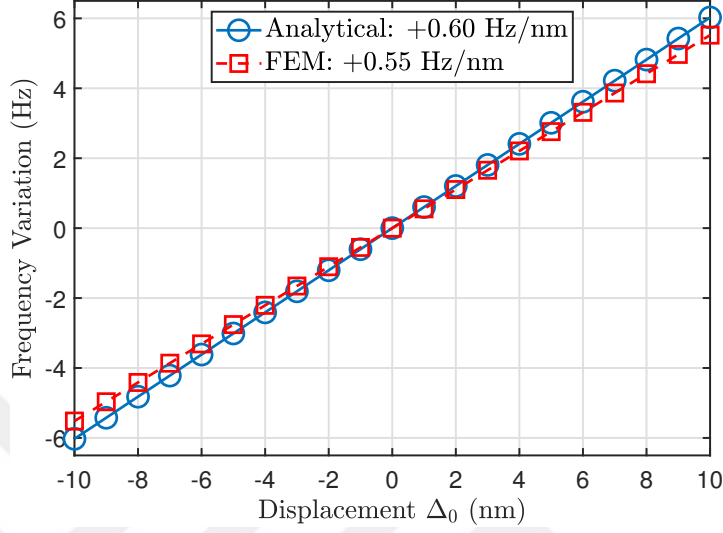


Figure 4.4: Frequency variation against supporting beams' displacement; analytical versus FEM.

- **Supporting beam# i , $i = 1 : 4n$:** By Proposition 4.3, we obtain the axial stress as

$$\sigma_{xx,i}^{sS}(x) = \frac{\Delta_j}{\Delta_0} E(T) \hat{\varepsilon}_{xx,i}^s(x), \quad (4.43)$$

where $\hat{\varepsilon}_{xx,i}^{c(s)}$ is the strain under the unit displacement Δ_0 :

$$\hat{\varepsilon}_{xx,i}^s(x) = \hat{\varepsilon}_{0,s}(x) = a_s^{*\top} \chi_s'(x). \quad (4.44)$$

By plugging (4.43) into (4.13), we have

$$k_{pq,s}^S = \sum_{i=1}^{4n} \frac{\Delta_j}{\Delta_0} \left\{ E(T) I_s \int_0^{l_s} \hat{\varepsilon}_{0,c(s)}(x) {}_p\varpi_{c(s),i}''(x) {}_q\varpi_{c(s),i}''(x) dx \right\}, \quad (4.45)$$

$$p, q \in \{1, 2\},$$

where ${}_p\varpi_{c(s),i}$ is the p -th mode shape of supporting beam# i .

The total stress stiffness is given by

$$k_{pq}^S = k_{pq,1}^S + k_{pq,2}^S + k_{pq,s}^S, \quad p, q \in \{1, 2\}. \quad (4.46)$$

Comparison against FEM. Figure 4.3 shows the calculated displacement fields in comparison with the FEM for $\Delta_1 = 10$ nm, which was done with the same setting as Section 4.3.1’s COMSOL simulation. Selecting $N_1 = N_2 = 4$ basis functions for the rings, we first solved the QP problem (4.38a) for the reference displacement $\Delta_0 = 1$ nm and then scaled the solution using (4.39). We observe that our method captures Ring#2’s displacement field very accurately.

To validate the stress stiffness calculation, we compare our model against FEM in terms of frequency changes for different supporting beams’ displacements. As shown in Figure 4.4, our model produces somewhat higher frequencies similar to the thermal stiffness. The least-squares slopes of the frequency graphs are +0.60 Hz/nm (+9.15 ppm/nm) for the analytical model and +0.55 Hz/nm (+8.78 ppm/nm) for the FEM. We can add the rigidity assumption of connecting beams to the error sources identified in the discussion of thermal stiffness.

4.4 Stiffness Matrix

The total stiffness matrix of the gyroscope is given by

$$K = K_m + K_T + K_S - K_e, \quad (4.47)$$

where

- $K = [k_{ij}] \in \mathbb{R}^{2 \times 2}$ is the nominal mechanical stiffness matrix in the absence of thermal stresses. If this stiffness is calculated at a reference Young’s modulus $E(\hat{T})$, it should be scaled by $E(T)/E(\hat{T})$ to account for the variations of material properties.
- $K_T = [k_{ij}^T] \in \mathbb{R}^{2 \times 2}$ is the thermal stiffness matrix that we derived in Section 4.3.1.
- $K_S = [k_{ij}^S] \in \mathbb{R}^{2 \times 2}$ is the stress stiffness matrix that we derived in Section 4.3.2.
- $K_e = [k_{ij}^e] \in \mathbb{R}^{2 \times 2}$ is the electrostatic stiffness, which its calculation details are discussed in Chapters 2 and 3. The thermal expansion and stresses alter

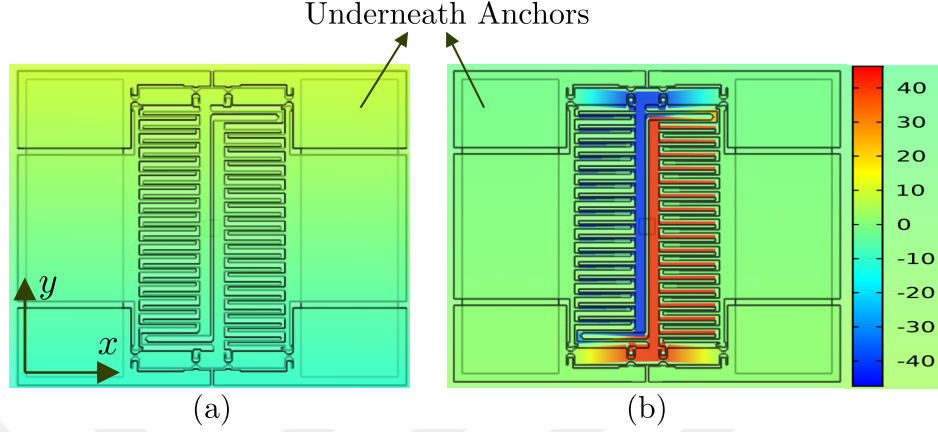


Figure 4.5: y -displacement (nm) distribution across a stress sensors under pure thermal load $\Delta T = 50^\circ\text{C}$ (a) and pure mechanical load $\varepsilon_{xx} = +42 \mu\text{m/m}$ (b).

ring-electrode gaps, which we can compute using the substrate's interpolated displacement field.

From a perturbation analysis viewpoint, temperature effects perturb the gyroscope's initial stiffness matrix $K_0 = K_m - K_{e,0}$ (is the initial electrostatic softening) as

$$\begin{aligned} K_0 &\mapsto K_0 + \delta K, \\ \delta K &= \left(\frac{E(T)}{E(\bar{T})} - 1 \right) K_m + K_T + K_s - (K_e - K_{e,0}). \end{aligned} \quad (4.48)$$

We can then proceed with the same analysis as Section 3.5.

4.5 Stress Sensors

In Section 3.4, we explained the working principle of stress sensors and how to use their outputs to interpolate the substrate's strain field. During the stress tests in Chapter 3, temperature variations were negligible ($<0.7^\circ\text{C}$), and hence, temperature was not a critical factor. A crucial difference with respect to those tests is that the stress sensors do not respond to pure thermal expansion. The FEM simulations visualized in Figure 4.5 demonstrate that pure thermal load

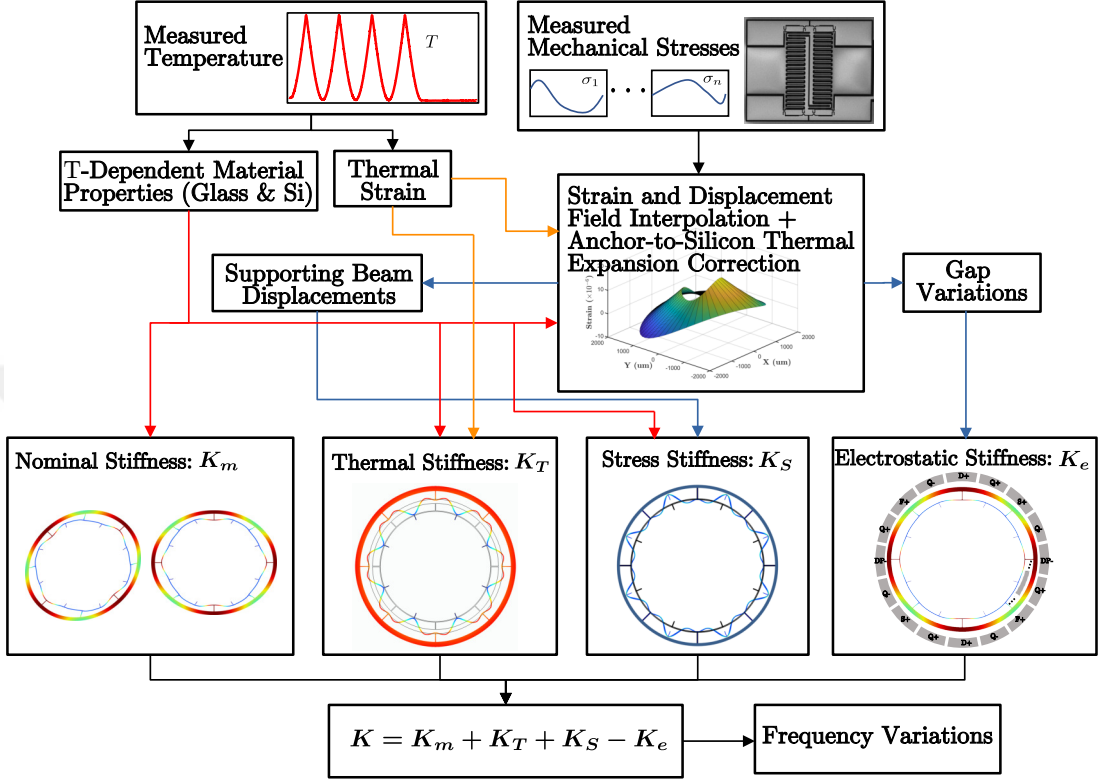


Figure 4.6: Flowchart of key steps of modeling thermal effects.

($\Delta T = 50^\circ\text{C}$) generates a negligible displacement gradient across the stress sensor's interdigitated fingers and has no appreciable effect on their gaps. On the contrary, the pure mechanical load $\varepsilon_{xx} = +42 \mu\text{m}/\text{m}$ (applied to the underneath anchors) produces a significantly more pronounced displacement gradient that the stress sensor can resolve and pick up. The mechanical strains picked up by the stress sensors primarily result from the CTE mismatches at the interface between the glass substrate and the die-attach epoxy. On this basis, the output $\tilde{\varepsilon}_{ij}(t)$ of the stress sensor located at $(\tilde{R}_i, \tilde{\theta}_j)$ is given by

$$\tilde{\varepsilon}_{ij}(t) = \frac{\partial \tilde{u}_r}{\partial r}(r, \theta, t) \Big|_{(\tilde{R}_i, \tilde{\theta}_j)} - \tilde{\varepsilon}_T, \quad (4.49)$$

where $\tilde{\varepsilon}_T$ is the thermal strain in the substrate. We then proceed by the interpolation algorithm explained in Section 3.4. An important distinction is that the calculated displacements at the silicon layer comprise two parts:

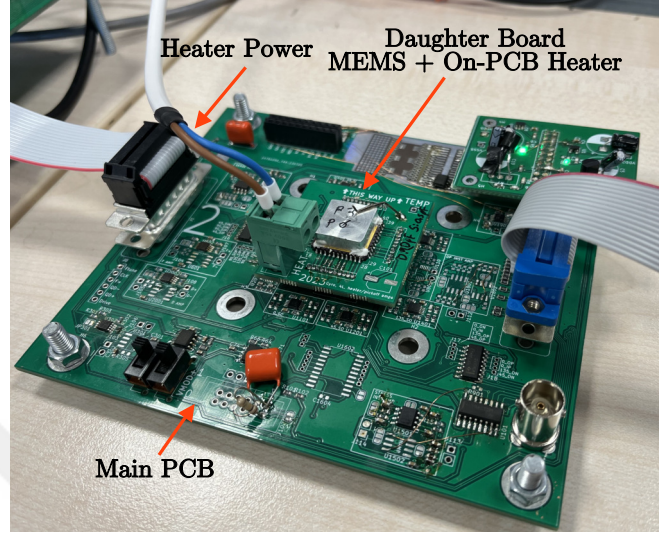


Figure 4.7: Temperature test setup; the PTAT temperature sensor is located under the daughter PCB.

1. The displacement imposed by the substrate via anchors. Similar to Section 3.4, we obtain this part based on the substrate’s interpolated strain field and glass CTE.
2. The displacement due to the silicon’s thermal expansion/contraction, calculated based on the silicon CTE and the distance from anchors to the points of interest (bases of supporting beams and edges of electrodes).

The flowchart in Figure 4.6 explains how strain interpolation is integrated with stiffness calculation in our modeling.

4.6 Results

4.6.1 Experimental setup

Figure 4.7 shows the setup for temperature tests. The daughter board houses the MEMS die with an on-PCB heater located under the MEMS die. The temperature sensor (PTAT) and the gyro front-ends are beneath the daughter board. This test setup allows us to heat the MEMS die without affecting most of the electronics. By controlling the heater’s power, we cycle the temperature and record

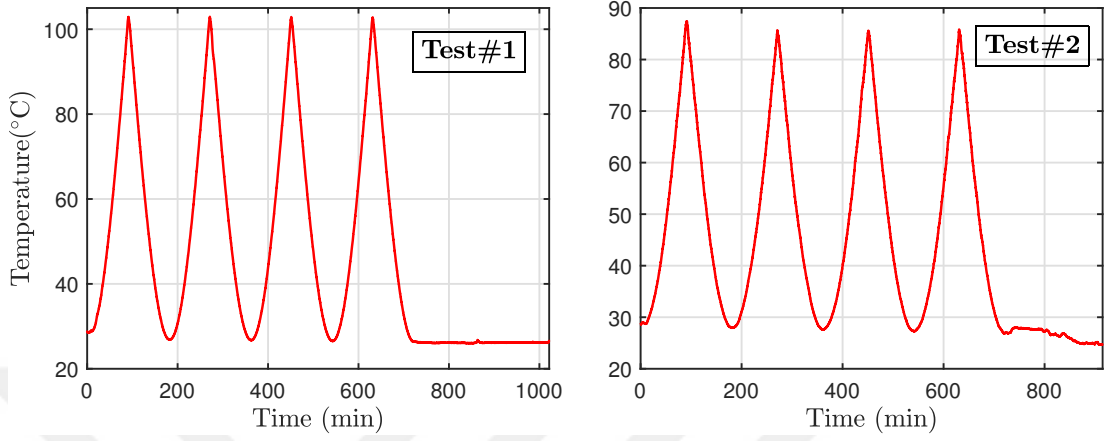


Figure 4.8: Temperature cycles during Tests#1 and 2.

the drive mode's frequency and the stress sensors' outputs. During these tests, a PI quadrature control loop was implemented, limiting the quadrature output to the $\pm 1^\circ/\text{s}$ range. The data were collected with the 1 Hz sampling rate. We here report two tests labeled as Tests#1 and 2 with their temperature cycles shown in Figure 4.8. We have provided the measured stresses during these tests (over time and temperature) in Figures 4.9-4.12. Some of the stress sensors were not functional and did not give meaningful outputs. In particular, we got outputs from only four outer stress sensors in Test#2. The stresses are mostly tensile (< 3 MPa for Test#1 and < 2 MPa for Test#2) since silicon is surrounded by materials with higher CTEs. A key difference between the gyroscopes used in these tests is the die-attach adhesives used in their packaging. For the devices in Tests#1 and 2, a hard silver-filled epoxy (Young's modulus: 198 MPa, CTE: 31-158 ppm/ $^\circ\text{C}$) and a soft silicon-based epoxy (Young's modulus: 6.92 MPa, CTE: 140 ppm/ $^\circ\text{C}$) were used, respectively [66]. As a result, we observe larger stresses during Test#1. According to Figures 4.10 and 4.12, hysteresis loops form over temperature cycles, hinting that residual stresses build up after each cycle. These residual stresses stem from the die-attach epoxy's larger CTE and viscoelastic characteristics.

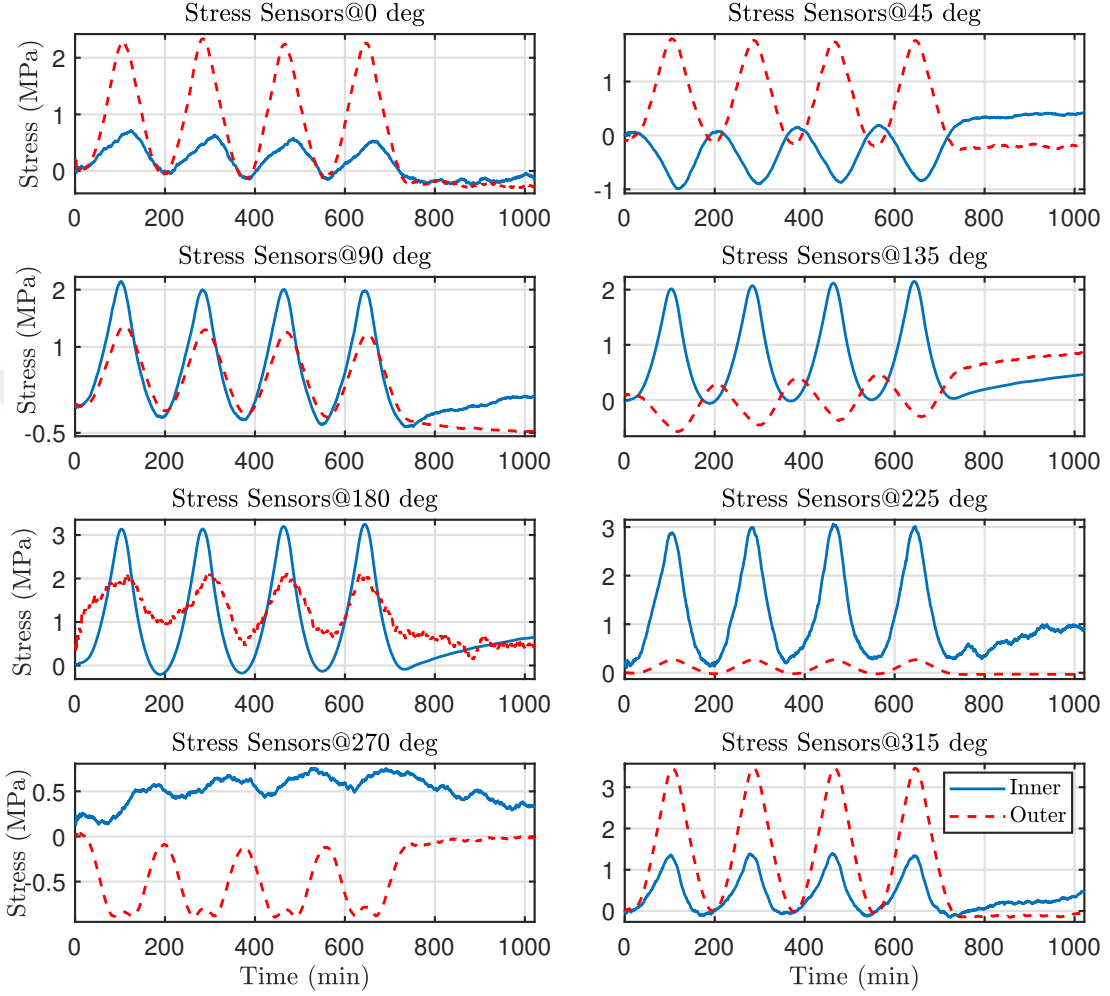


Figure 4.9: Measured stresses (over time) during Test#1.

4.6.2 Simulation setup

We implemented our model in MATLAB following the flowchart in Figure 4.6. The model takes the experimental temperature and stress data as the input, performs strain interpolation, obtains the supporting beams' displacements along with electrostatic gap variations, and, finally, calculates the stiffness matrices according to (4.47).

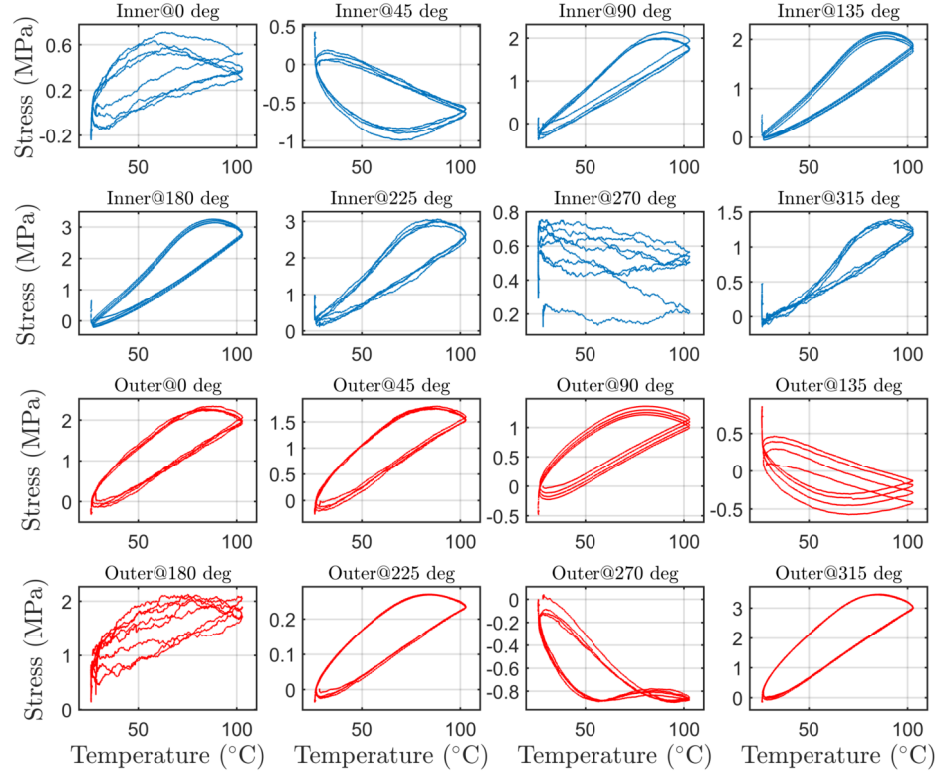


Figure 4.10: Measured stresses (over temperature) during Test#1.

4.6.3 Discussion

The analytical-versus-experimental graphs for the frequency variations over time and temperature are given in Figures 4.13 and 4.14, respectively. The analytical model's outputs are in good agreement with the experimental data and successfully track the variations of frequency. We also observe hysteresis loops in the analytical results, which confirm the veracity of our modeling's assumptions and overall framework. We pointed out that the residual mechanical strains originating from the die-attach epoxy are responsible for the hysteresis. Hence, the discrepancy between the predicted and experimental hysteresis loops is due to the (mechanical) strain interpolation errors. As discussed in Section 3.6.5, the missing or faulty stress sensor measurements limit the strain interpolation accuracy to a large extent. Figure 4.8 illustrates the contributions of two mechanisms of temperature effects—namely, change in Young's modulus, E , and thermal stresses—to

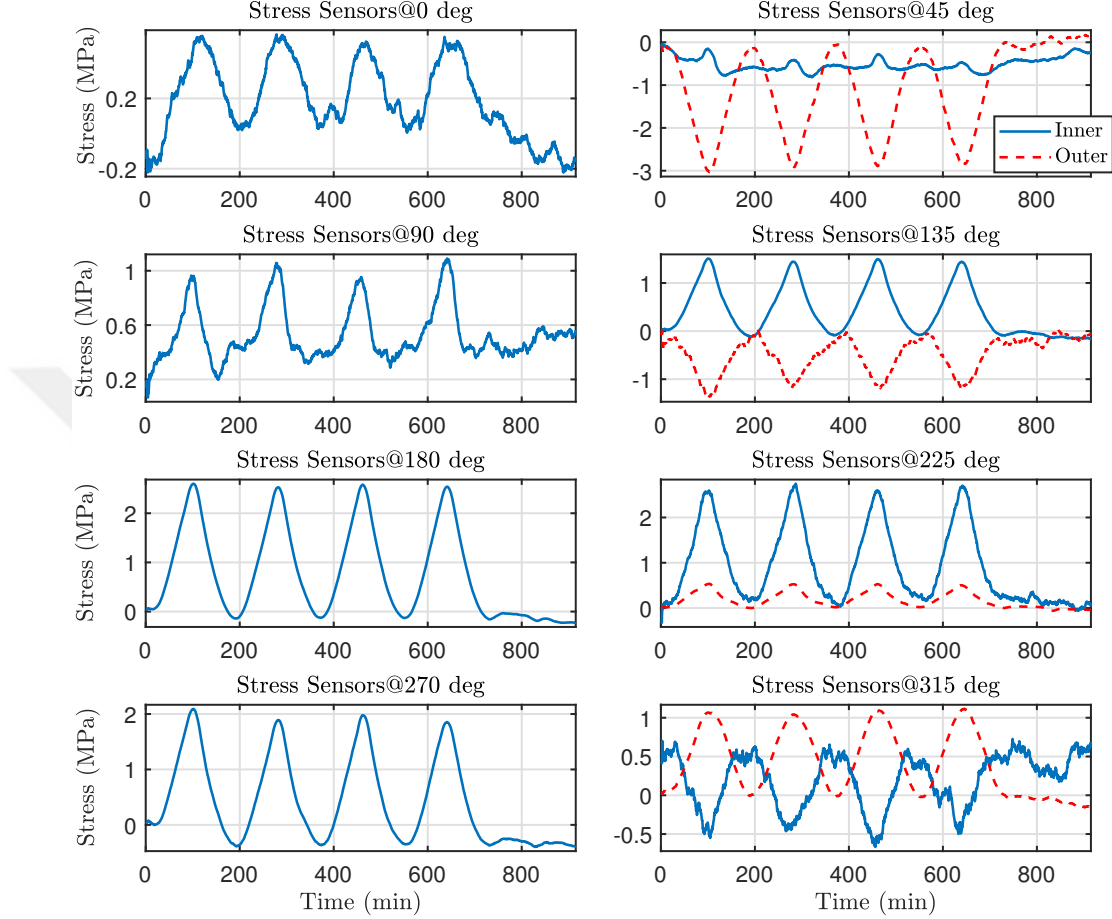


Figure 4.11: Measured stresses (over time) during Test#2.

the TCF for both tests. These results accentuate the countering role of thermal stresses against the drop in E . The predicted TCFs for Tests#1 and 2 are $-11.47 \text{ ppm}/^{\circ}\text{C}$ and $-14.62 \text{ ppm}/^{\circ}\text{C}$, respectively, which agree with the experimental values $-10.61 \text{ ppm}/^{\circ}\text{C}$ and $-14.13 \text{ ppm}/^{\circ}\text{C}$. The reported TCFs are the slopes of the least-squares lines fitted to the frequency-temperature plots. The difference between the TCFs of the tests is attributable to the the different epoxies. The hard epoxy of the device in Test#1 generates larger thermal stresses compared to the soft epoxy of Test#2's device.

Figure 4.16 plots variations of different stiffness types with temperature, as categorized in our model. In this figure, variations of the nominal stiffness are only due to the change in Young's modulus. We observe that the thermal stiffness, similar to the nominal stiffness, has a softening effect while the stress stiffness

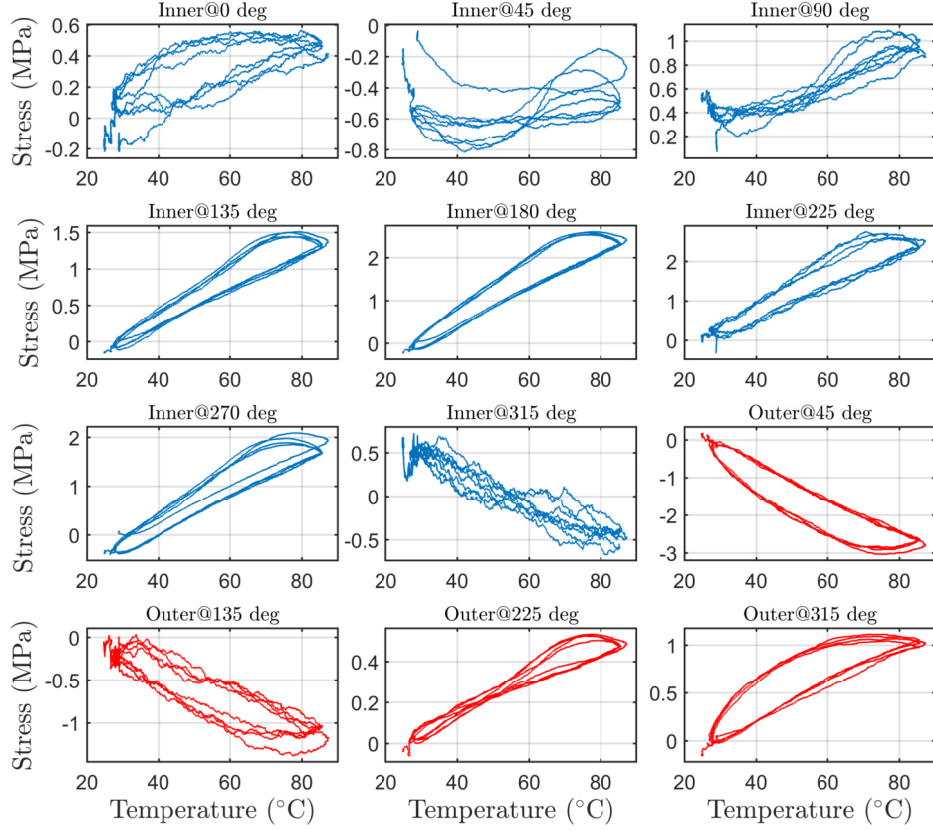


Figure 4.12: Measured stresses (over temperature) during Test#2.

produces a hardening effect. Hence, the stiffness components associated with the displacements of supporting beams are responsible for the positive push in the TCF. Moreover, there is no significant change in the electrostatic softening, mainly because of the differential arrangement of electrodes that balances the gap variations on both sides of the outer ring.

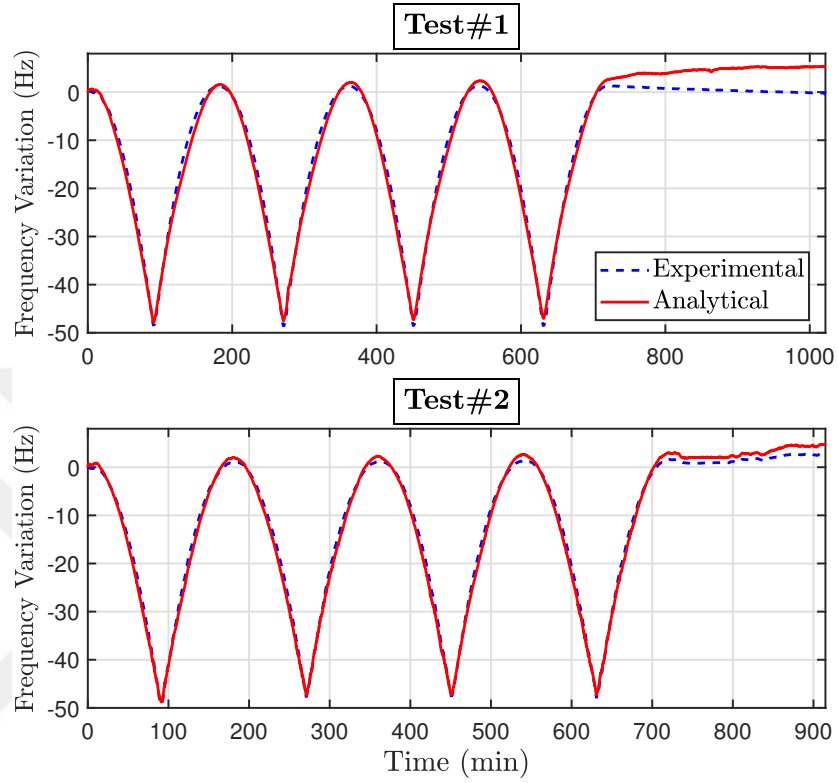


Figure 4.13: Frequency variations during Tests#1,2; experimental versus analytical.

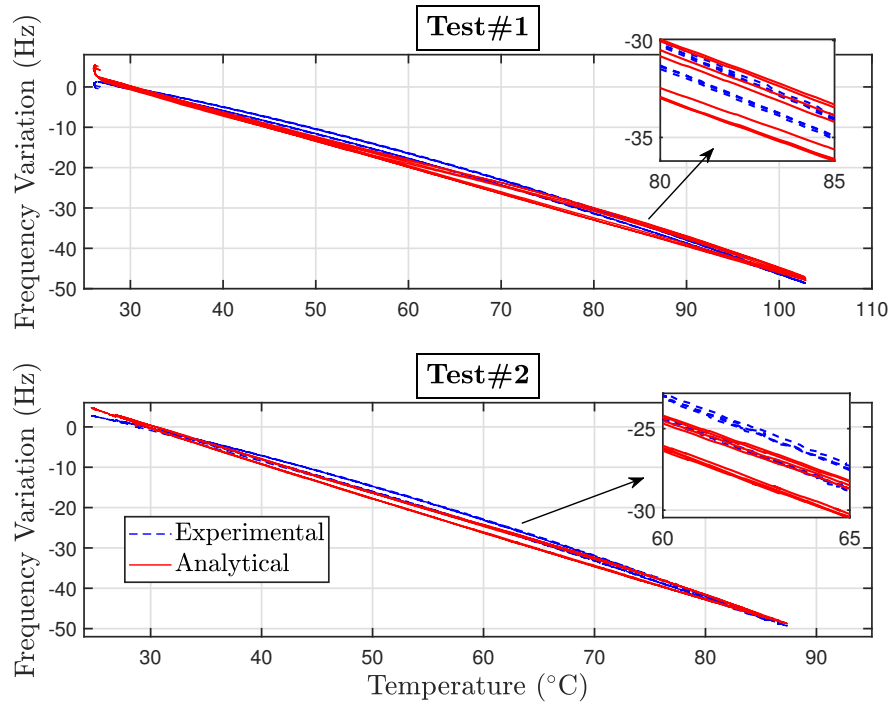


Figure 4.14: Frequency variations over temperature and formation of hysteresis loops during Tests#1,2; experimental versus analytical.

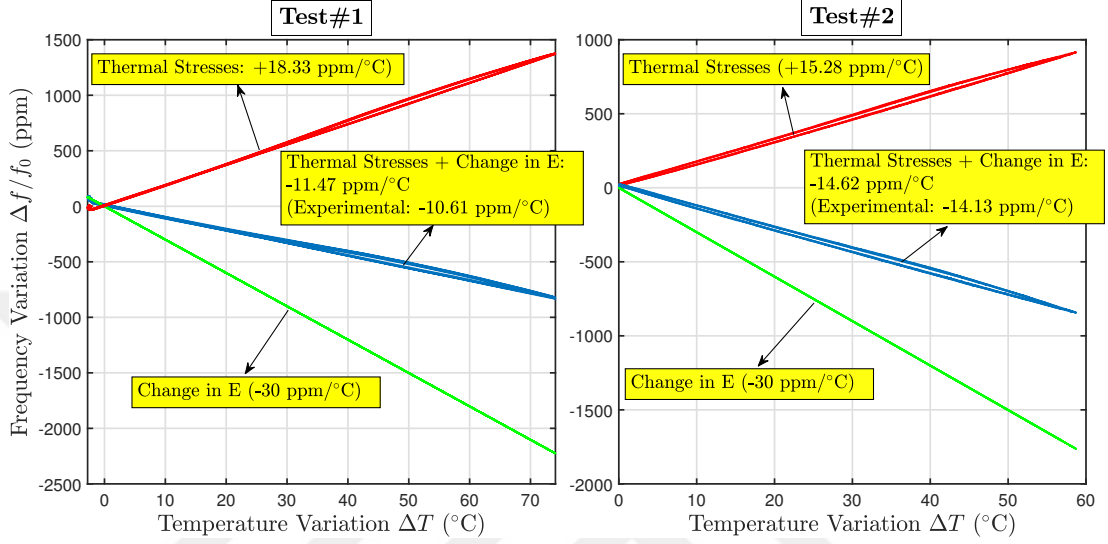


Figure 4.15: Contributions of two aspects of temperature effects, change in E (Young's modulus) and thermal stresses, to the TCF; reported values are the least-squares slopes.

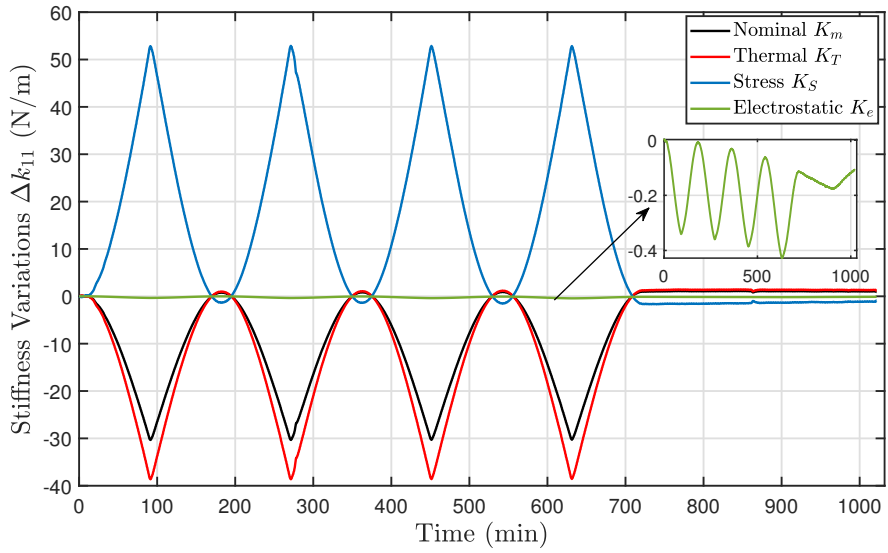


Figure 4.16: Variations of different stiffness types during Test#1 as predicted by our model; the plots are for the k_{11} component of the stiffness matrix.

Chapter 5

Conclusion and Future Direction

This thesis reported new results on the analytical modeling of ring-based MEMS gyroscopes with the aim of improving their performance. A summary of the key findings is as follows.

In Chapter 2, we presented a novel mathematical model for the effects of imperfections on the frequency split and orientation of mode shapes in multi-ring gyroscopes. Our approach involved calculating the mode shapes of the suspension system, which allowed us to incorporate imperfections as spatially dependent functions and study their effects on the mass and stiffness matrices. We extended this method to mode shape and modal properties calculation of DRGs with a large number of rings. We employed perturbation analysis to derive the nonideal transfer matrix of the gyroscope in the presence of imperfections, which then enabled us to analyze the quadrature and in-phase errors with the following conclusions:

- The off-diagonal stiffness, k_{12} and the mismatch between the diagonal stiffness components k_{11} , k_{22} —resulting in frequency split, mode shape rotation, and sense mode phase error—generate the quadrature error, which leaks into the in-phase channel because of the phase error.
- Matching the frequency and bandwidths of the modes can nullify this leakage; however, the part of the in-phase error caused by the damping coupling,

$\bar{d}_{12}$, would persist.

- Mismatch or coupling in the mass matrix results in the mode shapes having different rotation angles. Compared to the stiffness, mass matrix nonidealities are less likely to occur in ring gyroscopes.

Notably, we examined imperfections in different structural components of the gyroscope, categorizing them based on the types of errors they engender in the mass and stiffness matrices. Our findings indicate that rings are robust against structural imperfections due to their circular symmetry. In contrast, defects in supporting beams can result in significant quadrature errors. For example, 50 nm (0.5%) width variation in one of those beams leads to 4700 °/s quadrature error. This observation implies that a VRG with compliant anchor connections approaches the ideal ring structure, enhancing robustness against imperfections. However, such a design would be more sensitive to accelerations due to the reduced stiffness of the $n = 1$ rigid body mode. We validated our modeling using the experimental data from the fabricated 59 kHz VRG and 41 kHz DRG, along with rigorous comparative simulations against FEM. Our model can simulate various imperfections and accurately predict frequency split, mode shape orientation, and quadrature error. Accordingly, the proposed method goes beyond the commonly used FE simulation while entailing much less computational complexity.

In Chapter 3, for the first time in the literature, we introduced a model that mathematically describes the stress effects in ring gyroscopes. Our model revolved around the key observation that stress-induced anchor displacements result in variations of electrostatic gaps and nonhomogeneous boundary conditions at the interface between the gyroscope’s suspension system and the anchored internal structure. We had a gyroscope equipped with 16 capacitive stress sensors, which were distributed with 45° symmetry across its structure. We used these stress sensors to interpolate the strain field across the substrate and deduce the anchor displacements. To capture the stress effects, we showed that two fundamental assumptions in the existing literature should be amended:

1. **Linearity:** the linear engineering strain should be upgraded to the non-linear Green-Lagrange strain to reveal the stress-induced stiffness through

geometric nonlinearity;

2. **Inextensibility:** for a ring, this stress stiffness is determined by the extensional stress arising from centerline extensibility.

The stress stiffness perturbs the nominal stiffness of the gyroscopes, leading to variations of frequencies, mode shape rotations, and sense mode phase error. We analyzed the resultant quadrature/in-phase errors and experimentally validated our model using extensive bending tests performed on our 59 kHz, 3.2 mm diameter VRG.

In Chapter 4, we presented a model for temperature effects, which can be regarded as a generalization of the model in Chapter 3. We built the model upon the fundamental observation that temperature affects material properties and engenders thermal stresses. We elaborated and mathematically formulated how these factors manifest in the gyroscope stiffness matrix and, consequently, frequencies. Similar to Chapter 3, we showed the necessity of geometric nonlinearity and extensibility in describing the effects of thermal stresses. We discussed that stress sensors are only sensitive to mechanical strains and explained how to use their outputs to calculate the additional stiffness imposed on the gyroscope. We verified our modeling using the experimental data from temperature tests on our 57 kHz, 3.2 mm-diameter VRG. We showed that stiffness introduced by the thermal expansions pushing the innermost supporting beams outward counters the effect of the drop in Young’s modulus. The model was not only capable of predicting TCF accurately but also captured the formation of frequency-temperature hysteresis loops over temperature cycles.

5.1 Generalization and Design Considerations

Although the bulk of this thesis’s analytical results were geared toward our VRG’s particular design, the core tenets apply to general circular gyroscopes:

- **Modeling.** The derivations of the nonideal transfer function, quadrature/in-phase errors, and eigenvalue perturbation analysis apply to circular and any

other type of MEMS gyroscope.

- **Stress effects.** The identified mechanism–stress stiffness arising from the interaction of extensional and bending stresses in conjunction with geometric nonlinearity—is valid for any circular gyroscope. The anchor displacements are the root cause for inducing these extensional stresses. The limitations of the inextensibility assumption should be considered in the stress effects analysis as this assumption fails to capture the extensional strains.
- **Temperature effects.** Temperature dependency of silicon’s physical properties is well-understood in the literature. Our results on thermal stresses and their countering role against the changes in material properties hold for general circular gyroscopes. Thermal stresses in the gyroscope’s structure do not cause hysteresis because of silicon’s linear elastic behavior. However, the stresses that result from the CTE mismatches at the die-attach level and the viscoelastic characteristics of the adhesive material tend to leave residual effects that manifest as hysteresis.

The orientation of this thesis was primarily toward understanding the effects of imperfections, stress, and temperature. We recapitulate a number of the key findings as the following design guidelines:

- Due to the degenerate mode shapes, circularly symmetric structures are robust against imperfection. The structural elements lacking the mode shape degeneracy (e.g., beams) do not offer such an advantage and, therefore, should be carefully designed and, if possible, arranged so that their combined effects balance each other out. At the same time, one should be careful that the structural stiffness is high enough to preclude acceleration-sensitive modes.
- The anchor placement within the gyroscope’s suspension is critical regarding stress effects. Minimizing the displacements that movements of these anchors could induce in the gyroscope’s structure would reduce the stress stiffness.
- Interestingly, although minimizing the stress stiffness is desirable concerning pure mechanical stresses, it could curtail the TCF contribution of thermal

stresses in countering the effects of variations in Young’s modulus. In Figure 4.16, we showed that stress stiffness is the only positive push in the total stiffness of the gyroscope.

- The quality and thermophysical properties of the die-attach adhesive in MEMS packaging are critical factors in the generation and distribution of thermal stresses across the gyroscope’s structure. For example, we observed that epoxies with higher Young’s modulus produce larger stresses. The adhesive is susceptible to residual stresses that induce hysteresis nonlinearity in the gyroscope’s frequency characteristics.

5.2 Suggestions for Future Research

We discuss possible avenues for improving and expanding upon the results of this thesis:

1. **Modal analysis.** The following are suggestions to improve the calculation of mode shapes, mass, and stiffness matrices in Chapter 2:
 - Group theory is a powerful tool that is used in various branches of physics and chemistry to formulate and take advantage of the symmetries of the objects under study, such as molecules [13]. Such symmetries can significantly simplify the object’s analysis by identifying smaller subspaces that act as building blocks of the space of all functions of interest. We used symmetries rather informally to calculate the mode shapes and displacement fields of ring gyroscopes with the Ritz method. Since symmetries (often in the form of rotational or reflective) are inherent to circular MEMS gyroscopes, group theory can be a valuable tool for their analytical or FEM modeling.
 - Upgrading the Euler-Bernoulli to more comprehensive theories such as Timoshenko and geometrically exact beams [82] can improve the modeling accuracy. This point is particularly valid for DRGs employing *fat*

beams (i.e., spokes) that cannot be adequately described by the Euler-Bernoulli theory. In addition, it might be desirable to increase the width of rings to enhance the quality factor, which, in turn, will further limit the veracity of the narrow ring assumption.

- The ring-beam boundary conditions we considered were idealized, leading to slightly overestimated stiffness. A perturbation analysis can be carried out to examine more realistic boundary conditions. Such an approach can be coupled with an optimization algorithm (similar to what we proposed in Section 2.9.3) to quantify the model’s deviation from the measured gyroscope data. We remark that the accuracy of calculated mode shapes is crucial not only for the sake of modal analysis but also for the stress and thermal stiffness matrices that we derived in Chapters 3 and 4.

2. **Strain interpolation.** The strain interpolation we used in Chapters 3 and 4 was purely data-based, and its accuracy was very sensitive to missing or faulty stress measurements, especially along the principal stress directions. This issue leads us to think of physics-based, more robust interpolation techniques, hinting that the stress distribution across the gyroscope’s substrate requires further study. Moreover, the strain interpolation problem is entangled with the yet-open question of the optimal number and placement of stress sensors. In the practical aspect, the fabrication yield and metal routing quality of stress sensors are critical factors of high priority.

3. **Thermoelastic damping.** The temperature dependency of thermoelastic damping in MEMS resonators is well-established [62, 83]. Thermoelastic damping affects the long-term drift since, to a large extent, it determines the bandwidth and damping asymmetry between the modes. Hence, incorporating thermoelastic damping is an imperative extension of the modeling approach proposed in Chapter 4. Under the assumption of one-way thermoelastic coupling from mechanical deformation to thermal dissipation [84], one can plug the mode shape calculation results into the heat conduction equation and derive the subsequent energy loss.

Finally, we emphasize that physics-based drift compensation is the envisaged

culmination of the analytical methods this thesis set forth for modeling imperfections, stress, and temperature effects. Therefore, the long-term performance of gyroscopes is the leading direction for extending and continuing this research. Considering the relatively large number of stress sensors we currently implement, physics-aware data-driven techniques, as the confluence of analytical modeling and machine learning, are also an avenue worthy of investigation.



Bibliography

- [1] N. Yazdi, F. Ayazi, and K. Najafi, “Micromachined inertial sensors,” *Proceedings of the IEEE*, vol. 86, pp. 1640–1659, Aug. 1998.
- [2] G. Langfelder, M. Bestetti, and M. Gadola, “Silicon MEMS inertial sensors evolution over a quarter century,” *Journal of Micromechanics and Microengineering*, vol. 31, p. 084002, Aug. 2021.
- [3] D. K. Shaeffer, “MEMS inertial sensors: A tutorial overview,” *IEEE Communications Magazine*, vol. 51, pp. 100–109, Apr. 2013.
- [4] M. N. Armenise, C. Ciminelli, F. Dell’Olio, and V. M. N. Passaro, *Advances in Gyroscope Technologies*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011.
- [5] C. Lanczos, *The Variational Principles of Mechanics*. Toronto: University of Toronto Press, 2020.
- [6] “IEEE Standard Specification Format Guide and Test Procedure for Coriolis Vibratory Gyros,” *IEEE Std 1431-2004*, pp. 1–78, 2004.
- [7] “IEEE Standard for Sensor Performance Parameter Definitions,” *IEEE Std 2700-2017 (Revision of IEEE Std 2700-2014)*, pp. 1–64, 2018.
- [8] C. Acar and A. Shkel, *MEMS vibratory gyroscopes: structural approaches to improve robustness*. MEMS reference shelf, New York, NY: Springer, 2009.
- [9] E. Tatar, T. Mukherjee, and G. K. Fedder, “Stress Effects and Compensation of Bias Drift in a MEMS Vibratory-Rate Gyroscope,” *Journal of Microelectromechanical Systems*, vol. 26, pp. 569–579, June 2017.

- [10] B. E. Uzunoglu, D. Erkan, and E. Tatar, “A Ring Gyroscope With On-Chip Capacitive Stress Compensation,” *Journal of Microelectromechanical Systems*, vol. 31, no. 5, pp. 741–752, 2022.
- [11] V. Kempe, “Gyroscopes,” in *Inertial MEMS: Principles and Practice*, pp. 364–459, Cambridge: Cambridge University Press, 2011.
- [12] “IEEE Standard Specification Format Guide and Test Procedure for Single-Axis Interferometric Fiber Optic Gyros,” *IEEE Std 952-1997*, pp. 1–84, Feb. 1998.
- [13] A. Zingoni, “Group-theoretic exploitations of symmetry in computational solid and structural mechanics,” *International Journal for Numerical Methods in Engineering*, vol. 79, no. 3, pp. 253–289, 2009.
- [14] D. M. Rozelle, “The Hemispherical Resonator Gyro: From Wineglass to the Planets,” in *Proc. 19th AAS/AIAA Space Flight Mechanics Meeting, January 2009*, vol. 134, pp. 1157–1178, Jan. 2009.
- [15] J. Jia, X. Ding, Z. Qin, Z. Ruan, W. Li, X. Liu, and H. Li, “Overview and analysis of MEMS Coriolis vibratory ring gyroscope,” *Measurement*, vol. 182, p. 109704, 2021.
- [16] F. Ayazi and K. Najafi, “A HARPSS polysilicon vibrating ring gyroscope,” *Journal of Microelectromechanical Systems*, vol. 10, pp. 169–179, June 2001.
- [17] H. Johari and F. Ayazi, “Capacitive Bulk Acoustic Wave Silicon Disk Gyroscopes,” in *2006 International Electron Devices Meeting*, Dec. 2006.
- [18] T.-H. Su, S. H. Nitzan, P. Taheri-Tehrani, M. H. Kline, B. E. Boser, and D. A. Horsley, “Silicon MEMS Disk Resonator Gyroscope With an Integrated CMOS Analog Front-End,” *IEEE Sensors Journal*, vol. 14, pp. 3426–3432, Oct. 2014.
- [19] M. W. Putty, *A micromachined vibrating ring gyroscope*. Ph.D. Dissertation, University of Michigan, 1995.
- [20] “CRS39A Series: High Performance MEMS Gyroscope,” technical Datasheet, Silicon Sensing Systems, 2021.

- [21] Z. Hao, S. Pourkamali, and F. Ayazi, “VHF single-crystal silicon elliptic bulk-mode capacitive disk resonators-part I: design and modeling,” *Journal of Microelectromechanical Systems*, vol. 13, pp. 1043–1053, Dec. 2004.
- [22] Z. Liu, H. Wen, and F. Ayazi, “Multi-coefficient eigenmode operation—breaking through $10^\circ/\text{h}$ open-loop bias instability in wideband aluminum nitride piezoelectric BAW gyroscopes,” *Microsystems & Nanoengineering*, vol. 9, pp. 1–11, Feb. 2023.
- [23] M. Parajuli, G. Sobrevela, and A. A. Seshia, “A Silicon MEMS Quatrefoil Suspension Gyroscope,” *Journal of Microelectromechanical Systems*, vol. 32, pp. 416–425, Oct. 2023.
- [24] D. E. Serrano, A. Rahafrooz, D. Younkin, K. Nunan, M. Dalal, S. Pal, I. Jafri, and S. Hiraoka, “Acoustically Isolated MEMS Baw Gyroscopes,” in *2023 IEEE 36th International Conference on Micro Electro Mechanical Systems (MEMS)*, pp. 853–856, Jan. 2023.
- [25] J. Y. Cho, S. Singh, J.-K. Woo, G. He, and K. Najafi, “ $0.00016 \text{ deg}/\sqrt{\text{hr}}$ Angle Random Walk (ARW) and 0.0014 deg/hr Bias Instability (BI) from a 5.2M-Q and 1-cm Precision Shell Integrating (PSI) Gyroscope,” in *2020 IEEE International Symposium on Inertial Sensors and Systems (INERTIAL)*, pp. 1–4, Mar. 2020.
- [26] A. D. Challoner, H. H. Ge, and J. Y. Liu, “Boeing Disc Resonator Gyroscope,” in *2014 IEEE/ION Position, Location and Navigation Symposium - PLANS 2014*, pp. 504–514, May 2014.
- [27] Q. Li, D. Xiao, X. Zhou, Y. Xu, M. Zhuo, Z. Hou, K. He, Y. Zhang, and X. Wu, “ $0.04 \text{ degree-per-hour}$ MEMS disk resonator gyroscope with high-quality factor (510 k) and long decaying time constant (74.9 s),” *Microsystems & Nanoengineering*, vol. 4, pp. 1–11, Nov. 2018.
- [28] Y. Xu, Q. Li, P. Wang, Y. Zhang, X. Zhou, L. Yu, X. Wu, and D. Xiao, “ $0.015 \text{ Degree-Per-Hour}$ Honeycomb Disk Resonator Gyroscope,” *IEEE Sensors Journal*, vol. 21, pp. 7326–7338, Mar. 2021.

- [29] M. M. Torunbalci, S. E. Alper, and T. Akin, “Advanced MEMS Process for Wafer Level Hermetic Encapsulation of MEMS Devices Using SOI Cap Wafers With Vertical Feedthroughs,” *Journal of Microelectromechanical Systems*, vol. 24, pp. 556–564, June 2015.
- [30] E. Tatar, C. Guo, T. Mukherjee, and G. K. Fedder, “Interaction effects of temperature and stress on matched-mode gyroscope frequencies,” in *The 17th International Conference on Solid-State Sensors, Actuators and Microsystems (Transducers Eurosensors XXVII)*, pp. 2527–2530, 2013.
- [31] C. L. Dym and I. H. Shames, *Solid Mechanics: A Variational Approach*. New York, NY: Springer New York, augmented ed., 2013.
- [32] J. N. Reddy, *Energy Principles and Variational Methods in Applied Mechanics*. Hoboken Chichester, U.S.A: Wiley, third ed., 2017.
- [33] M. Hosseini-Pishrobat and E. Tatar, “Analysis of Quadrature and Frequency Split in a MEMS Vibrating Ring Gyroscope with Structural Imperfections,” in *21st International Conference on Solid-State Sensors, Actuators and Microsystems (Transducers)*, (Virtual Conference), pp. 1291–1294, June 2021.
- [34] M. Hosseini-Pishrobat and E. Tatar, “Modeling and Analysis of a MEMS Vibrating Ring Gyroscope Subject to Imperfections,” *Journal of Microelectromechanical Systems*, vol. 31, pp. 546–560, Aug. 2022.
- [35] M. Hosseini-Pishrobat, B. E. Uzunoglu, D. Erkan, and E. Tatar, “An Analytical Model for Vibration Analysis of Disk Resonator Gyroscopes,” in *2022 IEEE International Symposium on Inertial Sensors and Systems (INERTIAL)*, pp. 1–4, May 2022. ISSN: 2377-3480.
- [36] B. J. Gallacher, J. Hedley, J. S. Burdess, A. J. Harris, A. Rickard, and D. O. King, “Electrostatic correction of structural imperfections present in a microring gyroscope,” *Journal of Microelectromechanical Systems*, vol. 14, pp. 221–234, Apr. 2005.
- [37] P. Bisegna and G. Caruso, “Frequency split and vibration localization in imperfect rings,” *Journal of Sound and Vibration*, vol. 306, pp. 691–711, Oct. 2007.

- [38] J. Liu, S. Wang, Z. Wang, N. Gao, and D. Zhang, “Estimation and elimination of eigenvalue splitting and vibration instability of ring-shaped periodic structure subjected to three-axis angular velocity components,” *Meccanica*, vol. 54, pp. 2539–2563, Dec. 2019.
- [39] C.-O. Chang, G.-E. Chang, C.-S. Chou, W.-T. C. Chien, and P.-C. Chen, “In-plane free vibration of a single-crystal silicon ring,” *International Journal of Solids and Structures*, vol. 45, pp. 6114–6132, Dec. 2008.
- [40] R. Eley, C. H. J. Fox, and S. McWilliam, “Anisotropy Effects on the Vibration of Circular Rings Made from Crystalline Silicon,” *Journal of Sound and Vibration*, vol. 228, pp. 11–35, Nov. 1999.
- [41] C. H. Ahn, E. J. Ng, V. A. Hong, Y. Yang, B. J. Lee, I. Flader, and T. W. Kenny, “Mode-matching of wineglass mode disk resonator gyroscope in (100) single crystal silicon,” *Journal of Microelectromechanical Systems*, vol. 24, no. 2, pp. 343–350, 2015.
- [42] S. W. Yoon, S. Lee, and K. Najafi, “Vibration sensitivity analysis of MEMS vibratory ring gyroscopes,” *Sensors and Actuators A: Physical*, vol. 171, pp. 163–177, Nov. 2011.
- [43] D. Jie, “Simulating the performance of ring-based coriolis vibrating gyroscopic sensors,” *Microsystem Technologies*, vol. 25, pp. 139–149, Jan. 2019.
- [44] B. Chouvion, S. McWilliam, and A. Popov, “Effect of nonlinear electrostatic forces on the dynamic behaviour of a capacitive ring-based Coriolis Vibrating Gyroscope under severe shock,” *Mechanical Systems and Signal Processing*, vol. 106, pp. 395–412, 2018.
- [45] Z. Ma, X. Chen, X. Jin, Y. Jin, X. Zheng, and Z. Jin, “Effects of structural dimension variation on the vibration of mems ring-based gyroscopes,” *Micromachines*, vol. 12, no. 12, 2021.
- [46] J. Kirkhope, “In-plane vibration of a thick circular ring,” *Journal of Sound and Vibration*, vol. 50, no. 2, pp. 219–227, 1977.

- [47] S. S. Rao, *Vibration of Continuous Systems*. Hoboken, NJ, USA: John Wiley & Sons Ltd, 2nd ed., 2019.
- [48] G. Genta, ed., *Vibration Dynamics and Control*. Mechanical Engineering Series, Boston, MA: Springer US, 2009.
- [49] T. K. Moon and W. C. Stirling, *Mathematical methods and algorithms for signal processing*. Upper Saddle River, NJ: Prentice Hall, 2000.
- [50] T. Charnley, R. Perrin, V. Mohanan, and H. Banu, “Vibrations of thin rings of rectangular cross-section,” *Journal of Sound and Vibration*, vol. 134, pp. 455–488, Nov. 1989.
- [51] “Electric and magnetic fields,” in *Handbook of Physics* (W. Benenson, J. W. Harris, H. Stocker, and H. Lutz, eds.), pp. 439–500, New York, NY: Springer, 2002.
- [52] M. I. Younis, “Elements of lumped-parameter modeling in MEMS,” in *MEMS Linear and Nonlinear Statics and Dynamics*, pp. 97–153, Boston, MA: Springer US, 2011.
- [53] D. Schiwietz, E. M. Weig, and P. Degenfeld-Schonburg, “Thermoelastic damping in MEMS gyroscopes at high frequencies,” *Microsystems & Nanoengineering*, vol. 9, pp. 1–12, Jan. 2023.
- [54] Q. Li, D. Xiao, X. Zhou, Z. Hou, M. Zhuo, Y. Xu, and X. Wu, “Dynamic Modeling of the Multiring Disk Resonator Gyroscope,” *Micromachines*, vol. 10, p. 181, Mar. 2019.
- [55] C. Lin, J. Zhao, Z. Yao, Q. Fan, B. Mo, and Y. Su, “Vibration modeling of disk resonator gyroscope by wave propagation,” *Journal of Sound and Vibration*, vol. 444, pp. 85–107, Mar. 2019.
- [56] I. P. Prikhodko, S. A. Zotov, A. A. Trusov, and A. M. Shkel, “Foucault pendulum on a chip: Rate integrating silicon MEMS gyroscope,” *Sensors and Actuators A: Physical*, vol. 177, pp. 67–78, Apr. 2012.

- [57] B. Friedland and M. Hutton, “Theory and error analysis of vibrating-member gyroscope,” *IEEE Transactions on Automatic Control*, vol. 23, pp. 545–556, Aug. 1978.
- [58] A. Shkel, R. T. Howe, and R. Horowitz, “Modeling and simulation of micromachined gyroscopes in the presence of imperfections,” in *Technical Proceedings of the 1999 International Conference on Modeling and Simulation of Microsystems*, (San Juan, Puerto Rico, USA), pp. 605–608, Apr. 1999.
- [59] “HF2 User Manual: 50 MHz Lock-in Amplifier,” technical Datasheet, Zurich Instruments, 2024. Available: https://docs.zhinst.com/hf2_user_manual/.
- [60] M. Hosseini-Pishrobat, B. E. Uzunoglu, and E. Tatar, “Modeling Stress Effects on Frequencies of a MEMS Ring Gyroscope,” in *2023 IEEE 36th International Conference on Micro Electro Mechanical Systems (MEMS)*, (Munich, Germany), pp. 869–872, IEEE, Jan. 2023.
- [61] M. Hosseini-Pishrobat, D. Erkan, and E. Tatar, “Analytical and experimental study of stress effects in a MEMS ring gyroscope,” *Sensors and Actuators A: Physical*, vol. 362, p. 114639, Nov. 2023.
- [62] I. P. Prikhodko, A. A. Trusov, and A. M. Shkel, “Compensation of drifts in high-Q MEMS gyroscopes using temperature self-sensing,” *Sensors and Actuators A: Physical*, vol. 201, pp. 517–524, Oct. 2013.
- [63] A. Buffoli, P. Segala, M. Gadola, T. Verdot, P. Robert, and G. Langfelder, “Searching for the Origin of Zero-Rate Offset and Scale-Factor Drift in Nems-Based Nav-Grade Gyroscope,” in *2023 IEEE International Symposium on Inertial Sensors and Systems (INERTIAL)*, pp. 1–4, Mar. 2023.
- [64] “ADIS16488A: Tactical Grade, Ten Degrees of Freedom Inertial Sensor,” technical Datasheet, Analog Devices, 2018. Available: <https://www.analog.com/en/products/adis16488a.html>.

- [65] G. Langfelder, S. Dellea, F. Zaraga, D. Cucchi, and M. Azpeitia Urquia, “The Dependence of Fatigue in Microelectromechanical Systems on the Environment and the Industrial Packaging,” *IEEE Transactions on Industrial Electronics*, vol. 59, pp. 4938–4948, Dec. 2012.
- [66] T. Veske, D. Erkan, and E. Tatar, “Characterization of Packaging Stress with a Capacitive Stress Sensor Array,” in *2023 IEEE 36th International Conference on Micro Electro Mechanical Systems (MEMS)*, pp. 909–912, Jan. 2023.
- [67] S. Schröder, F. Niklaus, A. Nafari, E. R. Westby, A. C. Fischer, G. Stemme, and S. Haasl, “Stress-Minimized Packaging of Inertial Sensors by Double-Sided Bond Wire Attachment,” *Journal of Microelectromechanical Systems*, vol. 24, pp. 781–789, Aug. 2015.
- [68] C. Jurgschat, F. Roewer, I. Tóth, T. Ohms, and A. Zimmermann, “Integrated Stress Sensors for Humidity Performance Drift Analysis and Compensation in Inertial Measurement Units,” *Journal of Microelectromechanical Systems*, vol. 31, pp. 918–926, Dec. 2022.
- [69] W. Mayer, A. Küster, P. Tritschler, T. Hiller, D. Radović, and A. Zimmermann, “Modeling and Experimental Analysis of Low-Cost MEMS Gyroscopes Under PCB Bending Stress,” in *2023 IEEE International Symposium on Inertial Sensors and Systems (INERTIAL)*, pp. 1–4, Mar. 2023.
- [70] C. M. Wang, Y. Y. Zhang, and S. Kitipornchai, “Vibration of initially stressed micro- and nano-beams,” *International Journal of Structural Stability and Dynamics*, vol. 07, pp. 555–570, Dec. 2007.
- [71] H.-W. Ma, S.-M. Yao, L.-Q. Wang, and Z. Zhong, “Analysis of the displacement amplification ratio of bridge-type flexure hinge,” *Sensors and Actuators A: Physical*, vol. 132, pp. 730–736, Nov. 2006.
- [72] M. A. Hopcroft, W. D. Nix, and T. W. Kenny, “What is the Young’s Modulus of Silicon?,” *Journal of Microelectromechanical Systems*, vol. 19, pp. 229–238, Apr. 2010.

- [73] M. Wen, W. Wang, Z. Luo, Y. Xu, X. Wu, F. Hou, and S. Liu, “Modeling and analysis of temperature effect on MEMS gyroscope,” in *2014 IEEE 64th Electronic Components and Technology Conference (ECTC)*, pp. 2048–2052, May 2014.
- [74] M. Ferguson, D. Keymeulen, C. Peay, K. Yee, and D. Li, “Effect of Temperature on MEMS Vibratory Rate Gyroscope,” in *2005 IEEE Aerospace Conference*, pp. 1–6, Mar. 2005.
- [75] H. Cao, Y. Zhang, Z. Han, X. Shao, J. Gao, K. Huang, Y. Shi, J. Tang, C. Shen, and J. Liu, “Pole-Zero Temperature Compensation Circuit Design and Experiment for Dual-Mass MEMS Gyroscope Bandwidth Expansion,” *IEEE/ASME Transactions on Mechatronics*, vol. 24, pp. 677–688, Apr. 2019.
- [76] M. Vujadinović, T. Hiller, L. Blocher, T. Northemann, and B. Choubey, “Towards a Better Understanding of Offset Changes Across Temperature in Mode-Split Open-Loop MEMS Gyroscopes,” in *2023 IEEE International Symposium on Inertial Sensors and Systems (INERTIAL)*, pp. 1–4, Mar. 2023.
- [77] R. B. Hetnarski and M. R. Eslami, *Thermal Stresses-Advanced Theory and Applications*. No. 158 in Solid Mechanics and Its Applications, Cham: Springer International Publishing, 2nd ed., 2019.
- [78] J. Salençon, *Handbook of Continuum Mechanics*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2001.
- [79] Y. Okada and Y. Tokumaru, “Precise determination of lattice parameter and thermal expansion coefficient of silicon between 300 and 1500 K,” *Journal of Applied Physics*, vol. 56, pp. 314–320, July 1984.
- [80] “Schott borofloat® 33: Thermal properties,” technical Datasheet, SCHOTT Technical Glass Solutions GmbH. Available: <https://www.schott.com/en-au/products/borofloat-P1000314>.
- [81] J. N. Reddy, *An Introduction to Continuum Mechanics*. Cambridge: Cambridge University Press, 2nd ed., 2013.

- [82] S. R. Eugster, *Geometric Continuum Mechanics and Induced Beam Theories*, vol. 75 of *Lecture Notes in Applied and Computational Mechanics*. Cham: Springer International Publishing, 2015.
- [83] B. Kim, M. A. Hopcroft, R. N. Candler, C. M. Jha, M. Agarwal, R. Melamud, S. A. Chandorkar, G. Yama, and T. W. Kenny, “Temperature Dependence of Quality Factor in MEMS Resonators,” *Journal of Microelectromechanical Systems*, vol. 17, pp. 755–766, June 2008.
- [84] E. Serra and M. Bonaldi, “A finite element formulation for thermoelastic damping analysis,” *International Journal for Numerical Methods in Engineering*, vol. 78, no. 6, pp. 671–691, 2009.
- [85] O. T. Bruhns, “Curved Beams,” in *Advanced Mechanics of Solids* (O. T. Bruhns, ed.), pp. 95–102, Berlin, Heidelberg: Springer, 2003.
- [86] F. Yang, R. Sedaghati, and E. Esmailzadeh, “Free in-plane vibration of curved beam structures: A tutorial and the state of the art,” *Journal of Vibration and Control*, vol. 24, pp. 2400–2417, June 2018.
- [87] N. S. Bakhvalov and L. G. Vasil’eva, “Evaluation of the integrals of oscillating functions by interpolation at nodes of gaussian quadratures,” *USSR Computational Mathematics and Mathematical Physics*, vol. 8, pp. 241–249, Jan. 1968.
- [88] P. K. Kytke and M. R. Schäferkötter, *Handbook of Computational Methods for Integration*. New York: Chapman and Hall/CRC, Dec. 2004.
- [89] G. W. Stewart and J.-g. Sun, “Generalized Eigenvalue Problems,” in *Matrix Perturbation Theory*, Computer science and scientific computing, Boston: Academic Press, 1990.
- [90] R.-C. Li, “Matrix Perturbation Theory,” in *Handbook of Linear Algebra*, Edited by Leslie Hogben, pp. (Chapter 21) 1–20, New York: Chapman and Hall/CRC, 2nd ed., 2013.

Appendix A

Vibration of a Single Ring

A.1 Basic Ring Mechanics

Our goal is to establish the basic equations governing the in-plane bending of a ring. The schematics of the ring is depicted in Figure A.1. We use the cylindrical coordinate system $\{r, \theta, z\}$ as our reference frame. The ring has a fixed mean radius of R , but we allow a varying width, $w(\theta)$ to accommodate for possible geometric imperfections. Our fundamental assumptions are

1. **Narrow ring:** $\max_{\theta \in [0, 2\pi]} w(\theta)/R \ll 1$; as a result, the neutral axis of the ring almost coincides with its centerline [85]. We define the coordinate ξ as the distance from the centerline, $\xi = r - R$.
2. **Euler-Bernoulli beam theory:** we assume that the ring's cross-sections remain perpendicular to the centerline during the bending, and the shear deformation and the Poisson effect have a negligible contributions to the total elastic energy [31].

Let $u_r(\theta)$ and $u_\theta(\theta)$ be the radial and tangential displacements at the centerline. The rotation due to bending of the cross-section located at the angle θ is given

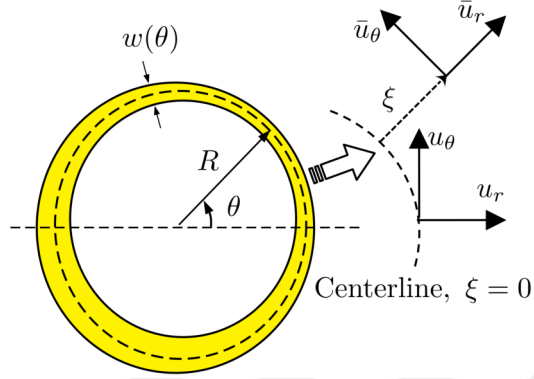


Figure A.1: Schematics of ring

by [86]

$$\gamma(\theta) = \frac{1}{R} \left(u_\theta(\theta) - \frac{\partial u_r(\theta)}{\partial \theta} \right), \quad (\text{A.1})$$

which has two components: the first one is the rotation due to the tangential displacement, and the second is the rotation caused by the gradient of the radial displacement around the cross-section. The displacement field $(\bar{u}_r, \bar{u}_\theta)$ across the whole ring is given by

$$\begin{aligned} \bar{u}_r(\xi, \theta) &= u_r(\theta), \\ \bar{u}_\theta(\xi, \theta) &= u_\theta(\theta) + \xi \gamma(\theta). \end{aligned} \quad (\text{A.2})$$

Note that $\bar{u}_z = 0$ as we consider only the in-plane deformations. Using (A.2), we obtain the extensional Green-Lagrange strain as

$$\mathcal{E}_{\theta\theta} = \varepsilon_{\theta\theta} + \eta_{\theta\theta} \quad (\text{A.3})$$

where $\varepsilon_{\theta\theta}$ and $\eta_{\theta\theta}$ —the linear and nonlinear parts of the strain, respectively—are given by

$$\begin{aligned} \varepsilon_{\theta\theta}(\xi, \theta) &= \frac{1}{R} \left(\bar{u}_r(\xi, \theta) + \frac{\partial \bar{u}_\theta(\xi, \theta)}{\partial \theta} \right) = \varepsilon_0(\theta) + \xi \kappa(\theta), \\ \eta_{\theta\theta}(\xi, \theta) &= \frac{1}{2R^2} \left[\left(\bar{u}_r(\xi, \theta) + \frac{\partial \bar{u}_\theta(\xi, \theta)}{\partial \theta} \right)^2 + \left(\bar{u}_\theta(\xi, \theta) - \frac{\partial \bar{u}_r(\xi, \theta)}{\partial \theta} \right)^2 \right] \\ &= \frac{1}{2} \left[(\varepsilon_0(\theta) + \xi \kappa(\theta))^2 + \left(1 + \frac{\xi}{R} \right)^2 \gamma^2(\theta) \right]. \end{aligned} \quad (\text{A.4})$$

Here, ε_0 and κ are the centerline's extensional strain and curvature change, respectively:

$$\begin{aligned}\varepsilon_0(\theta) &= \frac{1}{R} \left(u_r(\theta) + \frac{\partial u_\theta(\theta)}{\partial \theta} \right), \\ \kappa(\theta) &= \frac{1}{R} \frac{\partial \gamma(\theta)}{\partial \theta} \approx \frac{1}{R + \Delta R} - \frac{1}{R}.\end{aligned}\tag{A.5}$$

The details of calculating Green-Lagrange strain tensor in the cylindrical coordinate system can be found in [81, Chapter 3]. We now consider the constitutive equation

$$\sigma_{\theta\theta}(\xi, \theta) = E(\theta) \mathcal{E}_{\theta\theta}(\xi, \theta) \tag{A.6}$$

for the extensional stress $\sigma_{\theta\theta}$ associated with the strain $\mathcal{E}_{\theta\theta}$. E is the Young's modulus of the ring's material, and its dependence on θ accounts for possible elastic anisotropy. The volumetric density of the strain energy of the ring is

$$\frac{d\mathcal{U}}{dV} = \frac{1}{2} \sigma_{\theta\theta} \mathcal{E}_{\theta\theta} = \frac{1}{2} E \mathcal{E}_{\theta\theta}^2. \tag{A.7}$$

Assuming that the ring's out-of-plane thickness is h , we integrate (A.7) over the ring's volume ($dV = hRd\theta d\xi$) to obtain the total strain energy:

$$\begin{aligned}\mathcal{U} &= \mathcal{U}_1 + \mathcal{U}_2, \\ \mathcal{U}_1 &= \frac{1}{2} Rh \int_0^{2\pi} E(\theta) \int_{-w(\theta)/2}^{w(\theta)/2} \varepsilon_{\theta\theta}^2(\xi, \theta) d\xi d\theta, \\ \mathcal{U}_2 &= \frac{1}{2} Rh \int_0^{2\pi} E(\theta) \int_{-w(\theta)/2}^{w(\theta)/2} \{ \eta_{\theta\theta}^2(\xi, \theta) + 2\eta_{\theta\theta}(\xi, \theta) \varepsilon_{\theta\theta}(\xi, \theta) \} d\xi d\theta.\end{aligned}\tag{A.8}$$

$\mathcal{U}_1$ is the nominal potential energy corresponding to the linear strain, and $\mathcal{U}_2$ is the potential due to geometric nonlinearity, that is, the higher-order strain.

Inextensibility. The assumption that the centerline's circumference undergoes no variations, i.e., $\varepsilon_0(\theta) = 0$, further simplifies the ring's analysis by imposing the condition

$$u_r(\theta) + \frac{\partial u_\theta(\theta)}{\partial \theta} = 0. \tag{A.9}$$

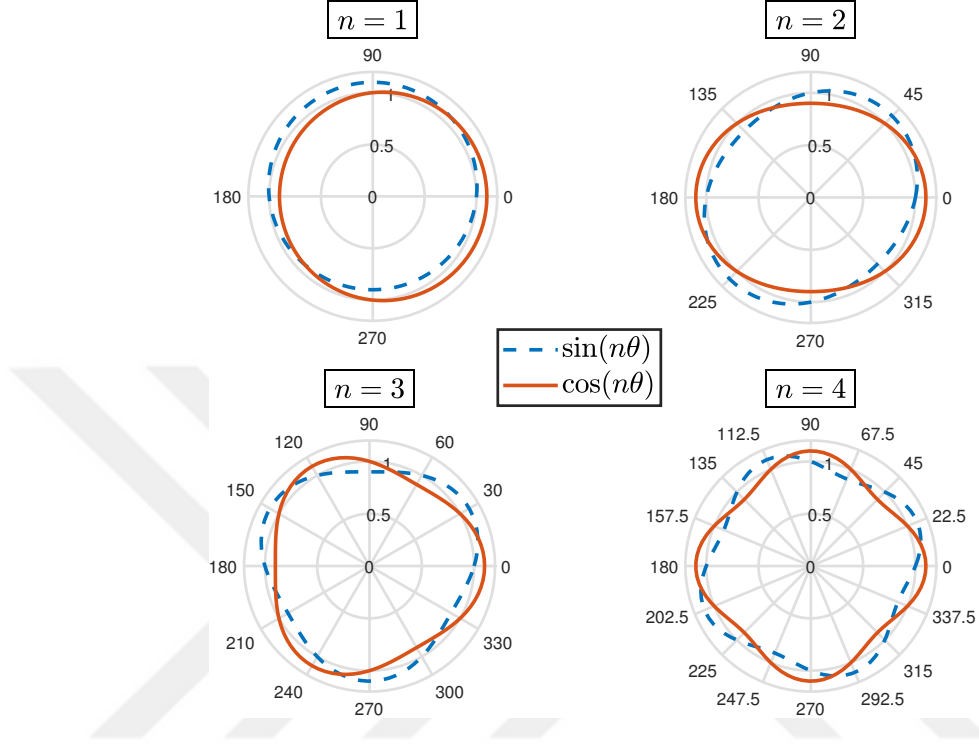


Figure A.2: The mode shape pairs $n = 1 : 4$ (for radial displacement) of the ring.

Kinetic energy. In the dynamic analysis where the displacement field (A.2) depends on time $t \geq 0$, the kinetic energy's volumetric density is given by

$$\frac{d\mathcal{T}}{dV} = \frac{1}{2}\rho \left(\dot{\bar{u}}_r^2(\xi, \theta, t) + \dot{\bar{u}}_\theta^2(\xi, \theta, t) \right), \quad (\text{A.10})$$

which gives us

$$\mathcal{T} = \frac{1}{2}\rho R h \int_0^{2\pi} \int_{-w(\theta)/2}^{w(\theta)/2} \left\{ \dot{u}_r^2(\theta, t) + \dot{u}_\theta^2(\theta, t) + \xi^2 \gamma^2(\theta, t) \right\} d\xi d\theta, \quad (\text{A.11})$$

where ρ is the mass density.

A.2 Mode Shapes and Natural Frequencies

Under the inextensibility condition (A.9) and assuming $u_\theta(\theta, t) = f(\theta)e^{j\omega t}$, with $\omega > 0$ being the natural frequency, the ring's kinetic and potential energies can be regarded as functionals $\mathcal{T}[f(\theta)]$ and $\mathcal{U}[f(\theta)]$ of the mode shape $f(\theta)$, respectively. The differential operators of mass $\mathcal{M}[\cdot]$ and stiffness $\mathcal{K}[\cdot]$ of the ring can

be obtained from

$$\begin{aligned}\delta\mathcal{T}[f; \delta f] &= -\omega^2 \int_0^{2\pi} \mathcal{M}[f] \delta f d\theta e^{2j\omega t} \\ \Rightarrow \mathcal{M}[f] &= \rho \left\{ \frac{I}{R} - \left(AR - 2\frac{I}{R} \right) \frac{\partial^2}{\partial \theta^2} + \frac{I}{R} \frac{\partial^4}{\partial \theta^4} \right\} f,\end{aligned}\quad (\text{A.12})$$

$$\begin{aligned}\delta\mathcal{U}[f; \delta f] &= \int_0^{2\pi} \mathcal{K}[f] \delta f d\theta e^{2j\omega t} \\ \Rightarrow \mathcal{K}[f] &= -\frac{EI}{R^3} \left\{ \frac{\partial^2}{\partial \theta^2} + 2\frac{\partial^4}{\partial \theta^4} + \frac{\partial^6}{\partial \theta^6} \right\} f,\end{aligned}\quad (\text{A.13})$$

where δ denotes the variation operator, $A = wh$ is the ring's cross-section area, and $I = w^3h/12$ is the cross-section's second moment of area. The mode shapes and natural frequencies are solutions of the eigenvalue problem

$$\mathcal{K}[f] = \omega^2 \mathcal{M}[f]. \quad (\text{A.14})$$

The circular symmetry condition $f(\theta + 2\pi) = f(\theta)$ implies that f belongs to the set of real-valued, 2π -periodic, continuous, nonconstant functions, which form a Hilbert space under the inner product $\langle f_1, f_2 \rangle = \int_0^{2\pi} f_1(\theta) f_2(\theta) d\theta$. Therefore, according to the Fourier expansion theorem, $f(\theta) \in \text{span}\{\cos(n\theta), \sin(n\theta)\}_{n=1}^\infty$ [49]. Since $\langle \cos(n\theta), \sin(n\theta) \rangle = 0$, for each $n = 1, 2, \dots$, $f_{n,1}(\theta) = \cos(n\theta)/n$ and $f_{n,2}(\theta) = -\sin(n\theta)/n$ —and their radial displacement counterparts $f'_{n,1}(\theta) = \cos(n\theta)$ and $f'_{n,2}(\theta) = \sin(n\theta)$ —constitute independent mode shapes (see Figure A.2). Moreover, their orthogonality is preserved under the mass and stiffness operators:

$$\begin{aligned}\langle f_{n,i}, (\mathcal{M},) \mathcal{K}[f_{n,j}] \rangle &= 0 \text{ for } i \neq j, \\ \langle f_{n,1}, (\mathcal{M},) \mathcal{K}[f_{n,1}] \rangle &= \langle f_{n,2}, (\mathcal{M},) \mathcal{K}[f_{n,2}] \rangle.\end{aligned}\quad (\text{A.15})$$

As a result,

$$\omega_n^2 = \frac{\langle f_{n,1}, \mathcal{K}[f_{n,1}] \rangle}{\langle f_{n,1}, \mathcal{M}[f_{n,1}] \rangle} = \frac{\langle f_{n,2}, \mathcal{K}[f_{n,2}] \rangle}{\langle f_{n,2}, \mathcal{M}[f_{n,2}] \rangle} \quad (\text{A.16})$$

implying that $f_{n,1}(\theta)$ and $f_{n,2}(\theta)$ form a degenerate pair of mode shape as they share the natural frequency

$$\omega_n = \frac{w}{\sqrt{12}R^2} \sqrt{\frac{E}{\rho} \frac{n(n^2 - 1)}{\sqrt{n^2 + 1 + \left(\frac{w}{R}\right)^2 (n^2 - 1)^2}}} \approx \frac{w}{\sqrt{12}R^2} \sqrt{\frac{E}{\rho} \frac{n(n^2 - 1)}{\sqrt{n^2 + 1}}}, \quad (\text{A.17})$$

where the last approximation is based on the narrow ring assumption.

Appendix B

Technical Results

B.1 Numerical Integration of Oscillatory Functions

As explained in Chapter 2, eigenvector computation is very sensitive to numerical errors, which could result in erroneous rotation angles for the mode shapes. This issue, in effect, necessitates an accurate numerical integration scheme for calculating a circular gyroscope's mass and stiffness matrices. Based on the particular oscillatory form of the integrals involved in these matrices, we focus on calculating the *oscillatory integrals*

$$\int_{-1}^1 f(x) e^{j\omega x} dx, \quad (\text{B.1})$$

where $f: [-1, 1] \rightarrow \mathbb{R}$ is an integrable function and $\omega \geq 0$ is the oscillation frequency of the integrand. Following the method proposed in [87], we approximate $f(x)$ by the following truncated Legendre expansion:

$$f(x) \approx \sum_{i=0}^3 \alpha_i P_i(x), \quad (\text{B.2})$$

where $P_i(x)$ is the i -th Legendre polynomial and α_i is its corresponding weight given by

$$\alpha_i = \frac{\int_{-1}^1 f(x) P_i(x) dx}{\int_{-1}^1 P_i^2(x) dx}. \quad (\text{B.3})$$

The Legendre polynomials satisfy the following properties [88]:

1. Recursion:

$$\begin{aligned} P_{i+1}(x) &= \frac{2i+1}{i+1} x P_i(x) - \frac{i}{i+1} P_{i-1}(x), \text{ for } i \geq 1, \\ P_0(x) &= 1, \quad P_1(x) = x. \end{aligned} \quad (\text{B.4})$$

2. Orthogonality:

$$\int_{-1}^1 P_i(x) P_j(x) dx = \frac{2}{2i+1} \delta_{i,j}. \quad (\text{B.5})$$

3. Oscillatory integral:

$$\int_{-1}^1 P_i(x) e^{j\omega x} dx = j^i \sqrt{\frac{2\pi}{\omega}} J_{i+\frac{1}{2}}(\omega), \quad (\text{B.6})$$

where $J_{i+\frac{1}{2}}(\cdot)$ is the Bessel function of the first kind.

Plugging (B.2) into (B.1) and using the formula (B.6), we obtain the following approximation:

$$\int_{-1}^1 f(x) e^{j\omega x} dx \approx \sum_{i=0}^3 \alpha_i j^i \sqrt{\frac{2\pi}{\omega}} J_{i+\frac{1}{2}}(\omega). \quad (\text{B.7})$$

To calculate α_i , we use the four-point Gauss-Legendre integration formula [88]:

$$\alpha_i \approx \frac{2i+1}{2} \sum_{l=1}^4 \varpi_l f(x_l) P_i(x_l) \quad (\text{B.8})$$

where

$$x_{1,4} = \pm \sqrt{\frac{3}{7} + \frac{2}{7} \sqrt{\frac{6}{5}}}, \quad \varpi_{1,4} = \frac{18 - \sqrt{30}}{36},$$

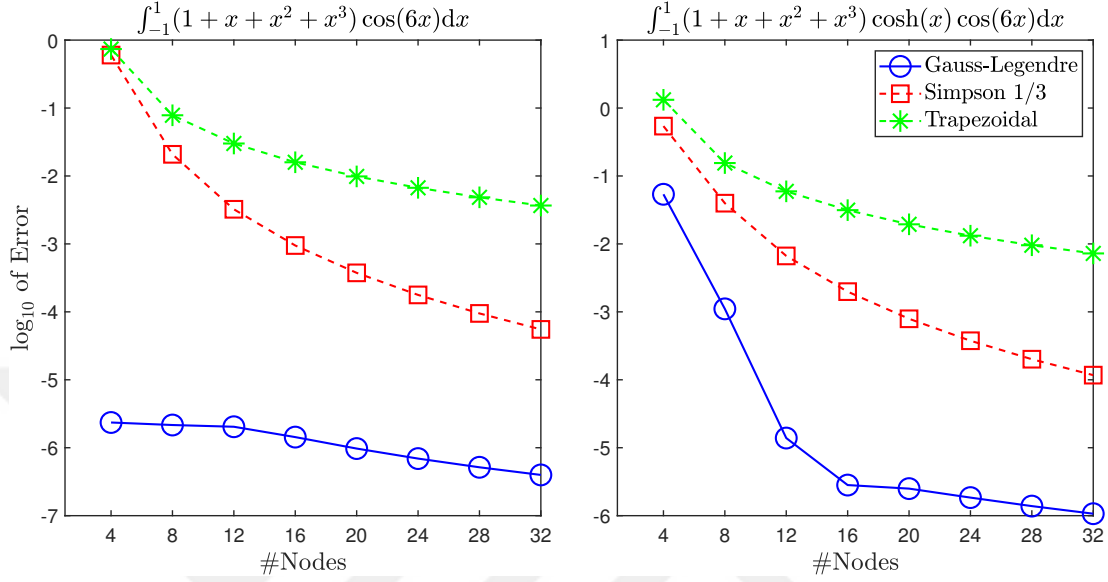


Figure B.1: Comparison of numerical integration errors of different methods.

$$x_{2,3} = \pm \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}, \quad \varpi_{2,3} = \frac{18 + \sqrt{30}}{36}.$$

We note that integration over any arbitrary finite interval can be converted to the integration on $[-1, 1]$ by the following change of variable:

$$\int_a^b f(y) e^{\gamma \omega y} dy = \frac{b-a}{2} e^{\gamma \omega \frac{a+b}{2}} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right) e^{\gamma \omega \frac{b-a}{2}x} dx. \quad (\text{B.9})$$

To increase accuracy, one can partition a given integration interval into smaller subintervals, apply the developed numerical integration algorithm over each subinterval, and take the summation of all of the obtained values.

To illustrate the proposed method's accuracy, we compared it with the popular trapezoidal and Simpson's 1/3 integration schemes. Figure B.1 shows the comparative graphs of the integration errors for a polynomial and nonpolynomial $f(x)$. Our method is highly accurate for the polynomial case since $f(x)$ lies in the subspace spanned by the Legendre polynomials. For the nonpolynomial case, the convergence rate of the proposed method is the highest, outperforming the other two algorithms.

B.1.1 MATLAB implementation

The implementation of the proposed numerical integration consists of two main MATLAB functions:

1. The first function receives the lower and upper bounds of the integration along with the user-specified number of subintervals and generates the required nodes for Gauss-Legendre integration:

```
1      function x = GaussLegendre.discretization(lower_bound,
2          upper_bound, N)
3
4          % Generating nodes for Gauss-Legendre quadrature over [
5              lower_bound, upper_bound] using N equal intervals.
6
7          % Zeros of Legendre polynomial P_4(x)
8          zer1 = sqrt(3/7-(2/7)*sqrt(6/5));
9          zer2 = sqrt(3/7+(2/7)*sqrt(6/5));
10
11          x_dummy = linspace(lower_bound, upper_bound, N+1);
12          A = [1 + zer2, 1 - zer2;
13              1 + zer1, 1 - zer1;
14              1 - zer1, 1 + zer1;
15              1 - zer2, 1 + zer2]/2;
16          x = zeros(4*N, 1);
17
18          for i = 1:N
19              x(4*i-3:4*i, 1) = A*[x_dummy(i);x_dummy(i+1)];
20          end
21      end
```

2. The second function carries out the integration using the nodes generated by the first function:

```
1      function [y] = legpoly_int(t,x,w)
2          % Numerical integration of x(t)*exp(lj*w*t) using Legendre
3              polynomials
4          %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
5          % Number of nodes should be 4*N (4,7,...)
6          N = floor(numel(t)/4);
```

```

6      %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
7      % Zeros of the Legendre polynomial P4(x)
8      zer1 = sqrt(3/7-(2/7)*sqrt(6/5));
9      zer2 = sqrt(3/7+(2/7)*sqrt(6/5));
10     %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
11     % Gauss-Legendre weights
12     weigh1 = (18+sqrt(30))/36;
13     weigh2 = (18-sqrt(30))/36;
14     %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
15     % Nodal values of Legendre polynomials
16     A = [1.0000    -0.8611     0.6123    -0.3047;
17          1.0000    -0.3400    -0.3266     0.4117;
18          1.0000     0.3400    -0.3266    -0.4117;
19          1.0000     0.8611     0.6123     0.3047];
20     %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
21     % Integration
22     y = 0;
23     for i = 1:N
24         da = (t(4*i-1)-t(4*i))/(zer1-zer2);
25         a_avg = (zer1*t(4*i)-zer2*t(4*i-1))/(zer1-zer2);
26         w1 = w*da;
27         % Integrals of legendre polynomial P0(t),...,P3(t)
28         if w==0
29             Jp = [2 0 0 0];
30         else
31             for kk = 1:4
32                 Jp(kk) = sqrt(2*pi/w1)*(1j)^(kk-1)*besselj(kk-0.5,w1);
33             end
34         end
35         % Coefficients (approximated by Gauss-Legendre quadrature)
36         xvec = [weigh2*x(4*i-3);weigh1*x(4*i-2);weigh1*x(4*i-1);
37                 weigh2*x(4*i)];
38         alpha = zeros(4,1);
39         for kj = 1:4
40             alpha(kj) = ((2*kj-1)/2)*dot(xvec,A(:,kj));
41         end
42         y = y + da*exp(1j*w*a_avg)*dot(alpha,Jp);
43     end

```

B.2 Eigenvalue and Eigenvector Perturbation Analysis

Detailed and mathematically rigorous treatment of general eigenvalue perturbation problems can be found in [89, 90]. Here, we present a version tailored to our specific application arising in stress stiffness analysis. We consider a gyroscope with the initial mass and stiffness matrices $M \in \mathbb{R}^{2 \times 2}$ and $K \in \mathbb{R}^{2 \times 2}$, respectively. The eigenproblem associated with the pair (M, K) is

$$Kv_i = \lambda_i Mv_i, \quad i = 1, 2, \quad (\text{B.10})$$

where $\lambda_i = \omega_i^2$ are the eigenvalues and $v_i \in \mathbb{R}^2$ are the eigenvectors, which—as shown in Figure B.2—are parametrized by the angles ψ_i :

$$\begin{aligned} v_1 &= [\cos(\psi_1), \sin(\psi_1)]^\top, \\ v_2 &= [-\sin(\psi_2), \cos(\psi_2)]^\top. \end{aligned} \quad (\text{B.11})$$

The eigenvectors are orthogonal with respect to the mass and stiffness matrices [48]:

$$\begin{aligned} v_i^\top Mv_j &= \bar{m}_i \delta_{i,j}, \\ v_i^\top Kv_j &= \bar{k}_i \delta_{i,j}, \quad i, j \in \{1, 2\}, \end{aligned} \quad (\text{B.12})$$

where $\bar{m}_i$ and $\bar{k}_i = \lambda_i \bar{m}_i$ are the modal mass and stiffness in the principal coordinates defined by the eigenvectors, respectively, and $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. Under an external effect, such as stress or temperature, the stiffness matrix is perturbed: $K \mapsto K + \delta K$, where $\delta K \in \mathbb{R}^{2 \times 2}$ is itself a stiffness matrix (i.e., it is symmetric positive definite), but it is small compared to K . As a result, we have the perturbed eigenvalue problem

$$(K + \delta K)(v_i + \delta v_i) = (\lambda_i + \delta \lambda_i)M(v_i + \delta v_i), \quad i = 1, 2. \quad (\text{B.13})$$

Our goal is to estimate the small perturbations $\delta \lambda_i$ and $\delta v_i \in \mathbb{R}^2$ in the eigenvalues and eigenvectors, respectively. For this, we first note that the initial eigenvectors form a (not necessarily orthogonal) basis for $\mathbb{R}^2$, and hence, we can express

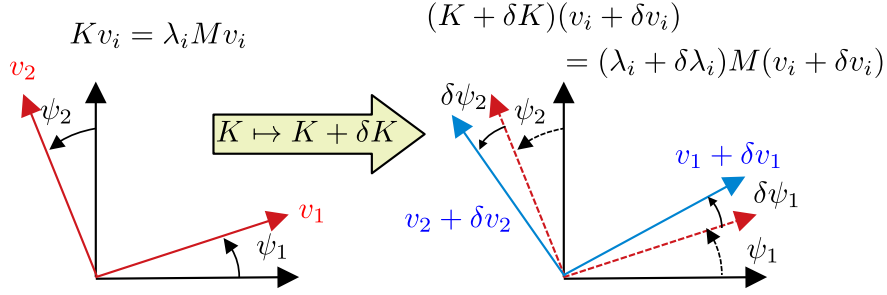


Figure B.2: Perturbation of eigenvectors.

δv_i as

$$\delta v_i = \sum_{j=1}^2 a_{ij} v_j, \quad (\text{B.14})$$

where a_{ij} are the pertinent coefficients. Equation (B.14) gives us the following key properties:

$$\begin{aligned} v_j^\top M \delta v_i &= a_{ij} \bar{m}_j, \\ v_j^\top M \delta v_i &= a_{ij} \bar{k}_j. \end{aligned} \quad (\text{B.15})$$

We further introduce the matrix $\bar{\delta K} \in \mathbb{R}^{2 \times 2}$ as the transformation of δK into the principal coordinates:

$$\begin{aligned} \bar{\delta K} &= \begin{bmatrix} v_1^\top \\ v_2^\top \end{bmatrix} K \begin{bmatrix} v_1 & v_2 \end{bmatrix} = [\bar{\delta k}_{ij}], \\ \bar{\delta k}_{11} &= \left(\frac{\delta k_{11} + \delta k_{22}}{2} \right) + \left(\frac{\delta k_{11} - \delta k_{22}}{2} \right) \cos(2\psi_1) + \delta k_{12} \sin(2\psi_1), \\ \bar{\delta k}_{22} &= \left(\frac{\delta k_{11} + \delta k_{22}}{2} \right) - \left(\frac{\delta k_{11} - \delta k_{22}}{2} \right) \cos(2\psi_2) - \delta k_{12} \sin(2\psi_2), \\ \bar{\delta k}_{12} &= \left(\frac{\delta k_{11} + \delta k_{22}}{2} \right) \sin(\psi_1 - \psi_2) - \left(\frac{\delta k_{11} - \delta k_{22}}{2} \right) \sin(\psi_1 + \psi_2) \\ &\quad + \delta k_{12} \cos(\psi_1 + \psi_2). \end{aligned} \quad (\text{B.16})$$

Eigenvalue perturbation. Expanding (B.13) and neglecting the second-order terms gives us

$$(K - \lambda_i M) \delta v_i + (\delta K - \delta \lambda_i M) v_i \approx 0, \quad i = 1, 2. \quad (\text{B.17})$$

By multiplying both sides of (B.17) by $v_i^\top$ and using the property (B.15), we obtain

$$\delta\lambda_i \approx \frac{\overline{\delta k_{ii}}}{\overline{m_{ii}}} \implies \delta\omega_i \approx \frac{\overline{\delta k_{ii}}}{2\overline{k_i}} \omega_i. \quad (\text{B.18})$$

Eigenvector perturbation. Introducing $\delta\psi_i$ as the perturbation of ψ_i and defining $\tilde{\psi} := \psi_1 - \psi_2$ (see Figure B.2), we can parametrize the a_{ij} coefficients as

$$\begin{aligned} a_{11} &= \frac{\cos(\tilde{\psi} + \delta\psi_1)}{\cos(\tilde{\psi})} - 1, \quad a_{12} = \frac{\sin(\delta\psi_1)}{\cos(\tilde{\psi})}, \\ a_{21} &= -\frac{\sin(\delta\psi_2)}{\cos(\tilde{\psi})}, \quad a_{22} = \frac{\cos(\tilde{\psi} - \delta\psi_2)}{\cos(\tilde{\psi})} - 1. \end{aligned} \quad (\text{B.19})$$

The orthogonality of the perturbed eigenvectors with respect to the perturbed stiffness matrix, i.e.,

$$(v_1 + \delta v_1)^\top (K + \delta K)(v_2 + \delta v_2) = 0, \quad (\text{B.20})$$

reduces to

$$a_{21}\overline{k_1} + a_{12}\overline{k_2} + (1 + a_{12}a_{21})\overline{\delta k_{12}} \approx 0. \quad (\text{B.21})$$

Assuming $\tilde{\psi} = 0$, we obtain the following small angle approximation from (B.21):

$$\delta\psi_i \approx \frac{\overline{\delta k_{12}}}{\overline{k_1} - \overline{k_2}}. \quad (\text{B.22})$$

B.3 Proofs of Results in Chapter 4

B.3.1 Proof of Proposition 4.1

We prove Proposition 4.1 for a more general case of an elastic solid with the temperature-dependent elasticity tensor $\mathbf{C}$ and thermal expansion tensor $\boldsymbol{\alpha}$ undergoing the temperature change $\Delta T = T - T_0$ with respect to a stress-free reference temperature T_0 . We first consider the following variational problem:

Problem 1. Find a displacement field u that minimizes the thermoelastic potential

$$\mathcal{U} = \int_{\Omega} \left\{ \frac{1}{2} \mathbf{C}(T) \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} - \mathbf{C}(T) \boldsymbol{\varepsilon}_T : \boldsymbol{\varepsilon} \right\} dV, \quad (\text{B.23})$$

subject to the nonhomogeneous Dirichlet boundary conditions

$$Au = b \text{ at } \Gamma_D \subseteq \partial\Omega. \quad (\text{B.24})$$

Here, $\Omega \subset \mathbb{R}^3$ is a compact set representing the domain occupied by the solid, $\partial\Omega$ is the boundary of Ω , and dV is the differential volume element in Ω ; $\boldsymbol{\varepsilon} = \nabla u$ is the strain tensor, $\boldsymbol{\varepsilon}_T = \boldsymbol{\alpha} \Delta T$ is the thermal strain, and $:$ denotes the tensor inner product [81]; A is the linear operator associated with the nonzero boundary conditions b at a subset Γ_D of the domain's boundary.

Next, we consider the following two subproblems:

Subproblem 1.1. Find a displacement field u_1 that solves Problem 1 under homogeneous boundary conditions, $b = 0$.

Subproblem 1.2. Find a displacement field u_2 that solves Problem 1 under zero thermal strain, $\boldsymbol{\varepsilon}_T = 0$.

Let u^* , u_1^* , and u_2^* be the solutions of Problem 1, Subproblem 1.1, and Subproblem 1.2, respectively. We now show that the superposition

$$u^* = u_1^* + u_2^* \quad (\text{B.25})$$

holds. To do this, using the multiplier λ , we form the Lagrangian

$$\mathfrak{L} = \mathcal{U} + \lambda^\top (Au - b) |_{\Gamma_D}, \quad (\text{B.26})$$

and apply the optimality condition $\delta \mathfrak{L} = 0$ (δ denotes the first variation) for each problem separately:

- **Problem 1:** u^* satisfies

$$\delta \mathfrak{L} = \int_{\Omega} (\mathbf{C}(T)\varepsilon^* - \varepsilon_T) : \delta \varepsilon \, dV + \delta \lambda^\top (Au^* - b) |_{\Gamma_D} = 0, \quad (\text{B.27})$$

for any admissible variations δu and $\delta \lambda$.

- **Subproblem 1.1:** u_1^* satisfies

$$\delta \mathfrak{L}_1 = \int_{\Omega} (\mathbf{C}(T)\varepsilon_1^* - \varepsilon_T) : \delta \varepsilon_1 \, dV + \delta \lambda_1^\top (Au_1^* - b) |_{\Gamma_D} = 0, \quad (\text{B.28})$$

for any admissible variations δu_1 and $\delta \lambda_1$.

- **Subproblem 1.2:** u_2^* satisfies

$$\delta \mathfrak{L}_2 = \int_{\Omega} \mathbf{C}(T)\varepsilon_2^* : \delta \varepsilon_2 \, dV + \delta \lambda_2^\top (Au_2^* - b) |_{\Gamma_D} = 0, \quad (\text{B.29})$$

for any admissible variations δu_2 and $\delta \lambda_2$.

We note that $\varepsilon^* = \nabla u^*$, $\delta \varepsilon = \nabla(\delta u)$, and so on. By setting $\delta u_1 = \delta u_2 = \delta u$ and $\delta \lambda_1 = \delta \lambda_2 = \delta \lambda$, we have

$$\begin{aligned} \delta \mathfrak{L}_1 + \delta \mathfrak{L}_2 &= \int_{\Omega} (\mathbf{C}(T)(\varepsilon_1^* + \varepsilon_2^*) - \varepsilon_T) : \delta \varepsilon \, dV \\ &+ \delta \lambda^\top (A(u_1^* + u_2^*) - b) |_{\Gamma_D} = 0 \implies \delta \mathfrak{L} = 0. \end{aligned} \quad (\text{B.30})$$

Therefore, $u_1^* + u_2^*$ is a solution to Problem 1, which, by virtue of its well-posedness and uniqueness of solution [77], implies (B.25).

B.3.2 A lemma on QP problems

Lemma B.1. *Consider the following QP problem parametrized by $\zeta_1, \zeta_2 \in \mathbb{R}$:*

$$\mathcal{QP} : \begin{cases} x^*(\zeta_1, \zeta_2) = \arg \min_x \frac{1}{2} x^\top P x + \zeta_1 q^\top x \\ \text{subject to } Ax = \zeta_2 b, \end{cases} \quad (\text{B.31})$$

where P is positive definite and A has full rank. Select two reference values $\hat{\zeta}_1 \neq 0$ and $\hat{\zeta}_2 \neq 0$, and consider the following two QPs:

$$\mathcal{QP}_1 : \begin{cases} x_1^*(\zeta_1) = \arg \min_x \frac{1}{2}x^\top Px + \zeta_1 q^\top x \\ \text{subject to } Ax = 0. \end{cases} \quad (\text{B.32})$$

$$\mathcal{QP}_2 : \begin{cases} x_2^*(\zeta_2) = \arg \min_x \frac{1}{2}x^\top Px \\ \text{subject to } Ax = \zeta_2 b. \end{cases} \quad (\text{B.33})$$

Then, the superposition relation

$$x^*(\zeta_1, \zeta_2) = \frac{\zeta_1}{\hat{\zeta}_1} x_1^*(\hat{\zeta}_1) + \frac{\zeta_2}{\hat{\zeta}_2} x_2^*(\hat{\zeta}_2) \quad (\text{B.34})$$

holds between the solutions of $\mathcal{QP}$, $\mathcal{QP}_1$ and $\mathcal{QP}_2$.

Proof. By forming the Lagrangian and applying the optimality condition, we obtain the following linear systems:

$$\mathcal{QP} : \begin{bmatrix} P & A^\top \\ A & O \end{bmatrix} \begin{bmatrix} x^* \\ \lambda \end{bmatrix} = \begin{bmatrix} -\zeta_1 q \\ \zeta_2 b \end{bmatrix}, \quad (\text{B.35a})$$

$$\mathcal{QP}_1 : \begin{bmatrix} P & A^\top \\ A & O \end{bmatrix} \begin{bmatrix} x_1^* \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} -\hat{\zeta}_1 q \\ 0 \end{bmatrix}, \quad (\text{B.35b})$$

$$\mathcal{QP}_2 : \begin{bmatrix} P & A^\top \\ A & O \end{bmatrix} \begin{bmatrix} x_2^* \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{\zeta}_2 b \end{bmatrix}, \quad (\text{B.35c})$$

where λ , λ_1 , and λ_2 are the Lagrange multipliers. By multiplying (B.35b) and (B.35c) by $\zeta_1/\hat{\zeta}_1$ and $\zeta_2/\hat{\zeta}_2$, respectively, summing the result and comparing it with (B.35a) we deduce (B.34). Note that the strict convexity of $\mathcal{QP}$ guarantees the uniqueness of x^* . $\square$