Analysis of Admission and Discharge Controls with Retrials in Intensive Care Units

by

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Koç University

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This is to certify that I have examined this copy of a doctoral dissertation by

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To my daughter Jülide...

ABSTRACT

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Faruk Akın
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Intensive Care Units (ICUs) are scarce resources and operate most of the time under high occupancy rates. When faced with limited bed availability, arriving patients are sometimes refused or admitted by discharging an existing patient early, which may result in both increased readmission rates and patient/hospital related costs. Yet, there is not a well-defined admission and discharge control policy to decrease such adverse consequences.

In this thesis, we consider a conceptual ICU setting that serves multiple types of patients, which reflects the trade-offs between first-time and recurring patients, between different health stages as well as between the early-discharge and rejection decisions. To do this, we define a so-called readmission orbit whose population consists of patients who are to be readmitted after a previous ICU discharge. There are two major approaches available to represent and analyze such a model: Markov Decision Processes (MDPs) and fluid approximations. Since the conceptual model suffers from the curse of dimensionality, we first consider a simple version of the conceptual model with only one patient type and one health condition, by which we isolate the effects various admission and discharge decisions on readmissions. We present a discrete-time MDP formulation for the simple model and investigate the structure of the optimal admission and discharge control policy under such a model. Next, we present discrete-time MDP formulation for the conceptual model, which aims to develop a well-performing and implementable policy, rather than finding an optimal policy. Then, we develop a deterministic fluid model to control admissions and discharges in such an environment and show that the optimal control of the fluid model in the steady state can be obtained by solving a nonlinear problem. We

derive a heuristic policy based on the solution of this nonlinear problem. Finally, we develop a discrete-event simulation platform that represents the ICU system. The platform is quite general, so that it can represent ICUs with different characteristics as well as a variety of control policies. We use this platform to benchmark the performance of the proposed heuristic policies, with those of the two state-of-the-art policies. Numerical results show that our proposed policies significantly outperform the the state-of-the-art policies.

ÖZETÇE

Yoğun Bakım Ünitelerinde Tekrar Denemeli Hasta Kabul ve Taburcu Etme Kontrollerin İncelenmesi

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Endüstri Mühendisliği ve Yöneylem Araştırması, Doktora

11 Ağustos 2020

Yoğun Bakım Üniteleri (YBÜ) kıt kaynaklardır ve çoğu zaman yüksek doluluk oranlarıyla çalışırlar. Sınırlı sayıda boş yatak bulunduğu durumlarda, yeni gelen hastalar bazen YBÜ'ye reddedilirler, bazen de mevcut bir hastanın erken taburcu edilmesiyle kabul edilirler ki, bu durum hem hasta geri kabullerini hem de hasta/hastaneler için maliyetleri artırabilir. Bununla birlikte, söz konusu olumsuz sonuçları azaltmaya yönelik, iyi tanımlanmış bir hasta kabul ve taburcu etme planı bulunmamaktadır.

Bu tezde, birçok hasta tipine hizmet veren ve ilk defa gelen hastalar ile tekrar kabul edilen hastalar arasındaki ilişkiyle, farklı hastalık seviyeleri arasındaki ilişkiyi yansıtan, aynı zamanda da yeni gelen hastaları reddetme ve onları kabul ederek başka hastaları erken taburcu etme kararlarları arasındaki ilişkiyi inceleyen kavramsal bir YBÜ tasarlıyoruz. Bunu yapabilmek için, daha önce YBÜ'den taburcu edilmiş hastalardan oluşan sanal bir yeniden-kabul yörüngesi tanımlıyoruz. Bu tarz modelleri temsil etmek ve incelemek için kullanılan iki ana yaklaşım vardır: Markov Karar Süreçleri (MKS) ve akışkan modelli yaklaşımlar. Kavramsal modelde, onu teorik olarak inceleyemeyecek kadar fazla boyut olduğundan, ilk olarak kavramsal modelden biraz uzaklaşıp, tek hasta tipi ve tek iyileşme sürecinden oluşan daha basit bir model ele alıyoruz ve böylelikle hasta kabul ve taburcu etme kararlarının hasta geri kabulleri üzerindeki etkisini arındırmış oluyoruz. Bu basit model için kesik zamanlı bir MKS sunuyoruz ve en iyi kabul ve taburcu etme planının yapısını inceliyoruz. Daha sonra, kavramsal model için en iyi kabul ve taburcu etme planı bulmaktan ziyade, başarılı sonuçlar üreten planlar bulmak üzere kesik zamanlı bir MKS sunuyoruz. Daha sonra, kavramsal modeldeki gibi bir ortamda hasta kabul ve taburcu etme işlemlerini kontrol etmek üzere belirleyici akışkan bir model geliştiriyoruz ve durağan durumdaki akışkan modelin en iyi planının doğrusal olmayan bir problemi çözerek elde edileceğini gösteriyoruz. Doğrusal olmayan problemin sayısal çözümünü kullanan sezgisel bir plan öneriyoruz. Son olarak, YBÜ ortamını temsil eden ayrık olaylı bir benzetim ortamı geliştiriyoruz. Benzetim ortamı YBÜ'deki değişik boyutları temsil ettiği gibi çeşitli kabul planlarının uygulanmasına da olanak sağlıyor. Benzetim ortamını, önerdiğimiz sezgisel planların verimlerini genel geçer planların verimleriyle karşılaştırmak üzere kullanıyoruz. Sayısal sonuçlar önerdiğimiz sezgisel planların genel geçer planlara önemli derecede üstünlük sağladığını gösteriyor.

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First and foremost, I would like to thank my principal advisor Prof. Lerzan Örmeci. Without her guidance and mentorship, this thesis would not be what it is now. Her continuous support, motivation and immense knowledge encouraged me to be better in my research. And she has always been like a family to me. Next, I would like to express my sincere gratitude to my co-advisor Asst. Prof. Erhun Özkan. Since we have started to collaborate, his energy and motivation pushed me one step further in my research and writing of this thesis. I hope I will have the opportunity to collaborate with both of my advisors in the future.

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ABBREVIATIONS

ICU	Intensive Care Unit
MDP	Markov Decision Process
LOS	Length of Stay

Chapter 1

INTRODUCTION

Intensive Care Units (ICUs) are among the most heavily-loaded departments of hospitals. Therefore, rationing ICU beds plays a crucial role in healthcare delivery. We have observed that this led to making very difficult decisions in Italy during the Covid-19 pandemic. In general, clinicians respond to bed scarcity either by rejecting an arriving patient or admitting him by discharging one of the existing patients early, i.e., discharging a patient before she completes her normal treatment time (Cooper et al. [2013]). Both actions may lead to certain adverse events and costs: On the one hand, rejecting the admission of a patient to an ICU may increase the risk of mortality for this patient (Robert et al. [2012]). Moreover, rejections also incur costs, e.g., McConnell et al. [2006] estimate the cost of an ambulance diversion due to a full ICU as $1,086 per hour. On the other hand, early discharges are observed to increase the readmissions back to the ICU (Chrusch et al. [2009], Chan et al. [2012], Town et al. [2014]). It is also well-known that readmitted patients remain in the ICU longer than in their first-time stays (Chan et al. [2012], KC and Terwiesch [2012]). Readmissions cause significant losses in hospital revenues. For example, Kim et al. [2015] show via simulation that a hospital could save $1.9 million per year by reducing readmissions.

This thesis aims to first model the admission and discharge control problem in ICUs and then develop a decision support tool for rationing and managing ICU capacity efficiently in order to minimize the total expected rejection and early-discharge costs over an infinite horizon. For this purpose, we consider an ICU that serves to multiple types of patients using a finite number of identical beds. Figure 3.1 presents the main model. An arriving patient may be admitted to an empty bed, rejected or admitted by discharging another patient early, where a corresponding one-time cost is incurred for the decisions of reject and early discharge. The recovery process of an admitted patient is assumed to have two stages: the patient is initially in the *critical stage*, where she cannot be discharged. Then she moves to the *semi-critical stage* upon which she is allowed to be discharged. A discharged patient may seek an ICU service within a short time period, e.g., within 14 days upon his discharge, with a certain probability, which we call the *readmission probability*. If a patient is discharged early, her risk of adverse events increases, which is accounted for by a higher readmission probability. To model this phenomenon, we use a so-called *readmission orbit*: we assume that a discharged patient joins the orbit with the corresponding readmission probability and seeks an ICU service after a random amount of time. We call a patient a *first-time patient* if it is her first stay at the ICU, and a *recurring patient* if she is readmitted to the ICU.

The resulting model is a realistic representation of a typical ICU, which reflects the trade-offs within different types of patients, between first-time and recurring patients, between the critical and semi-critical patients, as well as between the reject and early-discharge decisions. However, it is also intractable due to its high dimensionality. The aim of this thesis is first to analyze the corresponding admission and discharge control problem and then develop well-performing and implementable policies based on this model. There are two major approaches available to represent and analyze such a model: Markov Decision Processes (MDPs) and fluid approximations. We first present an MDP formulation for a much simpler version of the system described above, where we analyze the structure of the optimal control policy. Next, present an MDP formulation for the system described above, which aims to develop a well-performing and implementable policy, rather than finding an optimal policy. More specifically, we analyze a special case where the ICU has exactly one bed and the orbit size is limited by one. The analysis of this simpler system leads to our first heuristic policy: the so-called *one-bed policy*. Then, we develop a deterministic fluid model and derive its solution, which leads to our second heuristic policy, the so-called *fluid-based policy*.

Finally, we develop a discrete-event simulation platform that represents the ICU system. The platform is quite general, so that it can represent ICUs with different characteristics as well as a variety of control policies. We use this platform to benchmark the performance of the proposed heuristic policies, with those of the two state-of-the-art policies which we call the *reject policy* and the *early-discharge policy*. Both of these policies always accept patients whenever there is an available bed. Their decisions differ when all beds are occupied: the reject policy always rejects a new patient, and the early-discharge policy admits such a patient by discharging a patient early who is in semi-critical stage, if possible, and rejects if there is no such patient. It is reported in the literature that the former policy is commonly used in practice (Metcalfe et al. [1997], Robert et al. [2012]) and the latter is studied especially in operations research community (Dobson et al. [2010], Chan et al. [2012], Armony et al. [2018]).

Both state-of-the-art policies have certain downsides: on the one hand, the reject policy can perform poorly if the rejection costs are high relative to the early-discharge costs. On the other hand, the early-discharge policy does not necessarily perform well even when the early-discharge costs are less than the rejection costs because early discharges may increase the future patient arrivals which in turn increase the burden on the ICU in the long-run. Because both the one-bed and the fluid-based policies take into account the effect of early discharges on readmissions, they significantly outperform the state-of-art policies in the discrete-event simulation experiments.

The contributions of the thesis can be summarized as follows:

1. We develop a realistic conceptual model of a typical ICU that serves multiple patient types. A patient should be either accepted to the ICU or rejected upon her arrival, in other words, the patients cannot queue. The recovery process of each type is characterized by two stages: critical and semi-critical. Moreover, type i can have both first-time (denoted by $(i1)$) and recurring patients (denoted by $(i2)$). More explicitly, a first-time (recurring) type-i patient stays as a critical patient for a random amount of time with mean

$1/\mu^c_{i1}$ $(1/\mu^c_{i2})$. She becomes a semi-critical patient with probability p^0_{i1} (p^0_{i2}), while she leaves the system with probability $1-p^0_{i1}$ $(1-p^0_{i2})$, which corresponds to the death of the patient. She completes her recovery when she spends an additional random period with mean $1/\mu^{sc}_{i1}$ $(1/\mu^{sc}_{i2})$. She may be discharged early due to a new patient before she completes her recovery or discharged normally at the end of her recovery time. She joins the readmission orbit to seek further ICU care with probability p_{i1} (p_{i2}) if she is discharged normally, while this probability increases to p^e_{i1} (p^e_{i2}) if she is discharged early. With the remaining probabilities, the patient leaves the system, which may correspond to complete recovery or death. If the patient joins the orbit, she stays there for a random amount of time with mean $1/\gamma_i$, before she requests readmission to the ICU as a recurring patient in critical stage. Note that a rejected first-time patient leaves the system, while a rejected recurring patient returns to the orbit to request readmission again.

2. We model a simpler version of the conceptual model with an MDP, where all random durations are assumed to be exponential, and analyze the structure of the optimal policy.

3. We present an MDP model based on the conceptual model, where all random durations are assumed to be exponential. We analyze the model when the ICU has one bed and at most one patient can be in the readmission orbit. This analysis leads to the one-bed policy.

4. We introduce a deterministic fluid model based on the conceptual model and show that the optimal control of the fluid model in the steady state can be obtained by solving a nonlinear problem. We derive a heuristic policy based on the solution of the nonlinear problem.

5. We develop a discrete-event simulation platform to represent the ICU. Note that the platform explicitly accounts for all patients staying in the ICU as well as those in the readmission process. This platform allows us to compare the

performance of the four policies described above. Extensive numerical experiments show that both the one-bed policy and the fluid policy significantly outperform the state-of-the-art policies.

The rest of the thesis is organized as follows: Chapter 2 gives a review of the related literature. Chapter 3 introduces the admission and discharge control problem in an ICU. Chapter 4 simplifies the model and formulate it as an MDP and investigate the structure of the optimal admission and discharge control policy. Chapter 5 also formulates the problem as an MDP and presents the one-bed policy. Chapter 6 introduces the fluid approximation model, and the corresponding fluid-based policy. Chapter 7 explains the numerical setting and compares the performances of the proposed policies with the ones of the state-of-the-art policies via discrete-event-simulation. Chapter 8 presents concluding remarks.

Chapter 2

LITERATURE REVIEW

The conceptual model builds upon admission and discharge control of the ICU, where the future arrivals of recurring patients depend on the current decisions. This dependence is represented through a feedback mechanism. Moreover, each patient type is further categorized as first-time and recurring patients, and the LOS of a patient consists of two stages, each one represented by an exponential distribution (which we refer as two-stage exponential distribution). Accordingly, our research is related to various fields ranging from admission control to retrial or feedback queues.

Admission decisions to healthcare systems arise naturally in different contexts. Here are a few examples considering only admission control but not early discharge or readmissions: Helm et al. [2011] investigate the structure of an optimal admission control policy for elective surgeries in the presence of emergency surgeries. In Samiedaluie et al. [2017], a neurology ward is considered, where arriving patients may be admitted, queued, or rejected. Shmueli et al. [2003] compare the expected incremental survival benefit of the ICU patients under different admission control policies using a queueing theory perspective.

The MDP model we consider represents the ICU as a loss model. In this regard, it builds on the earlier studies that consider admission control in loss systems to characterize the structure of the optimal policies (Altman et al. [1998], Örmeci et al. [2001], Savin et al. [2005]). Ulukuş et al. [2011] extend the results of these studies to a two-class loss system where jobs may be terminated upon an arrival to provide service to the new job, which corresponds to an early discharge in our setting. All these studies assume exponential service times, whereas we represent the LOS of a patient by a two-stage exponential distribution (Chapter 5), inspired by Kurt [2014]. Kurt [2014] investigates the effects of admission, early discharges and overflow policies in a loss model with two patient types. Mild patients stay for an exponential amount of time, while the LOS of severe patients follow a two-stage exponential distribution. However, this study, as all the others, does not account for readmissions, in contrast to the model considered here. In fact, even the LOS in our model is much more complex than Kurt [2014], since each patient type is characterized by the stay type (first-time versus recurring) in addition to the health stage. Consequently, even one type of customer can behave in four different ways with respect to LOS and readmission probability. Further none of these studies consider early terminations of existing jobs and theirs effects on the workload.

In some healthcare settings, it is possible to control only discharges, rather than both admissions and discharges: Dobson et al. [2010] and Chan et al. [2012] consider an ICU setting where patients are always admitted either by assigning a free bed or in the absence of a free bed by discharging an existing patient early. Dobson et al. [2010] compare the performances of different early-discharge policies with respect to the probability of an early discharge and how early a discharge is on the average. This study aims to identify the policies that have the least negative effect on the patients, but does not explicitly account for readmissions. Chan et al. [2012], on the other hand, propose a greedy policy, where a patient with the least expected cost is discharged early and the cost accounts for the possible readmission due to the early discharge. However, the dynamics of the system does not include a flow for patients who seek readmission. Hence, the readmission phenomenon appears only as a cost metric, whereas our model represents the effects of early-discharge decisions as increased future arrival rates in addition to immediate costs. Shi et al. [2019] analyze a hospital setting that focuses on optimising how many patients to discharge and whom to discharge on a given day in order to minimize waiting and readmission costs. They assume that the probability that a patient will be readmitted depends on the time that she has spent in the system. That said, they treat readmission arrivals as exogenous variables in their analytical model. Further, Shi et al. [2019] do not distinguish first-time and recurring patients in terms of LOS.

Ouyang et al. [2020] is one of the closest studies to ours. The LOS of a patient in ICU is modeled with two health states as ours, but they allow the health state to improve or worsen, as opposed to ours where the health state can only improve. The objective is to minimize the mortality rate, rather than minimizing a more general cost function, by using the same actions, i.e., admit, reject or admit by discharging a patient early. We should note that in Ouyang et al. [2020] all patients, regardless of their health states, can be discharged, while we do not allow discharging patients in critical condition. Our main contribution with respect to Ouyang et al. [2020] is to explicitly model the readmissions through a feedback loop and to distinguish between first-time and recurring patients in terms of LOS.

Discharging a patient earlier than her full recovery is a special case of representing state-dependent service rate. Mallor et al. [2014] model an ICU as a queueing system where service times are adjusted with respect to the number of occupied beds with the objective of minimizing the rejection probabilities, with no reference to readmissions. Chan et al. [2014], on the other hand, combine state-dependent service rates and readmission probabilities by developing a deterministic fluid model in which the service rate can be adjusted depending on a pre-defined occupancy level. Patients finding all servers busy wait, thus there is no admission control. Discharged patients request readmission with respect to a probability depending on the applied service rate. Service times depend only on the occupancy level, so that first-time and recurring patients are not differentiated once they start service, which is in contrast to our model: the LOS of the first-time and the recurring patients may be be different.

To the best of our knowledge, Chan et al. [2014] is the only study in the area of retrial or feedback queue, which has state-dependent service rates. Majority of studies in retrial queues focus on evaluating system performance, rather than system control. One exception is Bhulai et al. [2014] who analyze the structural properties of the value function in a processor sharing retrial queue.

Örmeci et al. [2016] is another study that models the dependence between the current decisions and the future arrivals in the context of scheduling colonoscopy services. The main trade-off is between serving urgent diagnostic needs and providing screening service that may reduce the prevalence of the disease in the population. They use an orbit to represent the health state of the whole population, which improves with a certain probability when a colonoscopy service is performed. Moreover, the arrival rate for colonoscopy services decreases as the health state improves. However, the colonoscopy service is represented as a Markovian queue where all patient types have identical service times. Hence, except for the basic idea of the dependence between the current decisions and the future arrivals, the two systems are completely different.

A number of studies in the literature analyzes retrial systems (or retrial queues as they are sometimes called). In a classical retrial system, customers finding all servers busy, join a retrial group where they request another service after some amount of time or renege before getting service (Choi et al. [1995], Wang et al. [2001], Falin et al. [1993], Bhulai et al. [2014]). Likewise, customers may join a feedback queue with some probability after they complete service, in which case every time they leave the server the probability of joining the feedback queue decreases (van den Berg and Boxma [1991]). Despite the similarities of the retrial systems in the literature to our model, the main difference is that they evaluate system performance whereas we consider system control.

Chapter 3

AN ICU MODEL WITH READMISSIONS

The main purpose of a healthcare facility is to provide service to all patients who need it and the ideal is to provide the right service at the right time. However, for departments, such as ICUs, which generally operate under high occupancy levels, the meanings of “right service” and “right time” may not be always clear. The managers of such facilities frequently face the trade-offs such as discharging a patient to create room for a new patient or admitting a patient now versus keeping an idle bed for a more serious future patient. The consequences of admission and discharge decisions vary; some can be represented through appropriate costs whereas others may create additional service requests. To account for the latter type of consequences, we model the interactions between the admission-discharge decisions and the readmission requests. For this purpose, we define a virtual *readmission orbit* which contains patients who will seek readmission within a certain time period after being discharged from the ICU. The orbit includes all such patients, regardless of whether they stay in or out of the hospital. Hence, it should be regarded as a conceptual orbit, rather than a group of patients under observation.

Now we describe the process and the model, which is illustrated in Figure 3.1. Patients who need ICU services are classified into I different types based on their diseases, where $I \in \mathbb{N}_+ := \{1, 2, \ldots\}$. We use index $i \in \mathcal{I}$ to denote the patient types, where $\mathcal{I} := \{1, 2, \ldots, I\}$. A type-$i$ patient is further categorized as a *first-time* patient who stays in the ICU for the first-time (denoted by $k = 1$) or a *recurring patient* who seeks readmission within a specified number of days after a previous ICU discharge (denoted by $k = 2$). We assume that all ICU patients go through two health stages, namely *critical* and *semi-critical* stages (similar to Mathews and Long [2015] and Armony et al. [2018]). A patient should remain in the ICU till the end

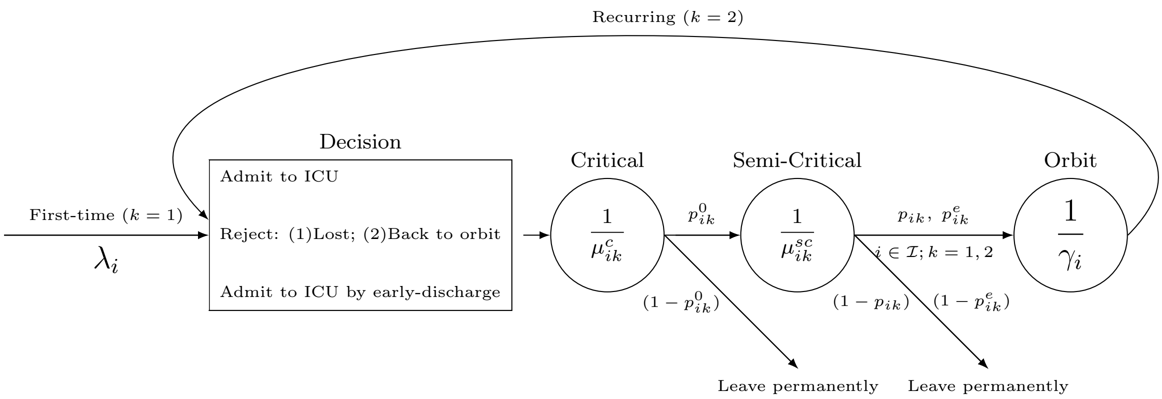


Figure 3.1: Representation of the System

of her critical stage, whereas a patient in the semi-critical stage can be discharged prematurely to make room for a new patient. A patient, regardless of whether she is a first-time or a recurring patient, is assumed to be in the critical health stage upon her arrival to the ICU. Hence, we assume that patients are too sick to wait, so that they are either immediately accepted to the ICU or transferred to another ICU. The ICU has a total number of B beds, $B \in \mathbb{N}_+$, all of which are identical and have sufficient equipment to treat all types of patients.

The first-time type-i patients arrive at the ICU with a constant rate λ_i and spend an average of $1/\mu^c_{i1}$ days in the critical stage. Upon completing the critical stage, they either move to the semi-critical stage with probability p^0_{i1} or leave the system permanently with probability $(1 - p^0_{i1})$ (which corresponds to death). The average time that first-time type-i patients spend in the semi-critical stage is $1/\mu^{sc}_{i1}$. When such patients complete their semi-critical stage, i.e., when their treatment in the ICU has ended, they are discharged: they either leave the system permanently with probability $(1 - p_{i1})$ or join the so-called orbit with probability p_{i1} to request readmission later on. We call such patients as *normally-discharged* patients, because they are not forced out of the ICU due to a new patient. Patients who are in the semi-critical stage may also be *discharged early* from the ICU before completing their normal LOS. Such patients have a higher probability of requesting readmission, compared to the normally discharged patients (Chrusch et al. [2009], Chan et al. [2012], Town et al. [2014]): Hence, an early-discharged semi-critical patient joins the orbit with probability $p^e_{i1} \geq p_{i1}$ or leaves the system permanently with probability $(1 - p^e_{i1})$.

Recurring patients arrive from the readmission orbit: patients who join the orbit stay there for an average of $1/\gamma_i$ days, which depends only on the patient type i, not on whether they join the orbit after a normal or early discharge or whether they are first-time or recurring. The time spent in the orbit represents the time between the discharge and the readmission request. Once admitted to the ICU, recurring patients go through the same process as the first-time patients, only with different parameters. In particular, recurring patients in semi-critical stage can be discharged early and those patients join the readmission orbit with probability p^e_{i2} such that $p^e_{i2} \geq p_{i2}$.

The system incurs the following one-time costs: r_{i1} (r_{i2}) for rejecting a first-time (recurring) patient and r^e_{i1} (r^e_{i2}) for early discharging a first-time (recurring) patient. Rejecting a patient is undesirable from both patient and hospital perspectives, so we assume that the rejection costs (r_{ik}) reflect all the patient and hospital-related costs. Discharging a patient before her treatment is completed increases the risk of physiological deterioration, as discussed in Chan et al. [2012]. Therefore, the early discharge costs (r^e_{ik}) represent the costs due to possible adverse effects of early discharge on patient health in addition to the hospital-related costs. Finally, because getting some care is better than having no care, we assume that cost of rejecting a patient is more than the cost of early discharging the same type of patient. That is, $r_{ik} \geq r^e_{ik}$ for all $i \in \mathcal{I}$ and $k \in \{1, 2\}$.

In the next chapter, we present a simplified version of the ICU model as a Markov Decision Process model and investigate the structural properties of the optimal admission and discharge policy.

Chapter 4

SINGLE PATIENT TYPE AND SINGLE HEALTH STAGE MDP FORMULATION

4.1 The Model

Each patient has unique characteristics that should be considered separately both by admission and while making a discharge decision. Or at least patients may be categorized into different types based on their physiological characteristics as we model in the next section. That said, we focus on a simpler setting in this section for the sake of isolating the effect of the readmission phenomenon. We consider a Markovian system and formulate the admission and discharge control problem as an MDP. We assume that patients are categorized as either *first-time* patients who stay in the ICU for the first-time or *recurring patients* who seek readmission after a previous ICU discharge (see Figure 4.1). We should note that first-time patient definition also covers patients who were previously discharged from the ICU, but a long enough time has passed since their last discharge. There are a total number of B beds, $B \in \mathbb{N}_+$, available in the ICU, all of which are identical and have sufficient equipment to treat any arriving patient.

Patients arrive in the system according to a Poisson process with rate λ and require a random service time in the ICU which is exponentially distributed with rate μ_1. We let the subscript $k = 1$ to denote the first-time patients and $k = 2$ denote the recurring patients. We assume that patients arrive in the ICU in critical condition and their health stage do not change throughout their stay. After first-time patients are discharged when their treatment is complete, i.e., their normal LOS is over, they either leave the system permanently with probability $(1 - p_1)$ or join the so-called orbit with probability p_1. First-time patients may also be discharged early

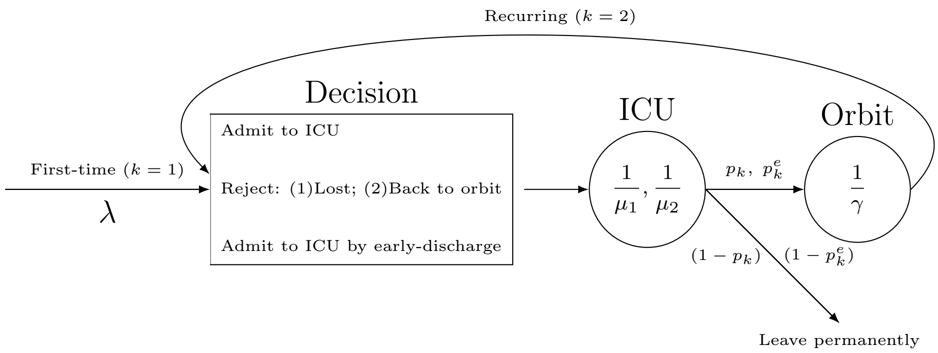


Figure 4.1: Representation of the Single-Patient Type System

from the ICU before completing their normal LOS, so that a new patient is admitted, in which case the early-discharged patient either leaves the system permanently with probability $(1 - p_1^e)$ or joins the orbit with probability p_1^e.

Patients who join the orbit after being discharged from the ICU (either normally- or early-discharged) spend a random time there, which is exponentially distributed with rate γ. Hence, the time spent in the orbit represents the time between the discharge and the readmission request. When readmitted to the ICU, recurring patients require a service time, which is exponentially distributed with rate μ_2. In accordance with the literature (Durbin and Kopel [1993], Chan et al. [2012], KC and Terwiesch [2012]), we assume that the rate at which first-time patients spend in the ICU is greater than or equal to the rate at which recurring patients spend in the ICU. That is, we assume that $\mu_1 \geq \mu_2$. Similar to the first-time patients, recurring patients are discharged either normally or early, after which they join the orbit with probabilities p_2 and p_2^e, respectively, and they leave the system permanently with the remaining probabilities. Since early-discharged patients are discharged before their normal LOS is complete, we assume that their readmission probability is higher than than that of the normally-discharged patients. More formally, we assume that $p_k^e \geq p_i$ for all $k \in \{1, 2\}$, which is consistent with a body of literature (Chrusch et al. [2009], Chan et al. [2012], Town et al. [2014]). Costs are associated with each rejection and early-discharge decisions. That is, the system incurs a cost of r_1 (r_2)

for rejecting a first-time (recurring) patient, and a cost of r_1^e (r_2^e) for discharging a first-time (recurring) patient early. Finally, we assume that the cost of rejecting a patient to the ICU is more than the cost of discharging the same type of patient early, since having some care is better than having no care. That is, $r_k \geq r_k^e$ for all $k \in \{1, 2\}$.

Remark 1 *In real life, rejection and early-discharge costs are difficult to estimate. Therefore, we use this parameters as tuning parameters in such a way that they help us in evaluating trade-offs associated with rejections and discharge decisions.*

Discharging patients early is similar to increasing service rate applied at times of high congestion in Chan et al. [2014]. They assume that patients served with higher service rates in station 1 (i.e. ICU in our setting) have a higher probability of going to station 2 (i.e. an orbit-like representation in our model). As stated in the previously, our work differs from that of Chan et al. [2014] in the following way: First, they control the system with a predefined-occupancy level whereas our system takes actions dynamically. Second, we model the problem as a two-class Markovian loss system, whereas they analyze the problem as a queuing model. Third, hey ignore the increased service time requirement of recurring patients in station 1, while we assume that recurring patients require more service time than first-time patients. Finally, we extend our model to multiple patient types with different health stages, whereas they consider only one type of patients with one stage (i.e., critical in our setting).

4.1.1 State Space

The state of the system is defined as (x_1, x_2, s), where x_1 and x_2 represent the number of first-time and recurring patients in the ICU, respectively, and s represents the number of patients currently in the orbit. We define the state space as $\mathcal{S} = \{(x_1, x_2, s) : \ x_1 \geq 0, \ x_2 \geq 0, \ 0 \leq x_1 + x_2 \leq B, \ 0 \leq s \leq S^{max}\}$ where S^{max} denotes the finite capacity of the orbit. A finite orbit capacity allows us to track the model without compromising the effectiveness of it. That is, the finite orbit capacity assumption is crucial in the sense that it guarantees a finite state space, by which we can uniformize the corresponding Markov chain. In our numerical analysis in this section, we choose the orbit size in a way such that hitting the orbit boundary occurs with a very low probability.

4.1.2 Action Space

An action is taken only at times of a patient arrival, i.e. either a first-time or recurring patient arrival. At normal discharge epochs, however, an additional action is not required since a patient completing his/her normal LOS is released from the ICU. When a first-time patient arrives, the following actions are available:

- Rejecting her to the ICU with a cost of r_1, and she is permanently removed from the system,
- Admitting her to a free bed in the ICU,
- Admitting her to a free bed in the ICU by discharging one of the existing patients early in the ICU. In that case, the system incurs a cost of r_k^e $(k = 1, 2)$, depending on the type of the discharged patient, and the early discharged patient either leaves the system or joins the orbit.

The first alternative is available in all states, while the second alternative is available only when there is a free bed to admit the new patient. Finally, the third alternative is available only when there is at least one patient staying in the ICU. In case of a recurring patient arrival, the same set of alternatives is available with the difference that a rejected recurring patient joins back to the orbit and the system incurs a cost of r_2. Imposing the assumption that rejected recurring patients go back to orbit ensures that it is the hospital's responsibility to treat the recurring patients and complete their service. Otherwise, the system would get rid of a recurring patient by rejecting him and incur a cost once and for all. The rejection cost r_k $(k = 1, 2)$ reflects all of the patient and system-related costs since rejecting a patient is undesirable from both patient and system perspective.

Similarly, the early-discharge cost r_k^e ($k = 1, 2$) represents possible adverse effects of early-discharge on patient health and system related costs. As discussed in Chan et al. [2012], these costs may include additional risks of physiological deterioration due to discharging the patient before his treatment is completed. We assume that rejecting a first-time (recurring) patient to the ICU is more costly than discharging first-time (recurring) early, i.e. $r_1 \geq r_1^e$ ($r_2 \geq r_2^e$), since receiving some care is better than not receiving any care.

4.1.3 Uniformization

We consider an infinite-horizon problem with the objective of minimizing the total expected β-discounted or the long-run average cost. Because we consider a Markovian system, we can use 304 the uniformization technique(Lippman [1975]). To be able to construct an MDP model, there must be a finite upper bound on the total transition rate. We define the total event rate as $\lambda + S^{max}\gamma + B\mu_1 + \beta = \Lambda$, which ensures that the system evolves with an exponential rate Λ and a transition occurs with a probability proportional to its rate over Λ. Without loss of generality, we set $\Lambda = 1$, so that the system will be observed with exponentially distributed intervals with rate 1 and a transition occurs with a probability equal to its rate. This, in return, transforms the system, which originally evolves in continuous time, into a discrete time equivalent. Then, in state (x_1, x_2, s), a first-time patient arrives with probability λ, a recurring patient arrives from the orbit with probability $s\gamma$, a first-time (recurring) patient is normally discharged with probability $x_1\mu_1$ ($x_2\mu_2$). At a potential transition epoch, there are also fictitious transitions, i.e. the system state remains the same with the corresponding probability. These are a fictitious recurring patient arrival with probability $(S^{max} - s)\gamma$ and a fictitious normal discharge with probability $(B\mu_1 - x_1\mu_1 - x_1\mu_1)$.

4.1.4 Event Operators

Using uniformization, we can now construct the value function, which is made up of various events such as arrival and discharges. Koole [2006] gives detailed information on the so-called EBDP. Let us define the value function $V_n(x_1, x_2, s)$ as the minimum total expected β-discounted cost starting in state (x_1, x_2, s) when there are n transitions remaining in the horizon. Results for the finite problem can be extended over the infinite horizon total expected β-discounted cost problem, as well as the long run average cost, since 1) both action and state spaces are finite, 2) costs are stationary and bounded for all actions, and 3) transition probabilities are stationary. That is: $V(x_1, x_2, s) = \lim_{n\to\infty} V_n(x_1, x_2, s)$ (Puterman [1994]). For the simplicity of the notation, we set $\mathbf{x} = (x_1, x_2)$ in the remainder of the thesis when necessary. With this, we present the event operators and their definitions as follows:

Arrival operator T_F defines the arrival process of the first-time patients, where e_k is the unit vector which has 1 in its kth component and 0 otherwise.

$$T_F V(\mathbf{x}, s) = \begin{cases} \min\left\{V(\mathbf{x}+e_1, s),\ r_1 + V(\mathbf{x}, s),\ r_k^e + E_k(\mathbf{x}, s)\right\} & x_k > 0\ (k = 1, 2),\ x_1 + x_2 < B \\ \min\left\{V(\mathbf{x}+e_1, s),\ r_1 + V(\mathbf{x}, s),\ r_k^e + E_k(\mathbf{x}, s)\right\} & x_k > 0,\ x_l = 0,\ x_1 + x_2 < B \\ \min\left\{V(\mathbf{x}+e_1, s),\ r_1 + V(\mathbf{x}, s)\right\} & x_k = 0\ (k = 1, 2),\ x_1 + x_2 < B \\ \min\left\{r_1 + V(\mathbf{x}, s),\ r_k^e + E_k(\mathbf{x}, s)\right\} & x_k > 0\ (k = 1, 2),\ x_1 + x_2 = B \\ \min\left\{r_1 + V(\mathbf{x}, s),\ r_k^e + E_k(\mathbf{x}, s)\right\} & x_k > 0,\ x_l = 0,\ x_1 + x_2 = B \end{cases} \tag{4.1}$$

Early-discharge function $E_k(\mathbf{x}, s)$ corresponds to discharging a type-k patient early (with $x_k > 0$) upon arrival of a first-time patient.

$$E_k(\mathbf{x}, s) = \begin{cases} p_k^e V(\mathbf{x} + e_1 - e_k, s+1) + (1 - p_k^e) V(\mathbf{x} + e_1 - e_k, s) & \text{if } s < S^{max} \\ V(\mathbf{x} + e_1 - e_k, s) & \text{if } s = S^{max} \end{cases} \tag{4.2}$$

Note that if the orbit population is $s = S^{max}$ in state $(\mathbf{x}, s)$, the early-discharged patient cannot join the orbit and the orbit state remains unchanged. When a type-k patient is discharged early and $s < S^{max}$, the system incurs a cost of r_k^e and the early discharged patient either joins the orbit with probability p_k^e, thus increasing the state of the orbit by 1 or leaves the system permanently with probability $(1-p_k^e)$, in which case the state of the orbit remains the same.

The first case in Equation 4.1 states that there are both types of patients in the ICU and at least one free bed, hence all actions are available. The second case corresponds to the states where there is only one type of patient in the ICU and early discharge of the other patient type is not possible. The third case represents an empty ICU, where the arriving patient can be either admitted or rejected. The fourth case corresponds to a ICU full with both types of patients and all actions except for regular admission are allowed. Finally, the last case corresponds to a full ICU with only type of patient, in which case only rejection and early discharge of an existing patient type is possible.

Arrival Operator T_R represents the arrival process of recurring patients from the orbit. $\tilde{E}_k(\mathbf{x}, s)$ corresponds to early discharge decisions of patient type-k $(k = 1, 2)$ in state $(\mathbf{x}, s)$ where $s > 0$ and $x_i > 0$ just before the arrival of a recurring patient.

$$T_R V(\mathbf{x}, s) = \begin{cases} \min\left\{V(\mathbf{x}+e_2, s-1),\ r_2 + V(\mathbf{x}, s),\ r^e_k + \tilde{E}_k(\mathbf{x}, s)\right\} & x_k > 0\ (k = 1, 2),\ \ x_1 + x_2 < B \\ \min\left\{V(\mathbf{x}+e_2, s-1),\ r_2 + V(\mathbf{x}, s),\ r^e_k + \tilde{E}_k(\mathbf{x}, s)\right\} & x_k > 0,\ x_l = 0\ (k \neq l),\ \ x_1 + x_2 < B \\ \min\left\{V(\mathbf{x}+e_2, s-1),\ r_2 + V(\mathbf{x}, s)\right\} & x_k = 0\ (k = 1, 2),\ \ x_1 + x_2 < B \\ \min\left\{r_2 + V(\mathbf{x}, s),\ r^e_k + \tilde{E}_k(\mathbf{x}, s)\right\} & x_k > 0\ (k = 1, 2),\ \ x_1 + x_2 = B \\ \min\left\{r_2 + V(\mathbf{x}, s),\ r^e_k + \tilde{E}_k(\mathbf{x}, s)\right\} & x_k > 0,\ x_l = 0\ (k \neq l),\ \ x_1 + x_2 = B \end{cases} \tag{4.3}$$

$$\tilde{E}_k(\mathbf{x}, s) = p^e_k V(\mathbf{x} + e_2 - e_k, s) + (1 - p^e_k) V(\mathbf{x} + e_2 - e_k, s - 1) \tag{4.4}$$

where the cases are similar to operator T_F. We should note that the orbit state is decreased by 1 if the recurring patient is admitted to the ICU upon arrival.

Service completion operators are defined as follows: when patients complete their normal LOS, they are discharged from the ICU, after which they either leave the system permanently or join the orbit with their corresponding probabilities. T_{DEP-k} $(k = 1, 2)$ defines these discharge operators:

$$T_{DEP-k} V(\mathbf{x}, s) = \begin{cases} p_k V(\mathbf{x} - e_k, s + 1) + (1 - p_k) V(\mathbf{x} - e_k, s) & \text{if } s < S^{max} \\ V(\mathbf{x} - e_k, s) & \text{if } s = S^{max} \end{cases} \tag{4.5}$$

Finally, *fictitious and uniformization event operators* are introduced as follows, respectively:

$$T_{UNIF}\Big\{\{f_1, \ldots, f_l\}; \{p(1) \ldots p(l)\}\Big\}(x) = \sum_j^l p(j) f_j(x) \quad \text{with } p(j) \leq 1 \ \text{ for all } j,$$

$$T_{FIC}V(\mathbf{x},s) = V(\mathbf{x},s) \tag{4.6}$$

where j denotes the type of the event and $p(j)$ denotes the probability of that event. By setting $\sum_j p(j) + \beta = 1$ with $\beta > 0$, the problem is a discounting problem, and by $\sum_j p(j) = 1$, the problem is a long-run average cost problem. Now we define the value function formally in terms of event operators:

$$\begin{aligned} V(\mathbf{x},s) = T_{UNIF}\Big\{ &\{T_F V,\ \ T_R V,\ \ T_{DEP-1}V,\ \ T_{DEP-2}V,\ \ T_{FIC}V\}; \\ &\{\lambda,\ \ s\gamma,\ \ x_1\mu_1,\ \ x_2\mu_2,\ \ \big((S-s)\gamma + B\mu_1 - x_1\mu_1 - x_2\mu_2\big)\}\Big\} \end{aligned} \tag{4.7}$$

We suppress $(\mathbf{x}, s)$ on the right hand side for the sake of simplicity. With $\lambda+S^{max}\gamma+B\mu_1+\beta = 1$, $V(\mathbf{x}, s)$ is defined as the total expected β-discounted cost of the system starting in state $(\mathbf{x}, s)$.

4.2 Structural Properties of the Value Function

In this part, we present simple monotonicity properties of the value function. These are: 1) monotonicity of the value function in the state variables and 2) the effect of changing a recurring patient to a first-time patient. These monotonicity results give some insights on the structure of the optimal admission control and discharge policies. We should note that deriving even the simplest form of monotonicity results is difficult due to the boundary effects and the existence of the orbit. Before we present the results, consider the following conditions:

Condition 1

a. *The rate of recurring patients who ask for readmission is higher than that of first-time patients, when all patients are normally discharged, i.e.* $\mu_2 p_2 \geq \mu_1 p_1$.

b. *Early-discharged recurring patients have a higher probability of requesting readmission than early-discharged first-time patients, that is* $p_2^e \geq p_1^e$.

c. *The cost of discharging a recurring patient early is higher than the cost of discharging a first-time patient early, i.e.* $r_2^e \geq r_1^e$.

Condition 1 states that recurring patients are undesirable. That is, they have higher readmission rates when discharged both normally and early (Condition $1.a$, $1.b$) and discharging them early is more costly than discharging first-time patients early (Condition $1.c$).

Proposition 1 *Under Condition 1, $V(x_1', x_2', s') \geq V(x_1, x_2, s)$ for all states (x_1, x_2, s), $(x_1, x_2, s) \in \mathcal{S}$, satisfying $x_1' \geq x_1$, $x_2' \geq x_2$ and $s' \geq s$. Further, $V(x_1, x_2 + 1, s) \geq V(x_1 + 1, x_2, s)$ for all $(x_1, x_2, s) \in \mathcal{S}$.*

Proof of 1 is provided in Appendix A.1.

Proposition 1 states that it is less costly to start a state with less first-time or recurring patients in the ICU and less patients in the orbit, respectively. Similarly, it is less costly to begin a state with one more first-time patient in the ICU rather than one more recurring patient. By Conditions 1a-c, this is easily justified since a recurring patient is more likely to seek another readmission when discharged either normally or early, and incurs more cost than the first-time patients when discharged early.

Proposition 2 *Suppose that Condition 1 holds and assume that $r_1^e = r_2^e$ and $p_1^e = p_2^e$ and a patient arrives in state (x_1, x_2, s) with $x_1 > 0$ and $x_2 > 0$. Then, first-time patients should not be discharged early.*

Proof: Consider a first-time patient arrival. She may be admitted, rejected or admitted by discharging one of the existing patients early. Let us compare the cost of discharging a first-time patient early with the cost of discharging a recurring patient early.

$$\begin{aligned}
&r_2^e + p_2^e V(x_1 + 1, x_2 - 1, s + 1) + (1 - p_2^e) V(x_1 + 1, x_2 - 1, s) \\
&= r_1^e + p_1^e V(x_1 + 1, x_2 - 1, s + 1) + (1 - p_1^e) V(x_1 + 1, x_2 - 1, s) \\
&\leq r_1^e + p_1^e V(x_1, x_2, s + 1) + (1 - p_1^e) V(x_1, x_2, s)
\end{aligned}$$

where the inequality follows from Proposition 1. The same result follows upon a recurring patient arrival:

$$r_2^e + p_2^e V(x_1, x_2, s) + (1 - p_2^e) V(x_1, x_2, s - 1)$$

$$= r_1^e + p_1^e V(x_1, x_2, s) + (1 - p_1^e) V(x_1, x_2, s-1)$$
$$\leq r_1^e + p_1^e V(x_1 - 1, x_2 + 1, s) + (1 - p_1^e) V(x_1 - 1, x_2 + 1, s - 1)$$

where the last inequality follows from Proposition 1. The case with $s = S^{max}$ trivially follows. □

Proposition 2 is a special case where the decision maker does not differentiate among readmission probabilities and early-discharge costs but know that recurring patients stay longer in the ICU. It trivially follows that new recurring patients are never rejected when there is at least one recurring patient in the ICU. That is:

$$r_2^e + p_2^e V(x_1, x_2, s) + (1 - p_2^e) V(x_1, x_2, s-1)$$
$$\leq r_2^e + p_2^e V(x_1, x_2, s) + (1 - p_2^e) V(x_1, x_2, s)$$
$$= r_2^e + V(x_1, x_2, s)$$
$$\leq r_2 + V(x_1, x_2, s)$$

By model assumption, we already have that $r_2 \geq r_2^e$. Further, Condition 1 states that recurring patients have a higher rate of being readmitted when discharged either normally or early and and bring a higher cost when discharged. Hence, the system incurs more cost by further keeping recurring patients in the orbit i.e., by rejecting them. However, violation of Condition 1 does not guarantee such an optimal policy. Example 1 presents a setting where the optimal policy is as described as in Proposition 2 even if the preceding conditions are relaxed. Example 2 presents a setting where the optimal policy proposed by Proposition 2 is not followed since Condition 1 is not satisfied.

Example 1 *Let $\lambda = 0.55$, $\mu_1 = 0.1$, $\mu_2 = 0.08$, $\gamma = 0.194$, $p_1 = 0.06$, $p_2 = 0.06$, $p_1^e = 0.26$, $p_2^e = 0.3$and $B = 5$. As a result, $\mu_2 p_2 < \mu_1 p_1$. The cost parameters are as follows: $r_1 = 2$, $r_2 = 4$, $r_1^e = 1$, $r_2^e = 1.1$ and the bed capacity is 5. Figure 4.2a shows the corresponding optimal admission and discharge control policy upon arrival of a first-time patient for orbit states which have a probability of being visited more than 0.0001. In this example, Condition 1a is not satisfied but the optimal policy is still as described in Proposition 2.*

Example 2 *Let the parameters be* $\mu_1 = 0.1$, $\mu_2 = 0.08$, $p_1 = 0.06$, $p_1^e = 0.26$, $p_2 = 0.14$, $p_2^e = 0.56$, $\lambda = 0.55$, $r_1^e = 24$, $r_1 = 25$, $r_2^e = 1$, $r_2 = 2$ *and* $B = 5$. *Figure 4.2c depicts the optimal admission and discharge control policy upon a recurring patient arrival in small orbit states and Figure 4.2d depicts the same optimal policy in higher orbit states. Let us focus on the optimal actions when a recurring patient arrives and the ICU is full with first-time patients (i.e. no recurring and 5 first-time patients). When there are less patients in the orbit (Figure 4.2c), arriving recurring patients are rejected when there is not a recurring patient to discharge early and thus are sent back to orbit. When the orbit size increases and there is not a recurring patient in the ICU to discharge early, arriving recurring patients are admitted by discharging a first-time patient early (Figure 4.2d). This is because increasing the orbit size by 1 in higher orbit states increases the recurring arrival rate more than as it would do in small orbit states.*

4.3 Existence of Preferred Patient Types

In this section, we prove the existence of a preferred patient type under some conditions. By preferred patient type, we mean that the new patients from a given type are always admitted to the ICU when there is a free bed. The existence of such classes, for example, are shown in a two-class loss system in Örmeci et al. [2001] and a two-class loss system with terminations in Ulukuş et al. [2011]. In Örmeci et al. [2001], early job terminations are not allowed and thus can be considered in a different context than our model. Ulukuş et al. [2011] allow early job terminations when there is an idle server and show that it is not optimal to do so when all servers are not busy. Consequently, they show the existence of preferred classes under some conditions. Although the model in Ulukuş et al. [2011] and ours seem to be similar, the major difference is the existence of a so-called orbit representation in our system. In their setting, the job leaves the system permanently once it is terminated and incurs a cost to the system, whereas early-discharged patients in our setting may join the orbit with some probability, where they seek readmission after a random

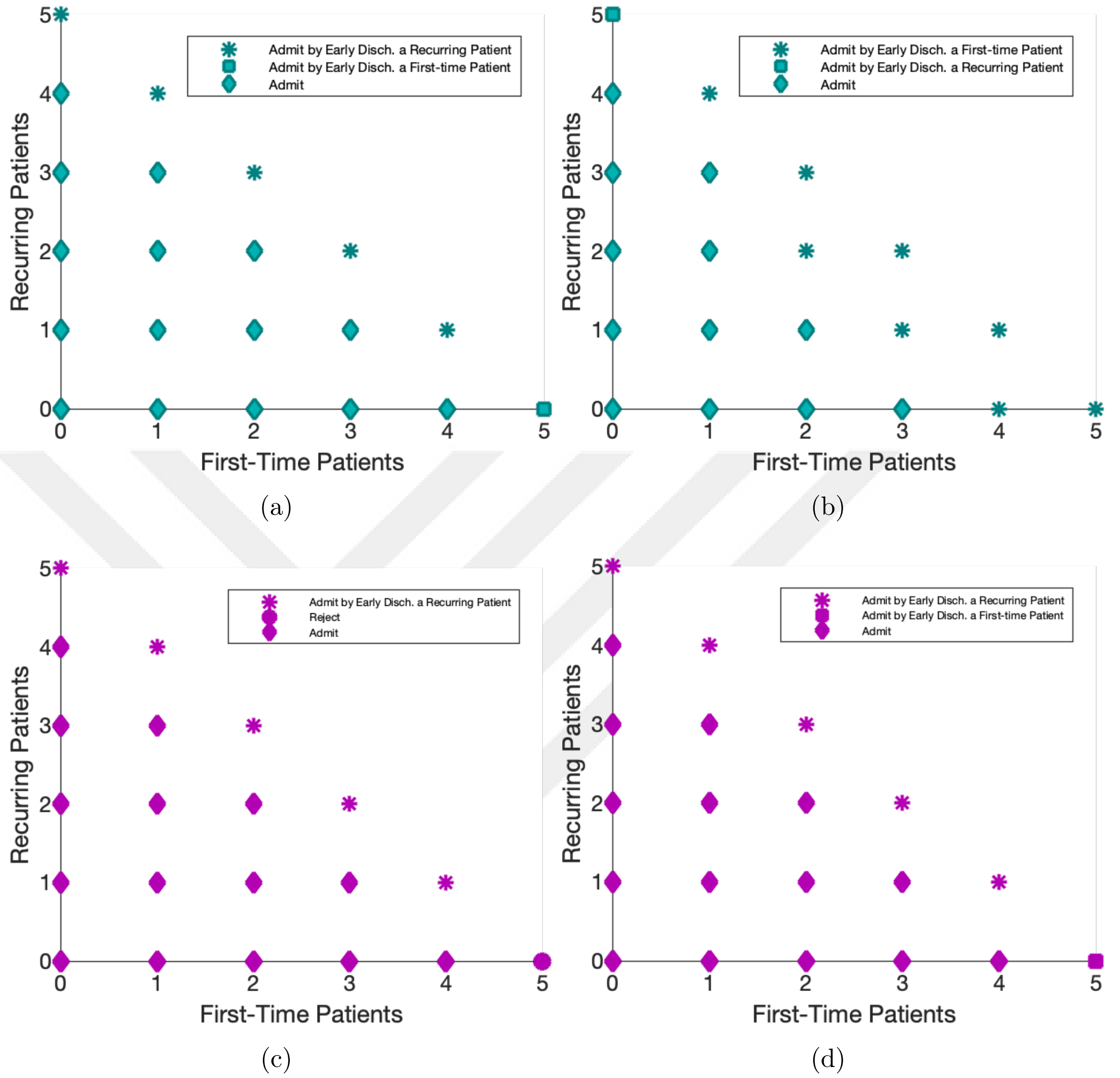


Figure 4.2: Figures for Examples 1-2-3

amount of time. As an example, consider an overloaded system where rejecting the new first-time patients costs much more than discharging patients from both types early. In such a setting, the optimal admission and discharge control policy would be expected not to reject a new patient but to admit him by early discharging a current patient, by which the system would avoid a high rejection cost. In fact, the following example and Figure 4.2b illustrates such a case.

Example 3 *Let the parameters be as follows:* $\mu_1 = 0.1$, $\mu_2 = 0.098$, $p_1 = 0.06$, $p_1^e = 0.06$, $p_2 = 0.06$, $p_2^e = 0.16$, $\lambda = 2.8$, $r_1^e = 1$, $r_1 = 2000$, $r_2^e = 1$, $r_2 = 2$ *and* *other parameters are as in the preceding examples. The optimal admission control policy upon a first-time patient arrival when there are six patients in the orbit is as depicted in Figure 4.2b.*

For a large set of parameters, we observe that the optimal admission and discharge control policy does not discharge patients early when there is a free bed, but we can not prove it theoretically. Therefore, we consider a simplified model where early-discharge decisions are not allowed when there is a free bed in the ICU. That is, when the ICU is not full, a new patient is either admitted to the ICU or rejected. In such an environment and under some conditions, we show that the new patients of a certain type are always admitted to the ICU when there is a free bed.

4.3.1 Sufficient Conditions for First-time Patients to Be Preferred:

For the simplified model that we described above, we modify the arrival operator for first-time patients as follows:

$$T_F^M V(\mathbf{x},s) = \begin{cases} \min\left\{V(\mathbf{x}+e_1,s),\ r_1+V(\mathbf{x},s)\right\} & x_k>0\ (k=1,2),\ \ x_1+x_2<B \\ \min\left\{r_1+V(\mathbf{x},s),\ r_e^k+E_k(\mathbf{x},s)\right\} & x_k>0\ (k=1,2),\ \ x_1+x_2=B \\ \min\left\{r_1+V(\mathbf{x},s),\ r_e^k+E_k(\mathbf{x},s)\right\} & x_k>0,\ x_l=0,\ k\neq l\ \ x_1+x_2=B \end{cases} \tag{4.8}$$

where the early-discharge function $E_k(\mathbf{x},s)$ preserves its meaning as in Equations 4.1 and 4.2. We replace the operator T_F with T_F^M, and the rest of the operators are as described before. Let us define $D(kl)(\mathbf{x},s)$ for all $(\mathbf{x},s)\in\mathcal{S}$ as the difference in the total expected discounted cost between system A and B if system A starts in state $(\mathbf{x},s)$ plus one patient type-k and system B starts in state $(\mathbf{x},s)$ plus one type-l patient ($k,l=0,1,2,s$ with s denoting the patients in the orbit). That is, $D(10)(\mathbf{x},s)=V(x_1{+}1,x_2,s){-}V(x_1{+}1,x_2,s)$, $D(02)(\mathbf{x},s)=V(x_1,x_2{+}1,s){-}V(x_1{+}1,x_2,s)$, $D(s0)(\mathbf{x},s)=V(x_1,x_2,s{+}1){-}V(x_1{+}1,x_2,s)$ and $D(21)(\mathbf{x},s)=V(x_1,x_2{+}1,s)-V(x_1+1,x_2,s)$. For the new first-time patients to be always admitted when there is a free bed, we need to find a non-negative upper bound for $D(10)(\mathbf{x},s)$. This requires not only to make additional parametric assumptions, but also to derive non-negative upper bounds on $D(20)(\mathbf{x}, s)$, $D(s0)(\mathbf{x}, s)$ and $D(21)(\mathbf{x}, s)$. Note that the results of Proposition 1 still hold where early-discharge are not allowed when there is a free bed. The following condition provides sufficient conditions for the first-time patients to be preferred:

Condition 2

i. $(\lambda r_2^e + \mu_1 r_1^e) \le \left(\frac{r_2^e}{1-p_2^e}\right)(\lambda(1-p_1^e) + \mu_2 - \mu_1 p_1^e)$

ii.

$$\frac{r_2^e}{1-p_2^e} \le \min\left\{\frac{(\mu_1+\lambda+\gamma)\,r_1 - \gamma r_1^e}{p_1},\right.$$

$$\frac{r_2^e\,(\lambda p_2^e + \mu_2)}{p_2^e(1-p_2^e)} - (1-\lambda-B\mu_1)\left(\frac{r_2 - r_2^e}{\lambda p_2^e + \mu_2 p_2}\right),$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\frac{(\mu_2 - \mu_1 p_1^e)}{(1-B\mu_1)\,(p_2^e - p_1^e) + (\mu_2 p_2 - \mu_1 p_1)}$$

$$\left. - \frac{((1-B\mu_1)\,r_2^e + \mu_1 r_1^e)}{\min\left\{\gamma\,(p_2^e - p_1^e),\,(\mu_2 p_2 - \mu_1 p_1)\right\}}\right\}$$

The following theorem immediately follows:

Theorem 1 *When Condition 1 and Condition 2 hold, first-time patients are always admitted to the ICU when there is a free bed and the corresponding upper bounds on differences in the total expected discounted costs are as follows:*

i. $V(x_1+1, x_2, s) - V(x_1, x_2, s) \le r_1$

ii. $V(x_1, x_2, s+1) - V(x_1, x_2, s) \le \overline{d}_s$

iii. $V(x_1, x_2+1, s) - V(x_1, x_2, s) \le \frac{r_2^e}{1-p_2^e}$

iv. $V(x_1, x_2+1, s) - V(x_1+1, x_2, s) \le r_2^e\left(\frac{1-p_1^e}{1-p_2^e}\right) - r_1^e$

where $\overline{d}_s$ is defined as follows:

$$\overline{d}_s = \min\Bigg\{ \frac{(\mu_1+\lambda+\gamma)\,r_1-\gamma r_1^e}{p_1},$$

$$\frac{r_2^e\,(\lambda p_2^e+\mu_2)}{p_2^e(1-p_2^e)} - (1-\lambda-B\mu_1)\left(\frac{r_2-r_2^e}{\lambda p_2^e+\mu_2 p_2}\right),$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\frac{(\mu_2-\mu_1 p_1^e)}{(1-B\mu_1)\,(p_2^e-p_1^e)+(\mu_2 p_2-\mu_1 p_1)}$$

$$- \frac{((1-B\mu_1)\,r_2^e+\mu_1 r_1^e)}{\min\left\{\gamma\,(p_2^e-p_1^e)\,,(\mu_2 p_2-\mu_1 p_1)\right\}}\Bigg\}$$

Please see Appendix A.2 for a complete proof.

4.3.2 Sufficient Conditions for Recurring Patients to be Preferred:

Similar to the previous case, we replace the the arrival operator for recurring patients T_R with T_R^M in a similar way as in Equation 4.8.

$$T_R^M V(\mathbf{x},s) = \begin{cases} \min\left\{V(\mathbf{x}+e_2,s-1),\ r_2+V(\mathbf{x},s)\right\} & x_1+x_2<B \\ \min\left\{r_2+V(\mathbf{x},s),\ r_e^i+\tilde{E}_i(\mathbf{x},s)\right\} & x_i>0\ (i=1,2),\ \ x_1+x_2=B \\ \min\left\{r_2+V(\mathbf{x},s),\ r_e^i+\tilde{E}_i(\mathbf{x},s)\right\} & x_i>0,\ x_j=0\ (i\neq j),\ \ x_1+x_2=B \end{cases} \tag{4.9}$$

where all definitions preserve their meaning as in Section 4.1.

We define $D(2s)(\mathbf{x},s) = V(x_1,x_2+1,s) - V(x_1,x_2,s+1)$ as the cost of having one more patient in the orbit than in the ICU as a recurring patient. Condition 3 provides sufficient conditions for recurring patients to be always admitted when there is a free bed (i.e., sufficient conditions for Theorem 2).

Condition 3

i. $(\lambda r_2^e+\mu_1 r_1^e) \leq \left(\frac{r_2^e}{1-p_2^e}\right)(\lambda\,(1-p_1^e)+\mu_2-\mu_1 p_1^e)$

ii.

$$\frac{r_2^e}{1-p_2^e} \leq \min\left\{\frac{(\lambda+\mu_2)\,r_2-\lambda r_1^e}{\lambda r_2^e+B\mu_2 p_2},\right.$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{\mu_2+(1-B\mu_1)\,p_2^e}{p_2^e}\right),$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{\mu_2-\mu_1 p_1^e}{(1-B\mu_1)\,(p_2^e-p_1^e)+(\mu_2 p_2-\mu_1 p_1)}\right)$$

$$\left.-\left(\frac{(1-B\mu_1)\,r_2^e+\mu_1 r_1^e}{\min\left\{\gamma\,(p_2^e-p_1^e)\,,(\mu_2 p_2-\mu_1 p_1)\right\}}\right)\right\}$$

Theorem 2 *When Condition 1 and Condition 3 hold, recurring patients are always admitted to the ICU when there is a free bed and the corresponding upper bounds on differences in the total expected discounted costs are as follows:*

i. $V(x_1, x_2+1, s) - V(x_1, x_2, s+1) \leq r_2$

ii. $V(x_1, x_2, s+1) - V(x_1, x_2, s) \leq \overline{d}_s$

iii. $V(x_1, x_2+1, s) - V(x_1, x_2, s) \leq \dfrac{r_2^e}{1-p_2^e}$

iv. $V(x_1, x_2+1, s) - V(x_1+1, x_2, s) \leq r_2^e\left(\dfrac{1-p_1^e}{1-p_2^e}\right) - r_1^e$

where $\overline{d}_s$ is defined as follows:

$$\overline{d}_s = \min\left\{\frac{(\lambda+\mu_2)\,r_2-\lambda r_1^e}{\lambda r_2^e+B\mu_2 p_2},\right.$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{\mu_2+(1-B\mu_1)\,p_2^e}{p_2^e}\right),$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{\mu_2-\mu_1 p_1^e}{(1-B\mu_1)\,(p_2^e-p_1^e)+(\mu_2 p_2-\mu_1 p_1)}\right)$$

$$\left.-\left(\frac{(1-B\mu_1)\,r_2^e+\mu_1 r_1^e}{\min\left\{\gamma\,(p_2^e-p_1^e)\,,(\mu_2 p_2-\mu_1 p_1)\right\}}\right)\right\}$$

Appendix A.3 provides the complete proof. When first-time (recurring) patients are preferred type, Theorem 1 (Theorem 2) states that the cost of having an extra recurring patient in the ICU is less than the expected cost of discharging a recurring patient early at each transition. Similarly, both of the theorems state that the cost of changing a first-time patient to a recurring patient is less than or equal to (*strictly* *less than* when $p_1^e > 0$) the difference between the expected cost of discharging a recurring patient at each transition and the expected cost of discharging a first-time patient at each transition. That is, $D(21)(\mathbf{x}, s) \leq (1 - p_1^e)\left(\frac{r_2^e}{1-p_2^e} - \frac{r_1^e}{1-p_1^e}\right)$ for all $(\mathbf{x}, s) \in \mathcal{S}$.

Note that when $\frac{r_2^e}{1-p_2^e}\frac{\mu_2-\mu_1 p_1^e}{(1-B\mu_1)\left(p_2^e-p_1^e\right)+(\mu_2 p_2-\mu_1 p_1)} - \frac{(1-B\mu_1)r_2^e+\mu_1 r_1^e}{\min\left\{\gamma\left(p_2^e-p_1^e\right),(\mu_2 p_2-\mu_1 p_1)\right\}}$ is the minimum of the terms inside the minimum operators in Condition 2 and Condition 3, then it immediately follows that both patient types are preferred types, that is, each arriving patient is admitted to the ICU when there is a free bed.

That said, the simple model analyzed in this chapter lacks some basic assumptions of the conceptual model described in Chapter 3. First, the simple model does not consider different health stages that patients go through throughout their stay. Recall that the conceptual model in Chapter 3 assumes that patients can be discharged early only when they are in semi-critical health condition. This assumption ensures that admitted patients receive sufficient treatment before they are discharged. However, in the numerical experiments with the simple model we observe that the optimal policy may lead to near-to-zero LOS for patients. Second, the simple model assumes that there is only one patient type, whereas the conceptual model considers multiple patient types, each having different arrival, service and readmission rates.

To be able to derive easy-to-implement and interpretable policies, we formulate the conceptual model described in Chapter 3 as an MDP.

Chapter 5

MULTI-PATIENT TYPE AND MULTI-HEALTH STAGE MDP FORMULATION

5.1 *The MDP Formulation*

In this chapter, we formulate the multi-patient and multi-health stage ICU model presented in Chapter 3 as a Markov Decision Process. Specifically, we assume that the first-time type-i patients arrive in the system with respect to a homogeneous Poisson process with rate λ_i and all patients go through both critical and semi-critical (i.e., if they survive the critical stage) stages, whose durations are independent and exponentially distributed random variables with the corresponding rates described in Chapter 3. Similarly, time spent in the orbit is also independent and identically distributed with the type-specific rates as described in Chapter 3.

To describe the state of the system, we need to track the number of patients in the ICU and in the orbit. Patients in the ICU vary with respect to their types ($i \in \mathcal{I}$), to the stage of their recovery ($m \in \{c, sc\}$), and whether they are first-time or recurring patients ($k \in \{1, 2\}$). Hence, we let x_{ik}^m denote the number of type-ik patients in stage m, while s_i denotes the number of type-i patients in the orbit. Accordingly, the state of the system is $(\mathbf{x}^c, \mathbf{x}^{sc}, \mathbf{s})$ where $\mathbf{x}^c = (x_{11}^c, \ldots, x_{I1}^c, x_{12}^c, \ldots, x_{I2}^c) \in \mathbb{N}^{2I}$, $\mathbf{x}^{sc} = (x_{11}^{sc}, \ldots, x_{I1}^{sc}, x_{12}^{sc}, \ldots, x_{I2}^{sc}) \in \mathbb{N}^{2I}$, and $\mathbf{s} = (s_1, \ldots, s_I) \in \mathbb{N}^I$. Because the ICU has B beds and no waiting room, the state should satisfy the capacity constraint $\sum_{i\in\mathcal{I}} \sum_{k\in\{1,2\}} \sum_{m\in\{c,sc\}} x_{ik}^m \leq B$. Furthermore, we assume that the orbit has finite capacity by imposing the constraint $\sum_{i\in\mathcal{I}} s_i \leq S^{max}$, where $S^{max} \in \mathbb{N}$. This allows us to uniformize the MDP model. When S^{max} is large enough, this assumption is not restrictive. When it is not, it is possible that the optimal policy uses early discharge decision to force a patient out of the system. This can be prevented by introducing an appropriate boundary cost. In the main formulation, we assume that S^{max} is large enough, but the next section, which considers a small system, introduces the boundary costs. Recall that patients can not be discharged before they complete their recovery in the critical stage. As described in Chapter 3 (Please see Figure 3.1), after their critical stage ends patients either leave the system permanently with probability $(1 - p^0_{ik})$ (which corresponds to death in our context) or move in to semi-critical stage with probability p^0_{ik}, where they can be discharged either normally or early.

When a patient completes his recovery (i.e., completing her recovery in both stages), he leaves the ICU, which does not call for any further decision since the system does not have a queue. Hence, ICU managers need to take an action only at patient arrival epochs.

Assume that a first-time type-i patient arrives. If the ICU has only critical patients at such an epoch, managers have only one option, which is rejecting the patient by incurring a rejection cost of r_{i1}. Otherwise, the following actions are available:

- Admitting her to the ICU (denoted by $a_i = 1$)
  - Admitting her to a free bed without discharging a patient early (denoted by $d_i = (0, 0)$): no further consequence is observed.
  - Admitting her by discharging a type-jk patient in semi-critical stage early (denoted by $d_i = (j, k)$): the system incurs a cost of r^e_{jk}, and the early-discharged patient joins the orbit with probability p^e_{jk} and leaves the system with probability $(1 - p^e_{jk})$.
- Rejecting her (denoted by $a_i = 0$ and $d_i = (0, 0)$): the system incurs a cost of r_{i1}, and the rejected patient is permanently removed from the system.

In case of a recurring patient arrival, the alternatives as well as their consequences are the same only with the following difference: We assume that once a patient is admitted to the ICU, she becomes a liability for the hospital. Hence, rejecting a recurring patient only delays the readmission request, so that we assume the rejected recurring patient joins back to the orbit while incurring a one-time reject cost of r_{i2}.

We define $(a_i, d_i) \in \mathcal{A}_i(\mathbf{x}^c, \mathbf{x}^{sc}, \mathbf{s})$ as the action taken when a type-i patient arrives, independent of whether she is a first-time or a recurring patient. The set of all possible actions in a given state $(\mathbf{x}^c, \mathbf{x}^{sc}, \mathbf{s})$ is as follows:

$$\begin{aligned}\mathcal{A}_i(\mathbf{x}^c, \mathbf{x}^{sc}, \mathbf{s}) = \Bigg\{ (a_i, d_i) :\ & a_i \in \{0, 1\}, \\ & d_i \in \Big\{ \big\{ (j,k) : j \in \mathcal{I}, k \in \{1,2\}, x^{sc}_{jk} \geq 1 \big\} \cup (0,0) \Big\}, \\ & \sum_{j \in \mathcal{I}} \sum_{k \in \{1,2\}} \mathbb{1}_{\{d_i = (j,k)\}} + \mathbb{1}_{\{d_i = (0,0)\}} = 1, \quad \mathbb{1}_{\{d_i \neq (0,0)\}} \leq a_i, \\ & \sum_{j \in \mathcal{I}} \sum_{k \in \{1,2\}} (x^c_{jk} + x^{sc}_{jk}) + a_i - \mathbb{1}_{\{d_i \neq (0,0)\}} \leq B \Bigg\}\end{aligned} \tag{5.1}$$

where $\mathbb{1}_{\{.\}}$ is the indicator function. Note that we suppress the arguments of $\mathcal{A}_i$ when the corresponding state is clearly identified.

We consider an infinite-horizon problem with the objective of minimizing the total expected β-discounted or the long-run average cost. We can use the uniformization technique (Lippman [1975]) to construct a discrete-time MDP model: Let $\gamma_{max} := \max_{i \in \mathcal{I}} \{\gamma_i\}$ and $\mu_{max} := \max\limits_{i \in \mathcal{I}, k \in \{1,2\}} \{\mu^c_{ik}, \mu^{sc}_{ik}\}$. Then the maximal event rate is $\Lambda = \sum_{i \in \mathcal{I}} \lambda_i + S^{max}\gamma_{max} + B\mu_{max} + \beta$. We set $\Lambda = 1$ without loss of generality, so that the system will be observed in exponentially distributed intervals with rate 1 and a transition occurs with a probability equal to its rate.

Let the value function $V_n(\mathbf{x}^c, \mathbf{x}^{sc}, \mathbf{s})$ denote the minimum total expected β-discounted cost starting in state $(\mathbf{x}^c, \mathbf{x}^{sc}, \mathbf{s})$ when there are n transitions remaining in the horizon. The value function of the MDP model defined over an infinite horizon also exists, since 1) both action and state spaces are finite, 2) costs are stationary and bounded for all actions, and 3) transition probabilities are stationary. Then, the value function for the infinite horizon β-discounted problem can be found as the limit: $V(\mathbf{x}^c, \mathbf{x}^{sc}, \mathbf{s}) := \lim_{n \to \infty} V_n(\mathbf{x}^c, \mathbf{x}^{sc}, \mathbf{s})$ (Puterman [1994]). Moreover, it satisfies the following Bellman equations:

$$V(\mathbf{x}^c, \mathbf{x}^{sc}, \mathbf{s})$$

$$
\begin{aligned}
&= \sum_{i\in\mathcal{I}} \lambda_i \min_{(a_i,d_i)\in\mathcal{A}_i} \Big\{ r_{i1}(1-a_i) \\
&\qquad + \sum_{j\in\mathcal{I}} \sum_{k\in\{1,2\}} \mathbb{1}_{\{d_i=(j,k)\}} \big(r^e_{jk} + (1-p^e_{jk}) V(\mathbf{x}^c + a_i e_{i1}, \mathbf{x}^{sc} - e_{jk}, \mathbf{s}) \\
&\qquad\qquad + p^e_{jk} V(\mathbf{x}^c + a_i e_{i1}, \mathbf{x}^{sc} - e_{jk}, \mathbf{s} + \mathbb{1}_{\{\sum_{m\in\mathcal{I}} s_m < S^{max}\}} \overline{e}_j) \big) \\
&\qquad + \mathbb{1}_{\{d_i=(0,0)\}} V(\mathbf{x}^c + a_i e_{i1}, \mathbf{x}^{sc}, \mathbf{s}) \Big\} \\
&+ \sum_{i\in\mathcal{I}} s_i \gamma_i \min_{(a_i,d_i)\in\mathcal{A}_i} \Big\{ r_{i2}(1-a_i) \\
&\qquad + \sum_{j\in\mathcal{I}} \sum_{k\in\{1,2\}} \mathbb{1}_{\{d_i=(j,k)\}} \big(r^e_{jk} + (1-p^e_{jk}) V(\mathbf{x}^c + a_i e_{i2}, \mathbf{x}^{sc} - e_{jk}, \mathbf{s} - a_i \overline{e}_i) \\
&\qquad\qquad + p^e_{jk} V(\mathbf{x}^c + a_i e_{i2}, \mathbf{x}^{sc} - e_{jk}, \mathbf{s} - a_i \overline{e}_i + \overline{e}_j) \\
&\qquad + \mathbb{1}_{\{d_i=(0,0)\}} V(\mathbf{x}^c + a_i e_{i2}, \mathbf{x}^{sc}, \mathbf{s} - a_i \overline{e}_i) \Big\} \qquad (5.2) \\
&+ \sum_{i\in\mathcal{I}} \sum_{k\in\{1,2\}} x^c_{ik} \mu^c_{ik} \Big\{ p^0_{ik} V(\mathbf{x}^c - e_{ik}, \mathbf{x}^{sc} + e_{ik}, \mathbf{s}) + (1-p^0_{ik}) V(\mathbf{x}^c - e_{ik}, \mathbf{x}^{sc}, \mathbf{s}) \Big\} \\
&+ \sum_{i\in\mathcal{I}} \sum_{k\in\{1,2\}} x^{sc}_{ik} \mu^{sc}_{ik} \big(p_{ik} V(\mathbf{x}^c, \mathbf{x}^{sc} - e_{ik}, \mathbf{s} + \mathbb{1}_{\{\sum_{m\in\mathcal{I}} s_m < S^{max}\}} \overline{e}_i) \\
&\qquad\qquad + (1-p_{ik}) V(\mathbf{x}^c, \mathbf{x}^{sc} - e_{ik}, \mathbf{s}) \big) \\
&+ \big(1 - \sum_{i\in\mathcal{I}} \lambda_i - \sum_{i\in\mathcal{I}} s_i \gamma_i - \sum_{i\in\mathcal{I}} \sum_{k\in\{1,2\}} (x^c_{ik}\mu^c_{ik} + x^{sc}_{ik}\mu^{sc}_{ik}) \big) V(\mathbf{x}^c, \mathbf{x}^{sc}, \mathbf{s}),
\end{aligned}
$$

where $\mathcal{A}_i = \mathcal{A}_i(\mathbf{x}^c, \mathbf{x}^{sc}, \mathbf{s})$ is defined in (5.1) and for all $i \in \mathcal{I}$ and $k \in \{1, 2\}$, e_{ik} and $\overline{e}_i$ denote $2I$- and I-dimensional vectors, whose $(i + I(k-1))$th and ith components are 1 and all other components are 0, respectively. The first four lines at the right hand side of (5.2) represent the arrival of the first-time patients, the the second four lines represent the arrival of the recurring patients, the ninth line represents the transitions when the patients complete their critical stage, the tenth and eleventh lines represent the normal discharge of semi-critical patients, and finally the last line represents the fictitious events, in which the system state remains the same with the corresponding probability. Note that we do not make any parametric assumptions in this chapter.

This model has two major drawbacks: First of all, it suffers from curse of dimensionality, as the system state has $5(I + 1)$ dimensions. Second, it assumes that the number of patients who will ask for readmission is known as the number of patients in the orbit, although this information is normally unavailable in practice. The main purpose of introducing the MDP model is to derive good and implementable policies, which calls for eliminating these drawbacks. For this purpose, we consider a simple version of the model in the next section.

5.2 One-Bed Policy Approximation

Our main goal is to evaluate the trade-offs between early-discharging and/or rejecting different patient types when the ICU capacity is scarce. Earlier studies on admission control show that certain results are obtained by focusing on system states with only one available server (Örmeci et al. [2001]). More recently, Ouyang et al. [2020] develop a good approximation based on the analysis of a one-server system in a similar setting. Hence, we simplify the MDP model that was formulated in the previous section by restricting the ICU bed capacity and the orbit size to one, while keeping the variety of patients represented by I types. This system can contain at most two patients, one in the ICU and one in the orbit. We believe that this simplified model is sufficient to extract the trade-offs between different patient types.

We first solve the simplified MDP model to find the corresponding optimal actions for each type of arrival in all possible states. In this model, the size of the orbit is very low so that we need to incur a boundary cost when a patient encounters a "full" orbit. Hence, we assume that when a semi-critical type-ik patient who wants to join the orbit upon her discharge finds the orbit full, the system incurs an additional cost of $r^e_{ik}/(1-p^e_{ik})$ $(r_{ik}/(1-p_{ik}))$ for an early (normal) discharge. These costs work as proxies to the cost of removing the patient permanently from the system.

If the ICU has a critical patient, the new patient will always be rejected. Moreover, we assume that a new patient is always accepted to the ICU when there is an available bed. This assumption aims to solve the ethical dilemma of denying ICU service to a patient currently in need because of anticipating a future patient in worse condition. Hence, we keep track of the actions when the ICU has exactly one patient in semi-critical stage, which corresponds to $2I \times (I + 1)$ states. We denote

$$
\begin{array}{c c}
 & \overbrace{\hphantom{\textit{Empty} \quad \textit{Type}\text{-}1 \quad \dots \quad \textit{Type}\text{-m} \quad \dots \quad \textit{Type}\text{-I}}}^{\text{State of the Orbit}} \\
 & \begin{array}{ccccccc} \textit{Empty} & \textit{Type}\text{-}1 & \dots & \textit{Type}\text{-m} & \dots & \textit{Type}\text{-I} \end{array} \\
\begin{array}{c} 1 \\ \vdots \\ b \\ \vdots \\ I \\ 1 \\ \vdots \\ c \\ \vdots \\ I \end{array} &
\left[\begin{array}{cccccc}
1 & 0 & \dots & 1 & \dots & 1 \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
1 & 1 & \dots & \mathbf{1} & \dots & 0 \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
0 & 0 & \dots & 1 & \dots & 0 \\
0 & 1 & \dots & 1 & \dots & 1 \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
1 & \mathbf{0} & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
0 & 0 & \dots & 1 & \dots & 0
\end{array}\right]
\end{array}
$$

Figure 5.1: Matrix $\textsc{Adm}(ik)$ shows the optimal actions when a type-ik patient arrives at an ICU that contains exactly one semi-critical patient. The first I rows denotes the type of the semi-critical first-time patient and the second set of $1 \dots I$ rows denotes the type of semi-critical recurring patient.

the optimal actions in each of these states upon an arrival of a type-ik patient by the matrix $\textsc{Adm}(ik)$, where all entries are either 0 or 1. Figure 5.2 shows an example: when the ICU has a semi-critical first-time type-b patient and the orbit has a type-m patient, the optimal action is to admit the new type-ik patient by early discharging this type-b patient, while the optimal action is to reject the new type-ik patient if there is one semi-critical recurring type-c patient in the ICU and one type-1 patient in the orbit.

We aim to use the matrix $\textsc{Adm}(ik)$, i.e., the optimal policy in a system with one bed and with an orbit of size 1, to derive a good admission-discharge policy for a system which has B beds and no information on the state of the orbit. To overcome the problem with the orbit state, we define a variable $\textsc{Orbit}(n)$, which serves as a proxy to the expected number of type-n patients in the orbit. The $\textsc{Orbit}$ vector is updated at all discharge and arrival epochs of recurring patients, where the details are given in Definition 1 below. Here, we illustrate how this estimation works using the matrix $\textsc{Adm}(ik)$ in Figure 5.2. Assume that a first-time type-ik patient arrives at the system, i.e., $k = 1$, and the ICU has exactly one first-time type-b patient in semi-critical state. We first find type m with the largest $\textsc{Orbit}$ value. If $\textsc{Orbit}(m)$ is positive, we choose the entry $(b, m + 1)$ in the matrix $\textsc{Adm}(ik)$ and implement the corresponding action. If $\textsc{Orbit}(m)$ is equal to 0, then the orbit is more likely to be empty, so that the action described by the entry $(b, 1)$ in the matrix $\textsc{Adm}(ik)$ is taken. If a recurring patient arrives so that $k = 2$, then we assume that the orbit is full with a type-i patient and set $m = i$.

When the ICU has $B > 1$ beds, there may be more than one type of semi-critical patients in the ICU. Hence, if $m > 0$, estimating the orbit size fixes the column $m + 1$, but the policy still needs to decide on the type of the patient to be discharged, i.e., decide on the row number which varies in $\{1, \ldots, 2I\}$. There are two major criteria for this decision: the cost of early discharge and the expected time an early discharged patient will spend in the ICU due to the readmission requests. The latter quantity can be computed in different ways. Here, we choose the expected time the early discharged patient spends in the ICU as a recurring patient until normal discharge. Another question is on how to combine these two criteria. After trying numerous combinations in our numerical study, we choose to use the product of these two quantities. Consequently, among the semi-critical patients who occupy a bed in the ICU and whose entry in the $(m+1)^{\text{st}}$ column of the $\textsc{Adm}(ik)$ matrix is equal to one are considered for early discharge. The types of these patients are denoted by the set $\mathcal{M}_{ik}$. Among these patient types, the one with the lowest $r^e_{jl} p^e_{jl} \left(\frac{1}{\mu^c_{j2}} + \frac{p^0_{j2}}{\mu^{sc}_{j2}} \right)$ index is early discharged. If $\mathcal{M}_{ik} = \emptyset$, then no patient is early discharged for the admission of a type-ik patient (and thus the type-ik patient is rejected).

Definition 1 describes the one-bed policy, while showing the details of how the variables $\textsc{Orbit}(n)$ are updated as well. Note that we set $a^+ := \max\{a, 0\}$. Below, $\textsc{Adm}(ik)$ for all $i \in \mathcal{I}$ and $k \in \{1,2\}$ is computed by numerically solving the simplified MDP model and $\textsc{Orbit}$: $\mathbb{Z}^I_+ \to \mathbb{R}^I_+$ is initialized with all zeros for all $i \in \mathcal{I}$.

Definition 1 *(One-bed policy)*

For a new type-ik patient $i \in \mathcal{I},\ k \in \{1,2\}$
  if $\sum_{j\in\mathcal{I}} \sum_{l\in\{1,2\}} (x^c_{jl} + x^{sc}_{jl}) < B$, *admit*
    if $k = 2$, *set* $\textsc{Orbit}(i) \to (\textsc{Orbit}(i) - 1)^+$
  else if *define* $m' = \arg\max_{n\in\mathcal{I}}\{\textsc{Orbit}(n)\}$
    if $k = 1$, *set* $m = m'$ *if* $\textsc{Orbit}(m') > 0$, *else set* $m = 0$
    else *set* $m = i$
    set $\mathcal{M}_{ik} = \left\{jl : \textsc{Adm}(ik)_{((j+I(l-1)),(1+m))} = 1, x^{sc}_{jl} > 0, j \in \mathcal{I}, l \in \{1,2\}\right\}$
    if $\mathcal{M}_{ik} \neq \emptyset$, *discharge a type-jl patient early s.t.* $jl \in \arg\min_{jl\in\mathcal{M}_{ik}} r^e_{jl} p^e_{jl} \left(\frac{1}{\mu^c_{j2}} + \frac{p^0_{j2}}{\mu^{sc}_{j2}}\right)$
      set $\textsc{Orbit}(j) \to \textsc{Orbit}(j) + p^e_{jl}$
      if $k = 2$, *set* $\textsc{Orbit}(i) \to (\textsc{Orbit}(i) - 1)^+$
    else if $\mathcal{M}_{ik} = \emptyset$ *reject*
Upon a normal discharge of a type-jl patient, set $\textsc{Orbit}(j) \to \textsc{Orbit}(j) + p_{jl}$

One-bed policy is easy to implement and computationally efficient. However, its performance depend on the number of patient types and the number of beds in the ICU. Hence, the next chapter introduces a different model, which results in a new policy. The performance of both policies will be compared in the numerical chapter.

Chapter 6

FLUID APPROXIMATION MODEL

In this chapter, we formulate the problem defined in Chapter 3 as a deterministic fluid model. We show that the optimal control of the fluid model in the steady state can be obtained by solving a nonlinear program (NLP). Then, we propose a fluid-based policy based on an optimal solution of the aforementioned NLP in Section 6.1.

Let a_{ik} denote the fraction of critical type-ik patients that are admitted to the system and b_{ik} denote the fraction of semi-critical type-ik patients that are discharged normally. We also let X_i denote the steady-state number of type-i patients in the orbit and $X_i(t)$ denote the number of type-i patients in the orbit at time t, $t \in \mathbb{R}_+ := [0, \infty)$. Hence, $X_i := \lim_{t\to\infty} X_i(t)$. To find a closed-form expression for these quantities, we use the rate-balance equations, which should hold for all $i \in \mathcal{I}$ and $t \in \mathbb{R}_+$:

$$\begin{aligned}
\dot{X}_i(t) &= a_{i1}p_{i1}^0\left(p_{i1}b_{i1} + p_{i1}^e(1-b_{i1})\right)\lambda_i \\
&\qquad + a_{i2}p_{i2}^0\left(p_{i2}b_{i2} + p_{i2}^e(1-b_{i2})\right)\gamma_i X_i(t) - a_{i2}\gamma_i X_i(t) \\
&= a_{i1}p_{i1}^0\left(p_{i1}b_{i1} + p_{i1}^e(1-b_{i1})\right)\lambda_i \\
&\qquad - a_{i2}\gamma_i\left(1 - p_{i2}^0\left(p_{i2}b_{i2} + p_{i2}^e(1-b_{i2})\right)\right)X_i(t)
\end{aligned}$$

where $\dot{X}_i(t)$ is defined as the derivative of X_i. Let $C_{i1} := a_{i1}p_{i1}^0\left(p_{i1}b_{i1} + p_{i1}^e(1-b_{i1})\right)\lambda_i$ and $C_{i2} := a_{i2}\gamma_i\left(1 - p_{i2}^0\left(p_{i2}b_{i2} + p_{i2}^e(1-b_{i2})\right)\right)$. Then, $\dot{X}_i(t) = C_{i1} - C_{i2}X_i(t)$. By algebra and assuming that $X_i(0) = 0$, we have the following solution:

$$X_i(t) = \frac{C_{i1}}{C_{i2}}\left(1 - \mathrm{e}^{-C_{i2}t}\right), \text{ which implies that:}$$

$$X_i = \frac{C_{i1}}{C_{i2}} = \frac{a_{i1}p_{i1}^0\left(p_{i1}b_{i1} + p_{i1}^e(1-b_{i1})\right)\lambda_i}{a_{i2}\gamma_i\left(1 - p_{i2}^0\left(p_{i2}b_{i2} + p_{i2}^e(1-b_{i2})\right)\right)}$$

Then in the steady-state system, Little's Law specifies the capacity constraint as follows:

$$\sum_{i\in\mathcal{I}}\left(\frac{a_{i1}\lambda_i}{\mu^c_{i1}}+\frac{a_{i2}\gamma_i X_i}{\mu^c_{i2}}+\frac{a_{i1}p^0_{ik}b_{i1}\lambda_i}{\mu^{sc}_{i1}}+\frac{a_{i2}p^0_{ik}b_{i2}\gamma_i X_i}{\mu^{sc}_{i2}}\right)\le B \tag{6.1}$$

The first two terms in (6.1) are resource requirements of critical first-time and recurring patients, respectively, and the last two terms are resource requirements of semi-critical first-time and recurring patients that are discharged normally, respectively. The total rate of patients that are discharged early and rejected are

$$\sum_{i\in\mathcal{I}}\left(a_{i1}p^0_{i1}(1-b_{i1})\lambda_i+a_{i2}p^0_{i2}(1-b_{i2})\gamma_i X_i\right), \quad \text{and}$$
$$\sum_{i\in\mathcal{I}}\left((1-a_{i1})\lambda_i+(1-a_{i2})\gamma_i X_i\right),$$

respectively. Recall that critical type-ik patients may leave the ICU due to death with probability $(1-p^0_{ik})$ for all $i\in\mathcal{I}$ and $k\in\{1,2\}$. Furthermore, rejected first-time patients are lost but rejected recurring patients return to orbit. Consequently, the related optimization problem is the following:

$$\min_{\substack{a_{ik},b_{ik},X_i\\ i\in\mathcal{I},k\in\{1,2\}}}\sum_{i\in\mathcal{I}}\left(r_{i1}(1-a_{i1})+r^e_{i1}a_{i1}p^0_{i1}(1-b_{i1})\right)\lambda_i$$
$$+\left(r_{i2}(1-a_{i2})+r^e_{i2}p^0_{i2}a_{i2}(1-b_{i2})\right)\gamma_i X_i \tag{6.2a}$$

$$\text{s.t.}\quad X_i=\frac{a_{i1}p^0_{i1}\left(p_{i1}b_{i1}+p^e_{i1}(1-b_{i1})\right)\lambda_i}{a_{i2}\gamma_i\left(1-p^0_{i2}\left(p_{i2}b_{i2}+p^e_{i2}(1-b_{i2})\right)\right)}\ge 0,\quad \forall i\in\mathcal{I} \tag{6.2b}$$

$$\sum_{i\in\mathcal{I}}\left(\frac{a_{i1}\lambda_i}{\mu^c_{i1}}+\frac{a_{i2}\gamma_i X_i}{\mu^c_{i2}}+\frac{a_{i1}p^0_{i1}b_{i1}\lambda_i}{\mu^{sc}_{i1}}+\frac{a_{i2}p^0_{i2}b_{i2}\gamma_i X_i}{\mu^{sc}_{i2}}\right)\le B \tag{6.2c}$$

$$0\le a_{ik}\le 1,\quad 0\le b_{ik}\le 1,\quad \forall i\in\mathcal{I},\ \forall k\in\{1,2\}. \tag{6.2d}$$

We can simplify the NLP (6.2) in the following way: let $w_{i1}:=a_{i1}b_{i1}$, $w_{i2}:=a_{i1}(1-b_{i1})$, $w_{i3}:=1-a_{i1}$, $y_{i1}:=a_{i2}b_{i2}$, $y_{i2}:=a_{i2}(1-b_{i2})$, $y_{i3}:=1-a_{i2}$, $\forall i\in\mathcal{I}$ and $\forall k\in\{1,2\}$. Then, $p^0_{i1}w_{i1}$ ($p^0_{i2}y_{i1}$) denotes the fraction of semi-critical first-time (recurring) type-i patients that are discharged normally, $p^0_{i1}w_{i2}$ ($p^0_{i2}y_{i2}$) denotes the fraction of semi-critical first-time (recurring) type-i patients that are discharged early, and w_{i3} (y_{i3}) denotes the fraction of critical first-time (recurring) type-i patients that are not admitted to the ICU. Consequently, NLP (6.2) is equivalent to the following:

$$\min_{\substack{w_{il},y_{il},X_i\\ i\in\mathcal{I},l\in\{1,2,3\}}} \sum_{i\in\mathcal{I}} \Big(\left(r_{i1}w_{i3} + r^e_{i1}p^0_{i1}w_{i2}\right)\lambda_i + \left(r_{i2}y_{i3} + r^e_{i2}p^0_{i2}y_{i2}\right)\gamma_i X_i \Big) \tag{6.3a}$$

$$\text{s.t.} \quad X_i = \frac{\left(p_{i1}w_{i1} + p^e_{i1}w_{i2}\right)p^0_{i1}\lambda_i}{\gamma_i\left((1-p^0_{i2}p_{i2})y_{i1} + (1-p^0_{i2}p^e_{i2})y_{i2}\right)} \geq 0, \quad \forall i\in\mathcal{I} \tag{6.3b}$$

$$\sum_{i\in\mathcal{I}} \left(\frac{(1-w_{i3})\lambda_i}{\mu^c_{i1}} + \frac{p^0_{i1}w_{i1}\lambda_i}{\mu^{sc}_{i1}} + \frac{(1-y_{i3})\gamma_i X_i}{\mu^c_{i2}} + \frac{p^0_{i2}y_{i1}\gamma_i X_i}{\mu^{sc}_{i2}} \right) \leq B \tag{6.3c}$$

$$\sum_{l=1}^{3} w_{il} = 1, \quad \sum_{l=1}^{3} y_{il} = 1, \quad \forall i\in\mathcal{I} \tag{6.3d}$$

$$w_{il} \geq 0, \quad y_{il} \geq 0, \quad \forall i\in\mathcal{I},\ l\in\{1,2,3\}. \tag{6.3e}$$

The objective (6.3a) minimizes the total patient rejection and early-discharge cost rate, (6.3b) is the rate-balance constraint, (6.3c) is the capacity constraint, (6.3d) and (6.3e) determine the action space.

Consider the policy, denoted by π^{All}, that admits all patients and discharges no one early, i.e., $w_{i1} = y_{i1} = 1$ and $w_{ij} = y_{ij} = 0$ for all $i \in \mathcal{I}$ and $j \in \{2,3\}$. The capacity that this policy requires can be found using (6.3c) and (6.3b) as follows:

$$\sum_{i\in\mathcal{I}} \lambda_i \left(\frac{1}{\mu^c_{i1}} + \frac{p^0_{i1}}{\mu^{sc}_{i1}} + \left(\frac{1}{\mu^c_{i2}} + \frac{p^0_{i2}}{\mu^{sc}_{i2}} \right) \frac{p_{i1}p^0_{i1}}{1-p^0_{i2}p_{i2}} \right). \tag{6.4}$$

If this quantity does not exceed the total capacity B, then policy π^{All} induces no costs, so that it is the optimal policy. Otherwise, the system does not have enough capacity to provide service to all patients till their complete recovery. We call such systems *overloaded systems.* In such systems, some patients must be either rejected or admitted by discharging another patient early, and finding an optimal solution to (6.3) is not trivial. However, we can still show that recurring patients should never be rejected under the following condition.

Condition 4 *For all* $i \in \mathcal{I}$, $\dfrac{1}{\mu^{sc}_{i2}} \geq \dfrac{p^e_{i2} - p_{i2}}{\mu^c_{i2}(1-p^0_{i2}p^e_{i2})}$

Condition 4 implies that the time that recurring type-i patients spend in semi-critical stage is greater than or equal to a certain ratio of the time they spend in critical stage. Then, we have the following result.

Lemma 1 *If Condition 4 holds, there exists an optimal solution of NLP* (6.3) *such that* $y_{i3} = 0$ *for all* $i \in \mathcal{I}$.

The proof of Lemma 1 is presented in E-Companion B.1. Lemma 1 states that the recurring patients are never rejected, that is, they are either admitted to a free bed or admitted by discharging another patient early in the ICU. This not only gives some insight on the structure of the optimal solution, but also significantly reduces the action space in the numerical computations. That is, we can set the decision variable y_{i3} in NLP (6.3) to zero. We further present some conditions under which the system operates at full capacity in overloaded regimes.

Condition 5 $r^e_{i2} \leq r_{i1}\left(\dfrac{1 - p^0_{i2}p^e_{i2}}{p_{i1}p^0_{i1}p^0_{i2}}\right)$, *for all* $i \in \mathcal{I}$.

Condition 5 implies that the cost of discharging a recurring type-i patient early is less than or equal to some ratio of the cost of rejecting a first-time patient from that given type. Condition 5 leads to the following result:

Lemma 2 *When Conditions 4 and 5 hold in an overloaded system, there exists an optimal solution that completely uses the bed capacity, i.e., the capacity constraint* (6.3c) *is binding.*

The proof of Lemma 2 is presented in E-Companion B.2.

Conditions 4 and 5 are not restrictive in the sense that they hold in all the parameter settings that we consider in the numerical experiments presented in Chapter 7 and are constructed with respect to studies from the ICU literature.

When these conditions are not satisfied, extreme situations can be encountered. The next example illustrates such a case, where Condition 5 is violated. In this case, it is optimal to completely shut down the system by rejecting all first-time patients:

Example 4 *Consider a system with single patient type and let the parameters be:* $\lambda = 75$, $\mu^c_{11} = \mu^{sc}_{11} = \mu^{sc}_{12} = 1$, $\mu^c_{12} = 0.1$, $\gamma = 0.5$, $p_{11} = 0.80$, $p^e_{11} = 0.90$,

$p_{12} = 0.99$, $p^e_{12} = 0.99$, $r^e_{11} = 1$, $r_{11} = 1$, $r^e_{12} = 10$, $r_{12} = 10$ *and* $B = 14$. *The optimal action proposed by solving NLP* (6.3) *is to reject all the first-time patients, that is, the system completely shuts down. Recurring critical patients' LOS is very high compared to other patients' LOS and readmission rates are very high for both normal and early discharges. In other words, most of the discharged patients go to the orbit, after which they will be recurring patients eventually. Because the system is overloaded and semi-critical patients spend very little time in the ICU, new first-time patients will either be rejected or admitted by discharging a recurring patient early, where the latter alternative is more costly than the former. Therefore, it is optimal for the system to completely shut down to avoid high early-discharge costs and a loop of recurring patients.*

In a way we can associate the extreme cases with systems that do not satisfy conditions 4 and 5. In Chapter 7, we consider different parameter settings, which are constructed based on the medical literature on ICU processes. Both conditions 4 and 5 are satisfied in all these settings, suggesting that they are not restrictive in the context of ICUs.

Next, we propose an easy-to-implement control policy based on the fluid model.

6.1 Fluid-Based Policy

In this section, we propose a control policy based on an optimal solution of the NLP (6.3). Due to the stochastic nature of the ICUs, there will be cases in which the NLP solution cannot be implemented directly. For example, consider an underloaded system in which the NLP solution is trivial and implies admitting and normally discharging all patients. However, in practice, even in underloaded systems, there can be patient arrival epochs when the ICU is full due to the stochastic nature of the system and thus not all patients can be admitted and discharged normally. To overcome this issue, we propose the following fluid-based policy that takes into account the stochastic nature of the ICUs. Note that once NLP (6.3) is solved offline, the policy described in Definition 2 is applied whenever a new patient arrives.

Definition 2 *(Fluid-based policy)*

1: For all $i \in \mathcal{I}$ and $k \in \{1,2\}$, update the following variables at all patient arrival and discharge epochs:

$TA_{ik} \leftarrow$ *Total number of type-ik patients requesting admission to the system*

$R_{ik} \leftarrow$ *Total number of type-ik patients rejected*

$ED_{ik} \leftarrow$ *Total number of type-ik patients discharged early*

2: Define $\mathcal{M} = \{jl : x^{sc}_{jl} > 0, j \in \mathcal{I}, l \in \{1,2\}\}$

3: If a type-ik patient arrives in the system,

4: ***if*** $\sum_{j\in\mathcal{I}} \sum_{l\in\{1,2\}} (x^c_{jl} + x^{sc}_{jl}) < B$, *admit*

5: ***else if*** $\left((\mathcal{M} = \emptyset) \text{ or } \left(k = 1 \text{ and } w_{i3} > \frac{R_{i1}}{TA_{i1}}\right)\right)$, *reject*

6: ***else if*** $\sum_{j\in\mathcal{I}} (w_{j2} x^{sc}_{j1} + y_{j2} x^{sc}_{j2}) > 0$, *set:*

$$j_1 \in \underset{j\in\mathcal{I}: w_{j2}>0, x^{sc}_{j1}>0}{\arg\max} \left(w_{j2} - \frac{ED_{j1}}{TA_{j1}} \right), \quad f_1 = w_{j_1 2} - \frac{ED_{j_1 1}}{TA_{j_1 1}},$$

$$j_2 \in \underset{j\in\mathcal{I}: y_{j2}>0, x^{sc}_{j2}>0}{\arg\max} \left(y_{j2} - \frac{ED_{j2}}{TA_{j2}} \right), \quad f_2 = y_{j_2 2} - \frac{ED_{j_2 2}}{TA_{j_2 2}}$$

7: ***if*** $(f_1 \geq f_2)$, *then discharge a type-$j_1 1$ patient early.*

8: ***else*** *discharge a type-$j_2 2$ patient early.*

9: ***else***

10: ***if*** $\left(k = 1 \quad \text{and} \quad r_{i1} < r^e_{jl} + \dfrac{\sum_{m\in\mathcal{I}} \lambda_m r_{m1} + \gamma_j r_{j2}}{\sum_{m\in\mathcal{I}} \lambda_m + B\mu^c_{j2} + \gamma_j}\right)$ *for all $jl \in \mathcal{M}$, reject*

11: ***else*** *discharge a type-jl patient early s.t.* $jl \in \arg\min_{jl\in\mathcal{M}} r^e_{jl} p^e_{jl} \left(\frac{1}{\mu^c_{j2}} + \frac{p^0_{j2}}{\mu^{sc}_{j2}} \right)$.

The fluid-based policy counts the cumulative number of admitted, rejected, and early-discharged patients at each patient arrival or discharge epochs. By considering Lemma 2, the fluid-based policy always admits a patient to the ICU if there is a free bed. When a patient arrives in a full ICU, if the fraction of rejected patients thus far is less than or equal to the rejection rate obtained from the NLP (6.3) solution (that is, w_{i3}), then the patient is rejected. Otherwise, the policy checks (in lines 10 and 11) whether there exists a patient in semi-critical condition and whose realized early-discharge rate is less than or equal to the early-discharge rate obtained from the NLP (6.3) solution. If such a patient exists, she is discharged early so that the new patient is admitted. If such a patient does not exist, then the fluid-based policy makes a decision myopically (see lines 12-14). Let $\mathcal{M}$ denote the set of patient types occupying an ICU bed in semi-critical condition. If there is not any semi-critical patients in the ICU (i.e., $\mathcal{M} = \emptyset$), the default action is to reject the arriving patient. Otherwise, the cost of rejecting her is compared with the cost of discharging a type-jl patient early, $jl \in \mathcal{M}$, plus the cost of having to reject all future arriving patients assuming the worst case, where the early-discharged type-jl patient joins the orbit and the ICU will be full with critical type-jl patients. If the corresponding rejection cost is lower than the corresponding early-discharge costs for all $jl \in \mathcal{M}$, the arriving patient is rejected, otherwise she is admitted by discharging one of the type-jl patients early, whose aforementioned myopic early-discharge cost is less than the rejection cost of the arriving patient.

Similar steps are followed upon a recurring patient arrival with the exception that rejecting recurring patients are not allowed when there is at least one semi-critical patient in the ICU, which is in consistent with Lemma 1.

In the following chapter, we compare the performances of the proposed policies with those of the state-of-the-art policies by a discrete-event simulation model.

Chapter 7

SIMULATION EXPERIMENTS

In this chapter, we compare the performances of the one-bed and fluid-based policies with those of the state-of-the-art policies by discrete-event simulation. We choose most of the system parameters from the ICU literature. We present numerical experiments with a single patient type ($I = 1$) in Section 7.1 and with two patient types ($I = 2$) in Section 7.2.

7.1 *Simulation with Single Patient Type*

We simulate an ICU with a single patient type, that is, $I = 1$. For convenience, we remove the index i from the notation and as usual, the subscript $k = 1$ denotes first-time patients and $k = 2$ denotes recurring patients. We choose most of the parameters from Long and Mathews [2017], where two ICUs at two different hospitals, denoted by H_A and H_B, are empirically analyzed. The parameters for the two hospitals are given in Table 7.1. We simulate each hospital separately.

Table 7.1: Parameter Setting For A Single Patient Type

	$1/\mu_{total}$	B	p_1^0	p_1	θ^{CR}	p_2^0	p_2	p_1^e	p_2^e	$1/\gamma$	θ^R	ρ
H_A	$73.3^\dagger$	51	0.92	0.05	0.67	0.92	0.05	0.14	0.14	$\{2, 4, 6\}^*$	$\{1, 1.25, 1.5\}$	$\{0.9, 1, 1.25, 1.5\}$
H_B	$116.9^\dagger$	14	0.79	0.08	0.67	0.79	0.08	0.21	0.21	$\{2, 4, 6\}^*$	$\{1, 1.25, 1.5\}$	$\{0.9, 1, 1.25, 1.5\}$

† in hours, ∗ in days

The first four parameters in Table 7.1 presents the parameters borrowed from Long and Mathews [2017] and the remaining refer to the parameters that are required for our analysis and are set as follows: θ^{CR} is defined as the ratio of the average time a patient spends in critical health condition to her overall average LOS and is adapted from Armony et al. [2018]. Because there is not any information about recurring patients in Long and Mathews [2017], we assume that $p_1 = p_2$ (a similar approach is used in Armony et al. [2018]). We scale the readmission probabilities (p_1^e and p_2^e) for early-discharged patients based on the findings of KC and Terwiesch [2012] and set $p_1^e = p_2^e$. Furthermore, because mean LOS of recurring patients is greater than or equal to that of the first-time patients (Durbin and Kopel [1993], KC and Terwiesch [2012]), we assume that the mean LOS of critical (semi-critical) recurring patients is θ^R times the one of the critical (semi-critical) first-time patients, where $\theta^R \in \{1, 1.25, 1.5\}$. Average time spent in the orbit is chosen such that $1/\gamma \in \{2, 4, 6\}$ days. We choose several different arrival rates for the first-time patients such that we consider the ICU in both underloaded and overloaded regimes. Specifically, we let ρ denote the ratio of the left-hand-side of the inequality (6.4) to the bed capacity (B) so that ρ denotes the system load. Then we choose the arrival rates such that $\rho \in \{0.9, 1, 1.25, 1.5\}$. Observe that $\rho = 0.9$ implies an underloaded system, $\rho = 1$ implies a critically-loaded system, and $\rho > 1$ implies an overloaded system.

For cost parameters, we fix $r_1^e = 1$ and $r_2 = 10$ throughout our analysis and consider the following two cases. In the first case, we set $r_2^e = 1$ and increase r_1 from 1 to 10 by increments of 1, which we denote as *cost case 1*. In the second case, we set $r_1 = 1$ and increase r_2^e from 1 to 10 by increments of 1, which we denote as *cost case 2*. The first case promotes early discharge by increasing the rejection cost of the first-time patients while the second case promotes rejecting the first-time patients by increasing the early-discharge cost of the recurring patients.

We simulate the ICU environment as a discrete-event-simulation in MATLAB, where all interarrival and service times are independently and exponentially distributed. Along with the one-bed and fluid-based policies, we test the performances of two state-of-the-art policies, namely the *reject* and *early-discharge* policies. Both policies admit new patients when there is a free bed. Under the reject policy, if the ICU is full upon a patient arrival, then the patient is rejected. Under the early-discharge policy, if the ICU is full upon a patient arrival and there is a patient occupying a bed in semi-critical health condition, then the patient is admitted by discharging a patient in semi-critical health condition early. If there are more than

Figure 7.1: Policy Performance with respect to the Reject Policy, Hospital A - Cost Case 1, $1/\gamma = 2, \theta^R = 1.25$

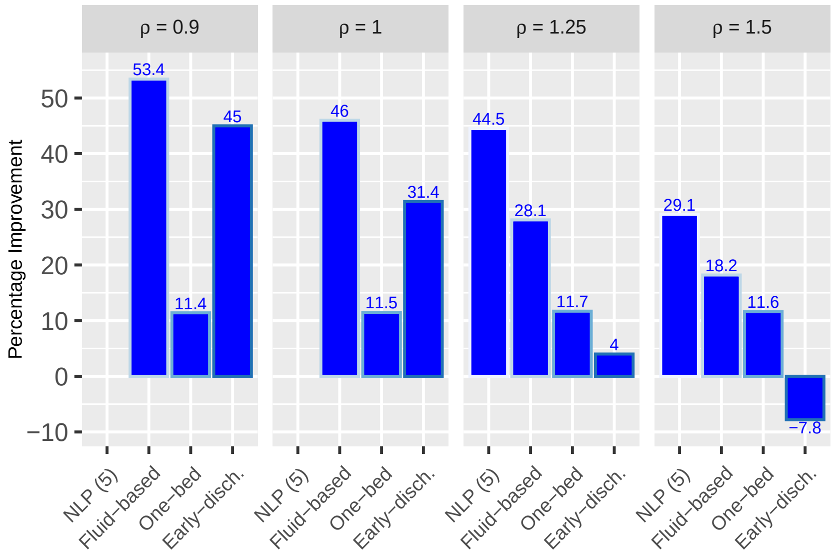
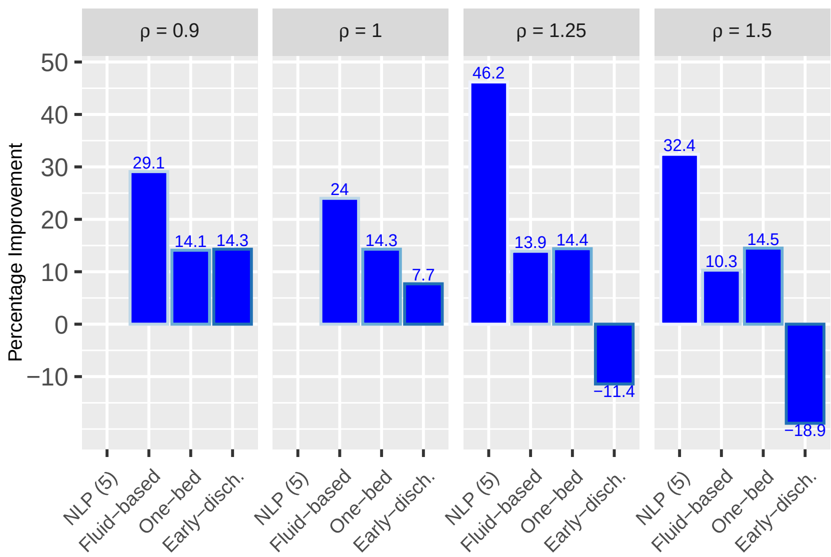


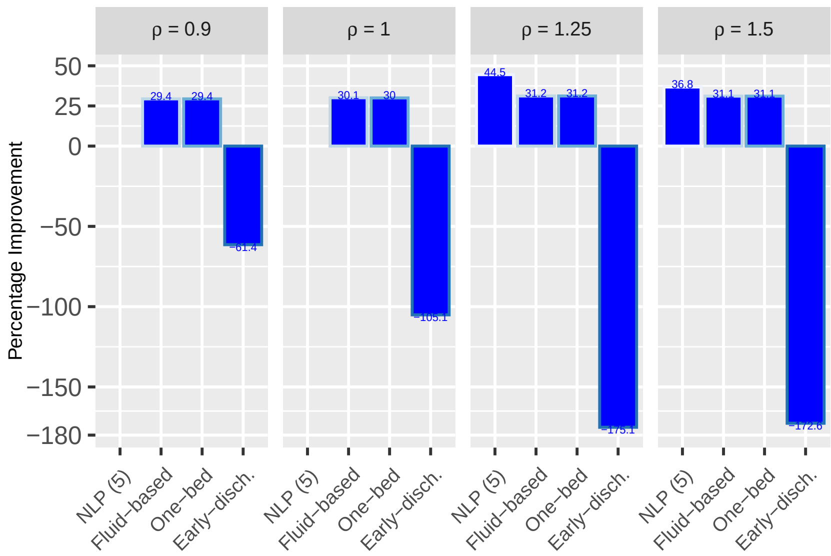
Figure 7.2: Policy Performance with respect to the Reject Policy, Hospital B - Cost Case 1, $1/\gamma = 2, \theta^R = 1.25$



one patient in semi-critical condition, a patient with the least early-discharge cost is discharged early. Otherwise, if there is no patient in semi-critical health condition, the arriving patient is rejected. Note that we let a first-time semi-critical patient to be early discharged when $r_1^e = r_2^e$ and there is a recurring patient in semi-critical condition in the ICU. For each policy and parameter setting, we run 100 replications in a way such that 2000 first-time patient arrivals are used as a warm-up period and 10000 first-time patient arrivals are used for statistical analyses. For a given parameter setting, we calculate the mean improvement of the early discharge, one-bed and fluid-based policies from the reject policy. We also report the improvement of the optimal NLP (6.3) objective function value over the reject policy to serve as a numerical upper bound when the system is overloaded. Because the optimal NLP (6.3) objective function value is non-zero only if the system is overloaded, we present the associated improvement only when the system is overloaded (that is, if $\rho \in \{1.25, 1.5\}$).

Figures 7.1 and 7.2 illustrate the performances of the fluid-based, one-bed, and early-discharge policies for both hospitals along with the optimal NLP (6.3) objective function value with respect to the reject policy under cost case 1 when $1/\gamma = 2$ and $\theta^R = 1.25$. Recall that the cost case 1 promotes early-discharge decisions by charging a high rejection cost for the new first-time patients. According to the results, the fluid-based policy performs the best among the four policies. We observe that when the system load is low, early-discharge decisions are more cost effective than the reject decisions. Observe also that the fluid-based policy performs better than the early-discharge policy when the system is the lowest, which shows that the fluid-based policy mitigates early-discharge decisions with reject decisions. For example, when the system load is low, the fluid-based policy performs around 50% and 30% on average better than the reject policy in Hospital A and Hospital B, respectively, and outperforms the early-discharge policy in all instances. One critical observation from Figures 7.1 and 7.2 is that the fluid-based policy performs better in large capacity systems than it does in the small capacity systems (that is, 51-bed vs. 14-bed ICU), which is expected because we assume that each patient and bed is infinitesimally

Figure 7.3: Policy Performance with respect to the Reject Policy, Hospital A - Cost Case 2, $1/\gamma = 2, \theta^R = 1.25$

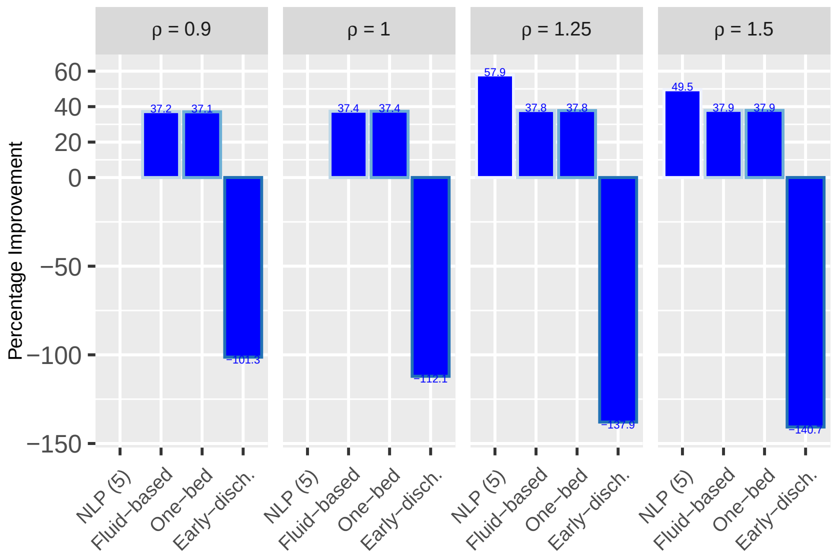


small in the fluid model and thus the fluid model approximates large ICUs better than the small ones.

The performance gaps between the policies decrease as the system load increases. This is because as the system load increases, more patients must be rejected and thus the rejection costs dominate the early-discharge costs and the performances of the policies converge to the one of the reject policy. Observe that as the system load increases, many patients will be rejected even under the early-discharge policy. This is because only patients in semi-critical condition can be early discharged and the early-discharges will lead to an ICU with a higher number of critical recurring patients which in turn leads to rejection of new first-time or recurrent patients.

According to the results, the performance of the early-discharge policy decreases dramatically as the system load increases in both Hospital A and Hospital B. The early-discharged patients have a higher probability of readmission and once readmitted they spend longer time in the ICU as either critical or semi-critical patients (recall that recurring patients' LOS is 25% higher than first-time patients' LOS in this setting). This will lead the early-discharge policy to early discharge semi-

Figure 7.4: Policy Performance with respect to the Reject Policy, Hospital B - Cost Case 2, $1/\gamma = 2, \theta^R = 1.25$



critical recurring patients when there are not any semi-critical first-time patients. The rejection rate of both the new first-time and recurring patients will increase by the same reasons. Consequently, if the early-discharge costs are relatively less than the rejection costs and the system is underloaded, early-discharge policy performs better than both the one-bed and reject policies do, but its performance decreases dramatically as the system load increases.

The performance of the one-bed policy is stable and higher than the performance of the reject policy under all system loads in both hospitals. We observe that the optimal solution in a one-bed environment is conservative in applying early-discharge decisions. Even with its very simple setting, the one-bed MDP model is able to capture adverse effects of consistently early discharging patients.

Figures 7.3 and 7.4 show the relative performances of the policies under cost case 2, where rejection of first-time patients is promoted by the relatively high early-discharge cost of recurring patients. As expected, the early-discharge policy performs by far the worst among the four policies. Surprisingly, the one-bed policy performs as good as the fluid-based policy and at least 30% better than the reject

Table 7.2: Parameter Setting For A Two Patient-Type ICU

	$1/\mu_{total}$	p^0_{i1}	p_{i1}	θ^{CR}	p^0_{i2}	p_{i2}	p^e_{i1}	p^e_{i2}	$1/\gamma$	θ^C	θ^{AR}
Type-1	$73.3^{\dagger}$	0.92	0.05	0.67	0.92	0.05	0.14	0.14	2^{*}	-	$\{50, 60, 70\}$
Type-2	$116.9^{\dagger}$	0.79	0.08	0.67	0.79	0.08	0.21	0.21	2^{*}	$\{2, 3, 4\}$	-

† in hours, ∗ in days

policy does. In both hospitals, the performances of those two policies are close to that of the optimal NLP (6.3) objective function value and does not change as the system load increases. We observe in our experiments that the one-bed policy follows a policy similar to the reject policy in a way such that it rejects the new first-time patients but always admits the recurring patients. We also observe that time spent in the orbit does not affect the results significantly. Table C.1 and Table C.2 in Appendix C present detailed and additional numerical results.

Recall from Lemma 2 that the recurring patients are never rejected under certain conditions in the deterministic fluid model. However, we are unable to prove the existence of such a property of the optimal policy in the MDP model with multiple patient types and recovery stages. For a simplified version of the model, where patients go through only one recovery stage and there is only one patient type, we show in Proposition 1 that recurring patients are not rejected when there is at least one recurring patient in the ICU. We are aware that the simplified model does not completely reflect the model dynamics described in Chapter 3, but we conjecture that it mimics that recurring patients are not rejected, which we observe in the deterministic fluid model and one-bed MDP model through an extensive numerical analysis.

7.2 Simulation with Two Patient Types

We consider an ICU with 32 beds and two patient types. Type 1 and 2 patients are based on the patients of the Hospital A and Hospital B in Table 7.1, respectively. The corresponding parameters are given in Table 7.2. The parameter θ^C denotes the ratio of the rejection (early-discharge) cost of a type-2 patient to the rejection

Table 7.3: Percentage Improvement over Reject Policy - Two-Patient-Type ICU

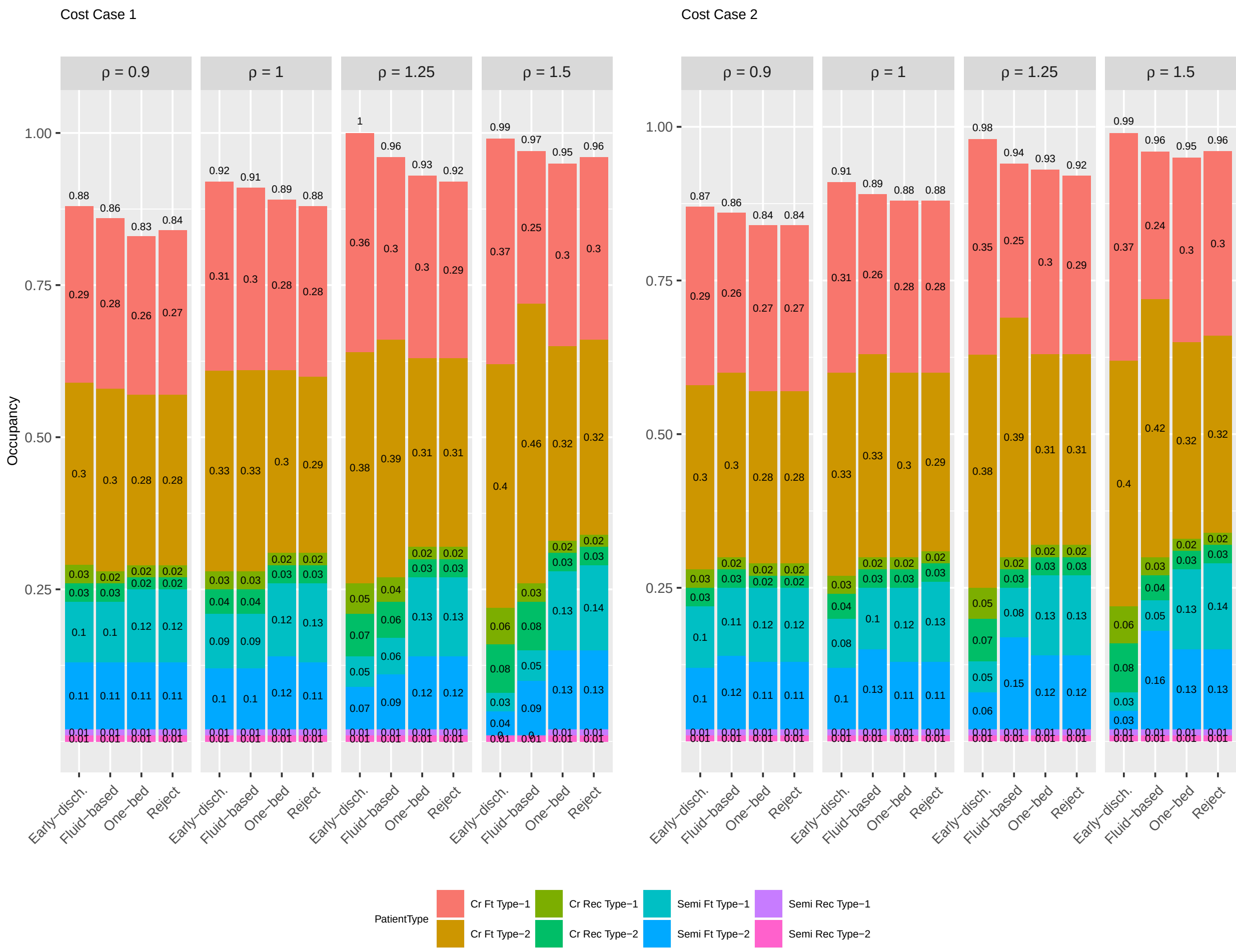
		Demand							
		$\rho = 0.9$	$\rho = 1$	$\rho = 1.25$	$\rho = 1.5$	$\rho = 0.9$	$\rho = 1$	$\rho = 1.25$	$\rho = 1.5$
		Cost Case 1				Cost Case 2			
(θ^C)	Policy	Type-A patients are 50% of all arriving patients (θ^{AR})							
2	Fluid NLP	-	-	50.8	36.2	-	-	59.2	51.5
	Early Discharge	39.5	25.5	-2.5	-14.8	-69.0	-106.0	-160.5	-163.5
	One-Bed Policy	14.3	13.7	14.0	13.8	36.0	36.2	36.9	36.6
	Fluid Based Heuristic	51.2	42.9	25.1	18.3	33.0	30.0	27.9	27.7
3	Fluid NLP	-	-	60.9	50.6	-	-	69.7	64.0
	Early Discharge	43.1	28.6	-1.2	-14.5	-59.6	-98.0	-157.3	-162.9
	One-Bed Policy	18.2	16.9	15.9	15.3	37.0	37.3	37.9	37.5
	Fluid Based Heuristic	59.0	52.4	36.0	30.4	45.5	42.1	37.7	35.5
4	Fluid NLP	-	-	68.6	60.4	-	-	75.9	71.3
	Early Discharge	45.3	30.4	-0.5	-14.3	-53.8	-93.2	-155.3	-162.5
	One-Bed Policy	20.4	19.6	17.5	16.5	37.7	38.0	38.5	38.1
	Fluid Based Heuristic	61.4	57.2	44.0	35.9	52.8	49.2	43.4	39.8
		Type-A patients are 60% of all arriving patients (θ^{AR})							
2	Fluid NLP	-	-	50.7	35.6	-	-	57.3	49.9
	Early Discharge	43.3	28.6	-0.3	-13.1	-60.1	-101.4	-160.2	-161.2
	One-Bed Policy	14.2	13.8	13.4	13.6	35.4	35.0	35.5	36.0
	Fluid Based Heuristic	53.9	45.0	26.9	19.8	34.0	30.1	27.0	27.0
3	Fluid NLP	-	-	59.8	48.6	-	-	67.1	61.5
	Early Discharge	47.5	32.2	1.3	-12.6	-48.9	-92.2	-156.5	-160.1
	One-Bed Policy	18.7	17.5	15.8	15.5	36.5	36.1	36.7	37.1
	Fluid Based Heuristic	60.9	53.8	38.6	32.7	47.1	43.5	38.7	36.7
4	Fluid NLP	-	-	67.0	57.9	-	-	73.3	68.7
	Early Discharge	50.2	34.4	2.2	-12.3	-41.7	-86.2	-154.0	-159.4
	One-Bed Policy	21.4	19.8	18.1	17.3	37.3	36.7	37.3	37.7
	Fluid Based Heuristic	64.6	58.4	46.7	42.0	55.3	51.9	45.9	42.7
		Type-A patients are 70% of all arriving patients (θ^{AR})							
2	Fluid NLP	-	-	51.0	35.0	-	-	55.7	47.9
	Early Discharge	43.3	29.5	1.3	-11.6	-62.3	-99.1	-157.1	-160.5
	One-Bed Policy	13.9	13.5	13.4	13.5	34.0	34.5	34.7	35.0
	Fluid Based Heuristic	53.7	45.6	27.9	19.6	34.0	31.9	28.0	27.0
3	Fluid NLP	-	-	58.8	46.2	-	-	64.5	58.2
	Early Discharge	47.8	33.3	3.0	-11.2	-50.1	-88.8	-153.0	-159.3
	One-Bed Policy	17.0	16.9	16.2	16.0	35.3	35.7	35.8	36.1
	Fluid Based Heuristic	59.9	53.6	39.4	31.8	46.0	44.0	39.9	37.8
4	Fluid NLP	-	-	65.3	54.7	-	-	70.3	65.1
	Early Discharge	50.8	36.0	4.1	-10.9	-41.7	-81.6	-150.0	-158.5
	One-Bed Policy	20.5	20.1	18.3	17.8	36.2	36.5	36.6	36.8
	Fluid Based Heuristic	64.8	59.0	47.4	40.7	55.0	52.9	48.1	45.0

(early-discharge) cost of a type-1 patient. The parameter θ^{AR} denotes the percentage of type-1 patients among all arriving first-time patients. We set the total first-time patient arrival rate for the ICU as in Section 7.1 such that the system operates in underloaded, critically loaded, and overloaded regimes. Observe that type-2 patients have higher LOS, higher death rate, higher probability of being readmitted, and higher rejection/early discharge costs than the type-1 patients have. We set the average time a patient spends in the orbit to 2 days because it does not affect the policy performances significantly as seen in Table C.1 and Table C.2. We also assume that recurring patients' total mean LOS is 25% higher than than the total mean LOS of first-time patients. As observed in Table C.1 and Table C.2, the results are robust to the increases in recurring patients' relative LOS.

We consider two different cost cases, namely cost case 1 and cost case 2. Specifically, cost parameters for type-1 patients are chosen as in Section 7.1 and the cost parameters of type-2 patients are adjusted accordingly with respect to the parameter θ^C. We compare the performances of the one-bed, fluid-based, and the early-discharge policies with the one of the reject policy for every combination of system load, cost case, θ^C and θ^{AR} and the results are presented in Table 7.3.

According to the results, in the cost case 1, the fluid-based policy performs the best among the four policies. However, its performance decreases as the system load increases. As the system load increases, more patients are early discharged under the fluid-based policy which in turn increases the number of critical recurring patients in the ICU. Because critical patients cannot be early discharged, new patients are rejected and thus the actions and so the performance of the fluid-based policy becomes similar to the ones of the reject policy. However, this decrease is not as dramatic as the decrease in the performance of the early-discharge policy. In the cost case 1, the performance of the one-bed policy is robust with respect to the system load and higher than the one of the reject policy. The performance of the early-discharge policy decreases dramatically as the system load increases. As the system load increases, many recurrent patients are rejected under the early discharge policy which in turn decreases the performance. In the cost case 2, the

Figure 7.5: Occupancy Rates Per Cost-Case and Policy for $\theta^C = 2$ and $\theta^{AR} = \%60$



one-bed policy performs as good as the fluid-based policy and much better than the reject and early-discharge policies. Consequently, our proposed policies significantly outperform the state-of-the-art policies.

Figure 7.5 shows how different policies manage the bed capacity for a particular set of parameters. The numbers over the bars represent the average ICU occupancy per policy while the ones inside the bars represent the average bed occupancy by the corresponding patient types. As expected, the early discharge policy leads to the highest occupancy rates and the reject policy achieves the least occupancy rates. We observe that the one-bed policy leads to almost the same occupancy rates as the reject policy, but achieve a higher cost performance than the reject policy. The fluid-based policy operates in between the early-discharge and reject policies and manages the bed capacity by both rejecting and early discharging type-1 patients. Specifically, as the load increases, the fluid-based policy gives priority to new type-2 patients, which can be observed by the increasing occupancy rate of type-2 patients as the load increases. For example, in the cost case 1 (cost case 2), percentage of beds occupied by type-2 patients increases from 45% to 64% (from 46% to 63%) as the load increases from 0.9 to 1.5.

We also check whether the performance of the policies change as the ratio θ^{CR}, the ratio of the average LOS of critical patients to their overall average LOS, changes. To do this, we consider a setting with $\theta^{CR} \in \{0.50, 0.75\}$ in both one-patient and two-patient ICUs. Table 7.4 presents the corresponding results. We observe that the early-discharge policy is affected the most among all policies. For example, with type-B patients only and under cost case 1, it performs around 48% better than the reject policy when the system load is 0.9 and $\theta^{CR} = 0.50$, but its relative performance drops down under zero when $\theta^{CR} = 0.75$. Note that the early-discharge policy performed better under cost case 1 that it does under cost case 2. For one-patient type ICUs, the performances of one-bed and fluid-based heuristics are relatively stable under cost case 2. This is because they follow a reject-like policy and make early-discharges only when a recurring patient is to be admitted. Under cost case 1 and one-patient type ICU, there is a significant decrease in fluid-based heuristic's performance as θ^{CR} changes from 0.50 to 0.75. However, its performance is still better than the reject policy, with 1.8% improvement for type-B patients only and $\rho = 1.5$. We observe the same pattern for fluid-based heuristic under two-patient type setting. The decrease in performances under cost case 2 is due to the fact that some patients are early discharged upon new first-time patient arrivals.

Table 7.4: Percentage Improvement over Reject Policy, $\theta^{CR} \in \{0.50, 0.75\}$, $\theta^{R} = 1.25$, $1/\gamma = 2$

		Demand							
		$\rho = 0.9$	$\rho = 1$	$\rho = 1.25$	$\rho = 1.5$	$\rho = 0.9$	$\rho = 1$	$\rho = 1.25$	$\rho = 1.5$
		Cost Case 1				Cost Case 2			
θ^{CR}	Policy	One-Patient-Type ICU - Type-A patients only, $B = 51$							
0.50	Fluid NLP	-	-	61.6	55.9	-	-	44.0	37.0
	Early Discharge	60.4	55.4	46.0	37.5	-14.7	-31.3	-66.8	-103.0
	One-Bed Policy	27.8	26.1	22.4	21.7	31.0	31.0	31.0	31.0
	Fluid Based Heuristic	64.7	61.1	55.1	48.3	31.0	31.0	31.0	31.0
0.75	Fluid NLP	-	-	30.2	19.0	-	-	44.0	37.0
	Early Discharge	30.0	9.0	-20.9	-26.0	-101.4	-158.0	-210.0	-185.9
	One-Bed Policy	10.2	10.6	11.0	11.3	29.0	30.0	30.1	31.0
	Fluid Based Heuristic	44.3	30.6	13.5	10.5	29.0	30.0	30.1	30.0
		One-Patient-Type ICU - Type-B patients only, $B = 14$							
0.50	Fluid NLP	-	-	67.0	60.3	-	-	58.0	50.0
	Early Discharge	48.1	43.5	25.7	15.1	-46.3	-58.4	-95.0	-108.3
	One-Bed Policy	18.0	18.1	17.1	16.6	38.0	38.1	39.0	39.0
	Fluid Based Heuristic	54.5	51.2	37.9	29.2	38.0	38.1	39.0	39.0
0.75	Fluid NLP	-	-	40.1	28.4	-	-	58.0	50.0
	Early Discharge	-0.7	-7.3	-24.1	-28.6	-113.0	-121.9	-142.0	-142.3
	One-Bed Policy	13.4	13.6	13.6	13.8	35.6	36.0	36.0	36.8
	Fluid Based Heuristic	19.2	14.9	3.4	1.8	35.7	36.0	36.0	36.8
		Two-Patient-Type ICU - Type-A,B patients, $B = 32$, $\theta^{C} = 2$, $\theta^{AR} = 60$							
0.50	Fluid NLP	-	-	69.5	60.4	-	-	58.0	50.0
	Early Discharge	66.0	40.4	45.7	31.3	2.2	-16.8	-70.1	-105.2
	One-Bed Policy	21.8	20.7	18.0	16.3	36.0	36.0	36.0	36.0
	Fluid Based Heuristic	68.5	64.1	52.8	43.6	39.1	36.0	31.9	30.8
0.75	Fluid NLP	-	-	42.2	30.9	-	-	57.0	49.0
	Early Discharge	23.4	6.0	-21.1	-28.8	-98.7	-137.7	-181.1	-171.8
	One-Bed Policy	12.9	12.9	13.1	13.3	36.0	35.0	35.0	36.0
	Fluid Based Heuristic	42.1	32.6	20.6	16.1	32.0	28.9	25.1	25.8

Chapter 8

CONCLUSION

We develop a realistic conceptual model of a typical ICU that serves multiple patient types. The recovery process of each type is characterized by two stages, critical and semi-critical. We model this system using a feedback mechanism. Hence, a so-called readmission orbit is embedded to the ICU model to keep track of the patients who will seek for ICU care after their discharge. The resulting model is a realistic representation of a typical ICU, which reflects the trade-offs within different types of patients, between first-time and recurring patients, as well as between the reject and early-discharge decisions.

In Chapter 4, we first consider a simple version of the conceptual model and formulate the admission and discharge control problem as a discrete-time MDP. This simple model assumes that there is only one patient type and one recovery stage. We show some monotonicity results on the value function and present sufficient conditions for each type of patient (namely first-time and recurring patients) to be a preferred type. However, the result of this simple MDP formulation lacks being interpretable for the generic model.

In Chapter 5, we develop a discrete-time Markov Decision Process to represent the conceptual model. However, it is intractable due to its high dimensionality. This is not only due to the large state and action spaces, but also mainly due to the feedback mechanism generating readmissions. Therefore, we considered a one-bed ICU model such that the ICU has only one bed and the orbit's state can be at most one. We then proposed an easy-to-implement policy based on the solution the one-bed model. In Chapter 6, we developed a deterministic fluid model and showed some structural properties of the corresponding optimal solution. More formally, we showed that recurring patients are never rejected and the system operates at full capacity under certain conditions. The former result is also consistent with our finding for the simplified version of the model, where patients go through only one recovery stage and there is only one patient type. The solution of the deterministic fluid model can not be directly applied due to the stochastic nature of the real system, therefore we fine-tuned the solution and proposed an easy-to-implement policy.

In Chapter 7, we implement a simulation analysis to evaluate the performances of the one-bed and fluid-based policies. We show that our proposed policies outperform the state-of-the-art policies. More specifically, they are flexible in the sense that they adapt to both early-discharge and reject motivated systems. The simulation results show that the reject policy outperforms the early discharge policy when the demand is higher than the bed capacity. That is, aggressive early discharge decisions put more pressure on the ICU in the long-run than they relieve the capacity in the short-run.

Although our formulation offers a unifying framework that covers main trade-offs in controlling admissions and discharges while providing a view on the effect of these decisions, it has some limitations. First of all, we assume that patients arrive to the ICU with a constant rate. The problem can be analyzed with time varying arrival rates. In our formulation, patients are attached a readmission probability when they are discharged and this probability is constant for each patient type. However, the readmission probability of a patient can be a combination of many things. Therefore, it may be interesting to incorporate a personalized readmission risk at time of discharge in to the model. One may also be interested in redefining the cost function. We used rejection and early-discharge costs only as tuning parameters since estimating them is non-trivial.

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Appendix A

We present the proofs of Proposition 1, Theorem 1 and Theorem 2.

A.1 Proof of Proposition 1

By Puterman [1994], $V(\mathbf{x}, \mathbf{s}) = \lim_{n\to\infty} V_n(\mathbf{x}, \mathbf{s})$. Assume that $V_0(x_1, x_2, s) = 0$ for all $(x_1, x_2, s) \in \mathcal{S}$. We will show that the following holds for all $n \geq 0$:

$$V_n(x_1 + 1, x_2, s) \geq V_n(x_1, x_2, s) \tag{A.1a}$$

$$V_n(x_1, x_2 + 1, s) \geq V_n(x_1, x_2, s) \tag{A.1b}$$

$$V_n(x_1, x_2, s + 1) \geq V_n(x_1, x_2, s) \tag{A.1c}$$

$$V_n(x_1, x_2 + 1, s) \geq V_n(x_1 + 1, x_2, s) \tag{A.1d}$$

Assume that (A.1) holds for $n \geq 0$. For $n = 0$, it is quite trivial. We will now show that (A.1) holds for $n + 1$. We start with (A.1a).

Proof of (A.1a)

We will use sample path argument throughout the proofs of (A.1). All else being equal, let system A and system B start in states $(x_1 + 1, x_2, s)$ and (x_1, x_2, s), respectively, such that system A follows the optimal policy in all transitions and system B imitates the decisions of system A whenever possible. We will show that system B has less cost than system A. More formally, let Π be the set of all admissible policies that satisfy (4.7). Denote the optimal policy $\pi^* \in \Pi$ and arbitrary policy $\pi' \in \Pi$. We let system A follow policy π^* and system B follow policy π'. Then, the following holds:

$$V^{\pi^*}_{n+1}(x_1 + 1, x_2, s) - V^{\pi^*}_{n+1}(x_1, x_2, s) \geq V^{\pi^*}_{n+1}(x_1 + 1, x_2, s) - V^{\pi'}_{n+1}(x_1, x_2, s) \tag{A.2a}$$

Our aim is to show that the following holds for $n+1$:

$$V_{n+1}^{\pi^*}(x_1+1, x_2, s) - V_{n+1}^{\pi'}(x_1, x_2, s) \geq 0 \tag{A.3a}$$

States where system B can imitate system A is trivial, i.e. the difference between the costs of the two systems is preserved. Therefore, we'll focus on the states where system B can not imitate system A, or the state-to-go's of the two systems are different. With a slight abuse of notation, we will drop the superscripts π^* and π' from here on but unless otherwise stated, system A will follow policy $\pi^* \in \Pi$ and system B will follow an arbitrary policy $\pi' \in \Pi$

Case 1: First-time Patient Arrival. System A admits the arriving patient by discharging the additional first-time patient in the ICU early. System B admits the new patient and the following holds by (A.1c):

$$\begin{aligned}
&r_1^e + p_1^e V_n(x_1+1, x_2, s+1) + (1-p_1^e) V_n(x_1+1, x_2, s) - V_n(x_1+1, x_2, s) \\
&r_1^e + p_1^e (V_n(x_1+1, x_2, s+1) - V_n(x_1+1, x_2, s)) \\
&\geq 0
\end{aligned}$$

If the orbit state is at the boundary, i.e., $s+1 = S^{max}$, the early discharged first-time patient leaves the system permanently, and the above inequality still holds since $r_1^e \geq 0$.

Case 2: Recurring Patient Arrival. With probability $\frac{(S^{max}-s)\gamma}{S^{max}\gamma}$, a fictitious recurring patient arrival occurs to both systems and they remain the same. With probability $\frac{s\gamma}{S^{max}\gamma}$, a recurring patient arrives to both systems and system A admits her by discharging the additional first-time patient early and system B admits her.

$$\begin{aligned}
&r_1^e + p_1^e V_n(x_1, x_2+1, s) + (1-p_1^e) V_n(x_1, x_2+1, s-1) - V_n(x_1, x_2+1, s-1) \\
&= r_1^e + p_1^e (V_n(x_1, x_2+1, s) - V_n(x_1, x_2+1, s-1)) \\
&\geq 0
\end{aligned}$$

where the inequality follows by (A.1c).

Case 3: Normal Discharge. Note that $x_1 < B$. With probability $\frac{(B\mu_1 - (x_1+1)\mu_1 - x_2\mu_2)}{B\mu_1}$ a fictitious normal-discharge occurs in both systems and they remain the same. With probability $\frac{\mu_1}{B\mu_1}$, additional first-time patient in system A is discharged normally and system B remains the same, and the following holds by (A.1c):

$$\begin{aligned}
&p_1 V_n(x_1, x_2, s+1) + (1-p_1) V_n(x_1, x_2, s) - V_n(x_1, x_2, s) \\
&= p_1 (V_n(x_1, x_2, s+1) - V_n(x_1, x_2, s)) \\
&\geq 0
\end{aligned}$$

If the orbit is at the boundary state, the normally-discharged additional patient in system A leaves the system with probability 1 and the above inequality still holds. With probability $\frac{(x_1\mu_1+x_2\mu_2)}{B\mu_1}$ a first-time or a recurring patient is discharged normally in both systems and (A.1a) still holds.

Proof of (A.1b)

All else being equal, let system A and system B start in states (x_1, x_2+1, s) and (x_1, x_2, s), respectively, such that system A follows the optimal policy in all transitions and system B imitates the decisions of system A whenever possible. Similar to the previous proof, we will show that the following holds:

$$V_{n+1}^{\pi^*}(x_1, x_2+1, s) - V_{n+1}^{\pi'}(x_1, x_2, s) \geq 0 \tag{A.6a}$$

We will focus on the case where system B can not imitate the decisions of system A.

Case 1: First-time Patient Arrival. System A admits the new patient by discharging the additional recurring patient in the ICU early, system B admits the new patient and the two systems move to the following:

$$\begin{aligned}
&r_2^e + p_2^e V_n(x_1+1, x_2, s+1) + (1-p_1^e) V_n(x_1+1, x_2, s) - V_n(x_1+1, x_2, s) \\
&r_2^e + p_2^e (V_n(x_1+1, x_2, s+1) - V_n(x_1+1, x_2, s)) \\
&\geq 0
\end{aligned}$$

where the inequality follows by (A.1c).

Case 2: Recurring Patient Arrival. With probability $\frac{(S^{max}-s)\gamma}{S^{max}\gamma}$, a fictitious recurring patient arrival occurs to both systems and they remain the same.

With probability $\frac{s\gamma}{s\gamma}$, a recurring patient arrives to both systems and system A admits her by discharging the additional first-time patient early and system B admits her.

$$\begin{aligned}
&r_2^e + p_2^e V_n\left(x_1, x_2+1, s\right) + \left(1-p_2^e\right) V_n\left(x_1, x_2+1, s-1\right) - V_n\left(x_1, x_2+1, s-1\right) \\
&= r_2^e + p_2^e\left(V_n\left(x_1, x_2+1, s\right) - V_n\left(x_1, x_2+1, s-1\right)\right) \\
&\geq 0
\end{aligned}$$

where the inequality follows by (A.1c).

Case 3: Normal Discharge. With probability $\frac{(B\mu_1 - x_1\mu_1 - (x_2+1)\mu_2)}{B\mu_1}$ fictitious normal discharge occurs in both systems and they remain the same. With probability $\frac{\mu_2}{B\mu_1}$, additional first-time patient in system A is discharged normally and system B remains the same, and the following holds by (A.1c):

$$\begin{aligned}
&p_2 V_n\left(x_1, x_2, s+1\right) + \left(1-p_2\right) V_n\left(x_1, x_2, s\right) - V_n\left(x_1, x_2, s\right) \\
&= p_2\left(V_n\left(x_1, x_2, s+1\right) - V_n\left(x_1, x_2, s\right)\right) \\
&\geq 0
\end{aligned}$$

If the orbit is at the boundary state, the normally-discharged additional patient in system A leaves the system with probability 1 and the above inequality still holds. With probability $\frac{(x_1\mu_1 + x_2\mu_2)}{B\mu_1}$ a first-time or a recurring patient is normally discharged in both systems and (A.1a) still holds.

Proof of (A.1c)

Let system A start in state $(x_1, x_2, s+1)$ ans system B in state (x_1, x_2, s). We will show that the following holds:

$$V_{n+1}^{\pi^*}\left(x_1, x_2, s+1\right) - V_{n+1}^{\pi'}\left(x_1, x_2, s\right) \geq 0 \tag{A.9a}$$

Consider the following cases where system B con not imitate system A.

Case 1: First-time Patient Arrival. Let orbit state $s = S^{max}$. System A admits the new patient by discharging the a type-k patient early and the early-discharged patient leaves the system permanently. System B admits the new patient by early discharging the same type-k patient and the two systems move to the following:

$$r_k^e + V_n(\mathbf{x} - e_k, s+1) - p_k^e V_n(\mathbf{x} - e_k, s+1) - (1 - p_k^e) V_n(\mathbf{x} - e_k, s)$$
$$r_k^e + (1 - p_k^e)(V_n(\mathbf{x} - e_k, s+1) - V_n(\mathbf{x} - e_k, s))$$
$$\geq 0$$

where e_k is the unit vector whose kth entry is equal to 1 and l lth entry ($l \neq k$) is equal to zero and the inequality follows by (A.1c).

CASE 2: RECURRING PATIENT ARRIVAL. With probability $\frac{(S^{max}-s-1)\gamma}{S^{max}\gamma}$, there is not a recurring patient arrival. With probability $\frac{\gamma}{S^{max}\gamma}$, the additional patient in the orbit of system A arrives. System B remains the same. If $x_1 + x_2 < B$ and system A admits her, system A still has more cost than system B by (A.1b). If system A rejects her, system A still incurs a cost of r_2 and still has more cost than system B by (A.1b). If system A admits the arriving recurring by discharging a first-time patient early, we have the following:

$$r_1^e + p_1^e V_n(x_1 - 1, x_2 + 1, s+1) + (1 - p_1^e) V_n(x_1 - 1, x_2 + 1, s) - V_n(x_1, x_2, s)$$
$$= r_1^e + p_1^e (V_n(x_1 - 1, x_2 + 1, s+1) - V_n(x_1 - 1, x_2 + 1, s))$$
$$+ V_n(x_1 - 1, x_2 + 1, s) - V_n(x_1, x_2, s)$$
$$\geq 0$$

where the inequality holds by (A.1c) and (A.1d). If system A admits the arriving recurring patient by discharging a recurring patient early, we have:

$$r_2^e + p_2^e V_n(x_1, x_2, s+1) + (1 - p_2^e) V_n(x_1, x_2, s) - V_n(x_1, x_2, s)$$
$$= r_2^e + p_2^e (V_n(x_1, x_2, s+1) - V_n(x_1, x_2, s))$$
$$\geq 0$$

which holds by A.1c.

CASE 3: NORMAL DISCHARGE. The only difference between the two systems occurs when the orbit state in system A is equal to S^{max}. More precisely, a type-k patient is normally discharged from both systems with probability $\frac{x_k \mu_k}{B\mu_1}$, and we have the following:

$$V_n(\mathbf{x} - e_k, s+1) - p_k V_n(\mathbf{x} - e_k, s+1) - (1 - p_k) V_n(\mathbf{x}_k, s)$$

$$
\begin{aligned}
&= (1 - p_k)V_n(\mathbf{x} - e_k, s+1) - V_n(\mathbf{x} - e_k, s) \\
&\geq 0
\end{aligned}
$$

where e_k is the unit vector with 1 and at entry k zero otherwise. The inequality is satisfied by (A.1c).

Proof of (A.1d)

. System A start at state (x_1, x_2+1, s) and system B starts at state (x_1+1, x_2, s). Consider the following cases where system B cannot imitate system A.

CASE 1: FIRST-TIME PATIENT ARRIVAL. System A admits the new first-time patient by discharging the additional recurring patient early. System B admits her by discharging the additional first-time patient patient early:

$$
\begin{aligned}
& r_2^e + p_2^e V_n(x_1+1, x_2, s+1) + (1-p_2^e)V_n(x_1+1, x_2, s) \\
& - r_1^e - p_1^e V_n(x_1+1, x_2, s+1) - (1-p_1^e)V_n(x_1+1, x_2, s) \\
& = (r_2^e - r_1^e) + (p_2^e - p_1^e)(V_n(x_1+1, x_2, s+1) - V_n(x_1+1, x_2, s)) \\
& \geq 0
\end{aligned}
$$

which holds by Condition 1 and induction hypothesis A.1c.

CASE 2: RECURRING PATIENT ARRIVAL. System A admits the arriving recurring patient by discharging the additional recurring patient early. System B admits her by discharging the additional first-time patient patient early:

$$
\begin{aligned}
& r_2^e + p_2^e V_n(x_1, x_2+1, s) + (1-p_2^e)V_n(x_1, x_2+1, s-1) \\
& - r_1^e - p_1^e V_n(x_1, x_2+1, s) - (1-p_1^e)V_n(x_1, x_2+1, s-1) \\
& = (r_2^e - r_1^e) + (p_2^e - p_1^e)(V_n(x_1, x_2+1, s) - V_n(x_1, x_2+1, s-1)) \\
& \geq 0
\end{aligned}
$$

which holds by Condition 1 and induction hypothesis A.1c.

CASE 3: NORMAL DISCHARGE. Note that $x_1 < B$. With probability $\frac{(B-(x_1+1)\mu_1 - x_2\mu_2)}{B\mu_1}$ a fictitious normal-discharge occurs in both systems and they remain the same. With probability $\frac{x_1\mu_1}{B\mu_1}$, first-time patients in both systems are discharged normally and with probability $\frac{x_2\mu_2}{B\mu_1}$ recurring patients in both systems are discharged normally. In both cases, system A and system B move together and the difference between them is preserved. Then remains the following cases on which we focus: with probability $\frac{\mu_2}{B\mu_1}$, additional patients in both systems are discharged normally and with probability $\frac{(\mu_1-\mu_2)}{B\mu_1}$ additional patient in system B is discharged normally and system A remains the same. We have the the following:

$$
\begin{aligned}
&\mu_2\Big(p_2V_n\left(x_1,x_2,s+1\right)-\left(1-p_2\right)V_n\left(x_1,x_2,s\right)\\
&\qquad\qquad\qquad\qquad -p_1V_n\left(x_1,x_2,s+1\right)-\left(1-p_1\right)V_n\left(x_1,x_2,s\right)\Big)\\
&+\left(\mu_1-\mu_2\right)\Big(V_n\left(x_1,x_2+1,s\right)-p_1V_n\left(x_1,x_2,s+1\right)-\left(1-p_1\right)V_n\left(x_1,x_2,s\right)\Big)\\
&=\mu_2\left(p_2-p_1\right)\left(V_n\left(x_1,x_2,s+1\right)-V_n\left(x_1,x_2,s\right)\right)\\
&+\left(\mu_1-\mu_2\right)\left(V_n\left(x_1,x_2+1,s\right)-V_n\left(x_1,x_2,s\right)\right)\\
&-\left(\mu_1-\mu_2\right)p_1\left(V_n\left(x_1,x_2,s+1\right)-V_n\left(x_1,x_2,s\right)\right)\\
&=\left(\mu_2p_2-\mu_1p_1\right)\left(V_n\left(x_1,x_2,s+1\right)-V_n\left(x_1,x_2,s\right)\right)\\
&+\left(\mu_1-\mu_2\right)\left(V_n\left(x_1,x_2+1,s\right)-V_n\left(x_1,x_2,s\right)\right)\\
&\geq 0
\end{aligned}
$$

where the inequality holds by A.1c and Condition 1.

A.2 Proof of Theorem 1

For the first-time patients to be a preferred type, we need to show that $V(x_1+1,x_2,s)-V(x_1,x_2,s)\leq r_1$ for all $(\mathbf{x},s)\in\mathcal{S}$. Proving this also requires us to show that the costs of having one more recurring patient in the ICU, one more patient in the orbit and changing a first-time patient to a recurring patient must be bounded with a finite cost.

Assume that the following hold for $n\geq 0$

$$V_n(x_1+1,x_2,s)-V_n(x_1,x_2,s)\leq \overline{d}_1 \tag{A.10a}$$

$$V_n(x_1,x_2+1,s)-V_n(x_1,x_2,s)\leq \overline{d}_2 \tag{A.10b}$$

$$V_n(x_1,x_2,s+1)-V_n(x_1,x_2,s)\leq \overline{d}_s \tag{A.10c}$$

$$V_n(x_1,x_2+1,s)-V_n(x_1+1,x_2,s)\leq \overline{d}_{21} \tag{A.10d}$$

where $\overline{d}_1$, $\overline{d}_2$, $\overline{d}_s$ and $\overline{d}_{21}$ are non-negative upper bounds on $D(10)(\mathbf{x}, s)$, $D(20)(\mathbf{x}, s)$, $D(s0)(\mathbf{x}, s)$ and $D(21)(\mathbf{x}, s)$, respectively (Please see Subsection 4.3.1 for definition of $D(ij)(\mathbf{x}, s)$). We will show that equations (A.10a), (A.10b), (A.10c) and (A.10d) hold for $n + 1$. Without loss of generality, we may set $\overline{d}_1 = r_1$.

Proof of Theorem 1.i

Let the system be in state (x_1, x_2, s) with $x_1 + x_2 < B$. Consider two systems, system A and B, such that system A in state $(x_1 + 1, x_2, s)$, and system B is in state (x_1, x_2, s). Let also couple these two systems such that from here on system B follows the optimal admission and discharge control policy (denoted by $\pi^* \in \Pi$, where Π is the set all admissible policies) and system A imitates the decisions of system B whenever possible to couple with him (an arbitrary policy denoted by $\pi' \in \Pi$). We will show that system A performs better than system B under some conditions.

Because of the boundary effect, we will evaluate two systems based on whether the system state is on the boundaries or not. Consider the following four cases:

CASE-1. $x_1 + 1 + x_2 < B$ AND $s < S^{max}$: System A can imitate system B for all actions of system B. Moreover, the additional patient in system A is discharged normally with probability μ_1. Two systems move as in the following:

$$
\begin{aligned}
&V_{n+1}^{\pi'}(x_1 + 1, x_2, s) - V_{n+1}^{\pi^*}(x_1, x_2, s) \\
&\leq \lambda r_1 + s\gamma r_1 + (x_1\mu_1 + x_2\mu_2)r_1 \\
&+ \mu_1(p_1 V_n(x_1, x_2, s + 1) + (1 - p_1)V_n(x_1, x_2, s) - V(x_1, x_2, s)) \\
&+ ((S^{max} - s)\gamma + B\mu_1 - (x_1 + 1)\mu_1 - x_2\mu_2)\, r_1 \\
&\leq \lambda r_1 + s\gamma r_1 + \mu_1 p_1 \overline{d}_s + (x_1\mu_1 + x_2\mu_2)r_1 \\
&+ ((S^{max} - s)\gamma + B\mu_1 - x_1\mu_1 - x_2\mu_2)\, r_1 - \mu_1 r_1 \\
&= (\lambda + S^{max}\gamma + B\mu_1)r_1 + \mu_1 p_1 \overline{d}_s - \mu_1 r_1 \\
&\leq r_1
\end{aligned}
\tag{A.11}
$$

which holds when $\overline{d}_s \leq \frac{r_1}{p_1}$. The first and second inequalities follow since system A can imitate all actions of system B and system B has still more cost (i.e. r_1) than system A, and the last inequality follows by $\overline{d}_s \leq \frac{r_1}{p_1}$ and $\lambda + S^{max}\gamma + B\mu_1 = 1$. Note that we use uniformization $\lambda + S^{max}\gamma + B\mu_1 = 1$ by letting $\beta \to 0$, which we will also use in the remaining parts of the proof.

CASE-2. $x_1 + 1 + x_2 < B$ AND $s = S^{max}$: The only difference from the previous case is that the normally-discharged additional patient in system A leaves the system with probability 1 (as well as other discharged patients), since she can not join the orbit. Two systems move as in the following:

$$\begin{aligned}
&V^{\pi'}_{n+1}(x_1+1, x_2, s) - V^{\pi^*}_{n+1}(x_1, x_2, s) \\
&\leq \lambda r_1 + s\gamma r_1 + (x_1\mu_1 + x_2\mu_2)r_1 + \mu_1(V_n(x_1, x_2, s) - V_n(x_1, x_2, s)) \qquad \text{(A.12)} \\
&+ ((S^{max} - s)\gamma + B\mu_1 - (x_1+1)\mu_1 - x_2\mu_2)\, r_1 \\
&\leq (\lambda + S^{max}\gamma + B\mu_1)r_1 - \mu_1 r_1 \\
&\leq r_1
\end{aligned}$$

which holds without any additional condition.

CASE-3. $x_1 + 1 + x_2 = B$ AND $s < S^{max}$: Since the ICU is full in system A, system A can not imitate system B when the latter admits new patients. Upon a first-time patient arrival, if system B admits the new patient, then system A rejects her and incurs an additional cost r_1 and two systems couple. For a recurring patient arrival, if system B rejects her, system A also rejects her and two systems preserve their difference. If system B admits her, let system A also admit her by discharging the additional first-time patient early. Together with the normal discharge of the additional first-time patient in system A, the two systems move to the following

$$\begin{aligned}
&V^{\pi'}_{n+1}(x_1+1, x_2, s) - V^{\pi^*}_{n+1}(x_1, x_2, s) \\
&\leq \mu_1\, (p_1 V_n(x_1, x_2, s+1) + (1-p_1)V_n(x_1, x_2, s) - V_n(x_1, x_2, s)) \\
&+ s\gamma\, (r_1^e + p_1^e V_n(x_1, x_2+1, s) + (1-p_1^e)V_n(x_1, x_2+1, s-1) - V_n(x_1, x_2+1, s-1)) \\
&+ x_1\mu_1 r_1 + x_2\mu_2 r_1 + ((S^{max} - s)\gamma + B\mu_1 - (x_1+1)\mu_1 - x_2\mu_2)\, r_1 \qquad \text{(A.13)} \\
&\leq \mu_1 p_1 \overline{d}_s + s\gamma \left(r_1^e + p_1^e \overline{d}_s\right) + (x_1\mu_1 + x_2\mu_2)r_1 \\
&+ ((S^{max} - s)\gamma + B\mu_1 - x_1\mu_1 - x_2\mu_2)\, r_1 - \mu_1 r_1 \\
&= (\lambda + S^{max}\gamma + B\mu_1)r_1 - \lambda r_1 - s\gamma r_1 - \mu_1 r_1 + \mu_1 p_1 \overline{d}_s + s\gamma(r_1^e + p_1^e \overline{d}_s) \\
&\leq r_1
\end{aligned}$$

which holds only if $\overline{d}_s \leq \frac{(\mu_1+\lambda)r_1+s\gamma(r_1-r_1^e)}{s\gamma p_1^e+\mu_1 p_1}$.

Case-4. $x_1+1+x_2 = B$ and $s = S^{max}$: Similar to the previous case, system A admits an arriving recurring patient by discharging the additional first-time patient early if system B admits her. Further, the normally-discharged additional first-time patient in system A (which happens with probability μ_1) leaves system permanently. That is:

$$\begin{aligned}
&V_{n+1}^{\pi'}(x_1+1,x_2,s) - V_{n+1}^{\pi^*}(x_1,x_2,s)\\
&\leq \mu_1\left(V_n(x_1,x_2,s) - V_n(x_1,x_2,s)\right) + (x_1\mu_1 + x_2\mu_2)r_1\\
&+ s\gamma\left(r_1^e + p_1^e V_n(x_1,x_2+1,s) + (1-p_1^e)V_n(x_1,x_2+1,s-1) - V_n(x_1,x_2+1,s-1)\right)\\
&+ \left((S^{max}-s)\gamma + B\mu_1 - (x_1+1)\mu_1 - x_2\mu_2\right) r_1 \qquad \text{(A.14)}\\
&\leq \left(\lambda + S^{max}\gamma + B\mu_1\right) r_1 - \lambda r_1 - \mu_1 r_1 - s\gamma r_1 + s\gamma\left(r_1^e + p_1^e \overline{d}_s\right)\\
&\leq r_1
\end{aligned}$$

which holds when $\overline{d}_s \leq \frac{(\mu_1+\lambda)r_1+S^{max}\gamma(r_1-r_1^e)}{S^{max}\gamma p_1^e}$ since $s = S^{max}$ by case assumption. Consequently, we have the following from A.11, A.13 and A.14 to prove that first-time patients are preferred type:

$$\overline{d}_s \leq \min\left\{\frac{r_1}{p_1},\ \frac{(\mu_1+\lambda)\,r_1 + s\gamma(r_1-r_1^e)}{s\gamma p_1^e + \mu_1 p_1},\ \frac{(\mu_1+\lambda)\,r_1 + S^{max}\gamma(r_1-r_1^e)}{S^{max}\gamma p_1^e}\right\} \qquad \text{(A.15)}$$

Note that

$$\frac{1}{S^{max}\gamma p_1^e + \mu_1 p_1^e} \leq \frac{1}{S^{max}\gamma p_1^e + \mu_1 p_1} \leq \frac{1}{s\gamma p_1 + \mu_1 p_1} \leq \frac{1}{p_1}$$

where $\mu_1, s\gamma, \lambda$ etc. are probabilities by $\lambda + S^{max}\gamma + B\mu_1 = 1$. Note also that

$$(\mu_1+\lambda)\,r_1 + \gamma(r_1-r_1^e) \leq (\mu_1+\lambda)\,r_1 + s\gamma(r_1-r_1^e) \leq (\mu_1+\lambda)\,r_1 + S^{max}\gamma(r_1-r_1^e) \leq r_1$$

Therefore, we let

$$\begin{aligned}
\overline{d}_s &\leq \frac{(\mu_1+\lambda+\gamma)\,r_1 - \gamma r_1^e}{p_1} \qquad \text{(A.16)}\\
&\leq \min\left\{\frac{r_1}{p_1},\ \frac{(\mu_1+\lambda)\,r_1 + s\gamma(r_1-r_1^e)}{s\gamma p_1^e + \mu_1 p_1},\ \frac{(\mu_1+\lambda)\,r_1 + S^{max}\gamma(r_1-r_1^e)}{S^{max}\gamma p_1^e}\right\}
\end{aligned}$$

A.2.1 Proof of Theorem 1.ii

In this part, we will prove that there exists a non-negative upper bound $\overline{d}_s$ for $D(s,0)(\mathbf{x},s)$ for all $(\mathbf{x},s) \in \mathcal{S}$. Let system A be in state $(x_1,x_2,s+1)$ and system

B in state (x_1, x_2, s). We will show that under some conditions the difference in the total expected discounted cost between system A and B is less than or equal to $\overline{d}_s$, whose closed form we will derive once we prove all of the arguments in Theorem 1. We couple system A and system B such that system B follows the optimal strategy, and system A imitates system B whenever possible to couple with it. We consider the following cases:

CASE-1. $x_1 + x_2 < B$ AND $s + 1 < S^{max}$: Additional patient in the orbit of system A arrives and we let it admit her. For other operators, system A can imitate system B and the difference in both systems is preserved.

$$\begin{aligned}
& V_{n+1}^{\pi'}(x_1, x_2, s+1) - V_{n+1}^{\pi^*}(x_1, x_2, s) \\
& \leq \lambda \overline{d}_s + s\gamma \overline{d}_s + \gamma \left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2+1, s)\right) + (x_1\mu_1 + x_2\mu_2)\, \overline{d}_s \qquad \text{(A.17)} \\
& + \left((S^{max} - s - 1)\,\gamma + B\mu_1 - x_1\mu_1 - x_2\mu_2\right) \overline{d}_s \\
& \leq (\lambda + S^{max}\gamma + B\mu_1)\overline{d}_s - \gamma\overline{d}_s + \gamma\overline{d}_2 \\
& \leq \overline{d}_s
\end{aligned}$$

which is true only if $\overline{d}_2 \leq \overline{d}_s$.

CASE-2. $x_1 + x_2 < B$ AND $s + 1 = S^{max}$: Normally-discharged patients in system A leave the system permanently. That is:

$$\begin{aligned}
& V_{n+1}^{\pi'}(x_1, x_2, s+1) - V_{n+1}^{\pi^*}(x_1, x_2, s) \\
& \leq \lambda \overline{d}_s + s\gamma \overline{d}_s + \gamma \left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2+1, s)\right) \\
& + x_1\mu_1 \left(V_n(x_1-1, x_2, s+1) - p_1 V_n(x_1-1, x_2, s+1) - (1-p_1)V_n(x_1-1, x_2, s)\right) \\
& + x_2\mu_2 \left(V_n(x_1, x_2-1, s+1) - p_2 V_n(x_1, x_2-1, s+1) - (1-p_2)V_n(x_1, x_2-1, s)\right) \\
& + \left((S^{max} - s - 1)\,\gamma + B\mu_1 - x_1\mu_1 - x_2\mu_2\right) \overline{d}_s \\
& \leq \lambda \overline{d}_s + s\gamma \overline{d}_s + \gamma \overline{d}_2 + x_1\mu_1(1-p_1)\overline{d}_s + x_2\mu_2(1-p_2)\overline{d}_s \qquad \text{(A.18)} \\
& + \left((S^{max} - s)\gamma + B\mu_1 - x_1\mu_1 - x_2\mu_2\right) \overline{d}_s - \gamma\overline{d}_s \\
& \leq (\lambda + S^{max}\gamma + B\mu_1)\overline{d}_s + \gamma\overline{d}_2 - \gamma\overline{d}_s \\
& \leq \overline{d}_s
\end{aligned}$$

which holds when $\overline{d}_2 \leq \overline{d}_s$.

CASE-3. $x_1 + x_2 = B$ AND $s + 1 < S^{max}$: When the additional patient in the orbit of system A arrives, she may be rejected or admitted by discharging one of the existing patients currently staying in the ICU early. In other situations, the two systems move as in the previous cases. There may be only first-time, only recurring or both types of patients in the ICU upon the arrival of additional recurring patient. Considering “subcases" where there are only one type of patient in the ICU will be sufficient.

SUBCASE-3.A. $x_2 = B$: There are only recurring patients in the ICU:

$$\begin{aligned}
&V_{n+1}^{\pi'}(x_1, x_2, s+1) - V_{n+1}^{\pi^*}(x_1, x_2, s) \\
&\leq \lambda \overline{d}_s + (x_1\mu_1 + x_2\mu_2)\,\overline{d}_s + s\gamma\overline{d}_s \\
&+ \gamma\left(r_2^e + p_2^e V_n(x_1, x_2, s+1) + (1-p_2^e)V_n(x_1, x_2, s) - V_n(x_1, x_2, s)\right) \\
&+ \left((S^{max} - s - 1)\gamma + B\mu_1 - x_1\mu_1 - x_2\mu_2\right)\overline{d}_s \\
&\leq (\lambda + S^{max}\gamma + B\mu_1)\,\overline{d}_s - \gamma\overline{d}_s + \gamma\left(r_2^e + p_2^e\overline{d}_s\right) \\
&\leq \overline{d}_s
\end{aligned} \tag{A.19}$$

which holds when $r_2^e \leq \overline{d}_s(1 - p_2^e)$.

SUBCASE-3.B. $x_1 = B$: System A admits the arriving additional patient in the orbit by early discharging a first-time patient

$$\begin{aligned}
&V_{n+1}^{\pi'}(x_1, x_2, s+1) - V_{n+1}^{\pi^*}(x_1, x_2, s) \\
&\leq (\lambda + S^{max} + B\mu_1)\overline{d}_s - \gamma\overline{d}_s \\
&+ \gamma\left(r_1^e + p_1^e V_n(x_1 - 1, x_2 + 1, s+1) + (1-p_1^e)V_n(x_1 - 1, x_2 + 1, s)\right) - \gamma V_n(x_1, x_2, s) \\
&\leq (\lambda + S^{max} + B\mu_1)\overline{d}_s - \gamma\overline{d}_s + \gamma\left(r_1^e + p_1^e\overline{d}_s + \overline{d}_{21}\right) \\
&\leq (\lambda + S^{max} + B\mu_1)\overline{d}_s + \gamma\left(r_1^e + p_1^e\overline{d}_s + \overline{d}_{21}\right) - \gamma\overline{d}_s \\
&\leq \overline{d}_s
\end{aligned} \tag{A.20}$$

which holds when $\overline{d}_{21} \leq \overline{d}_s(1 - p_1^e) - r_1^e$.

CASE-4. $x_1 + x_2 = B$ AND $s + 1 = S^{max}$: If system B admits a new patient by discharging an existing patient early, system A imitates system B. However, early-discharged patients in system A leave the system permanently. Additionally, normally-discharged patients in system A leave the system permanently. We consider the following two “subcases":

Subcase-4.a. $x_2 = B$: There are only recurring patients in the ICU. That is:

$$\begin{aligned}
&V^{\pi'}_{n+1}(x_1, x_2, s+1) - V^{\pi^*}_{n+1}(x_1, x_2, s) \\
&\leq \lambda\left(1-p^e_2\right)\left(V_n(x_1+1, x_2-1, s+1) - V_n(x_1+1, x_2-1, s)\right) + s\gamma\overline{d}_s \\
&+ \gamma\left(r^e_2 + p^e_2 V_n(x_1, x_2, s+1) + (1-p^e_2)V_n(x_1, x_2, s) - V_n(x_1, x_2, s)\right) \qquad \text{(A.21)} \\
&+ x_2\mu_2\left(V(x_1, x_2-1, s+1) - p_2 V(x_1, x_2-1, s+1) - (1-p_2)V(x_1, x_2-1, s)\right) \\
&+ \left((S^{max} - s - 1)\gamma + B\mu_1 - x_2\mu_2\right)\overline{d}_s \\
&\leq \left(\lambda + S^{max}\gamma + B\mu_1\right)\overline{d}_s - \lambda p^e_2\overline{d}_s + \gamma\left(r^e_2 + p^e_2\overline{d}_s\right) - \gamma\overline{d}_s \\
&\leq \overline{d}_s
\end{aligned}$$

which holds when $r^e_2 \leq \overline{d}_s\left(1 - p^e_2(1-\frac{\lambda}{\gamma})\right)$.

Subcase-4.b. $x_1 = B$: There are only first-time patients in the ICU. Following the same procedures in Equation A.21, we have the following:

$$\begin{aligned}
&V^{\pi'}_{n+1}(x_1, x_2, s+1) - V^{\pi^*}_{n+1}(x_1, x_2, s) \\
&\leq \lambda\left(1-p^e_1\right)\overline{d}_s + s\gamma\overline{d}_s + x_1\mu_1(1-p_1)\overline{d}_s + \left((S^{max} - s - 1)\gamma + B\mu_1 - x_1\mu_1\right)\overline{d}_s \\
&+ \gamma\left(r^e_1 + p^e_1 V_n(x_1-1, x_2+1, s+1) + (1-p^e_1)V_n(x_1-1, x_2+1, s) - V_n(x_1, x_2, s)\right) \\
&\leq \lambda\left(1-p^e_1\right)\overline{d}_s + s\gamma\overline{d}_s + x_1\mu_1\overline{d}_s + \left((S^{max} - s)\gamma + B\mu_1 - x_1\mu_1\right)\overline{d}_s - \gamma\overline{d}_s \\
&+ \gamma\left(r^e_1 + p^e_1\overline{d}_s + \overline{d}_{21}\right) \\
&= \left(\lambda + S^{max}\gamma + B\mu_1\right)\overline{d}_s - \lambda p^e_1\overline{d}_s - \gamma\overline{d}_s + \gamma\left(r^e_1 + p^e_1\overline{d}_s + \overline{d}_{21}\right) \qquad \text{(A.22)} \\
&\leq \overline{d}_s \qquad \text{(A.23)}
\end{aligned}$$

which holds when $\overline{d}_{21} \leq \overline{d}_s\left(1 - p^e_1(1-\dfrac{\lambda}{\gamma})\right) - r^e_1$.

When there are both types of patients in the ICU of of system B, system A can imitate all early discharge decisions of system B. Consequently, we have the following to further prove:

$$\overline{d}_s \geq \max\left\{\overline{d}_2,\ \frac{r^e_2}{(1-p^e_2)},\ \frac{\overline{d}_{21} + r^e_1}{1-p^e_1},\ \frac{r^e_2}{(\frac{\lambda}{\gamma}p^e_2 + 1 - p^e_2)},\ \frac{\overline{d}_{21} + r^e_1}{\frac{\lambda}{\gamma}p^e_1 + 1 - p^e_1}\right\} \qquad \text{(A.24)}$$

Without loss of generality, we let $\overline{d}_2 = \frac{\overline{d}_{21}+r^e_1}{1-p^e_1} = \frac{r^e_2}{(1-p^e_2)}$. Then,

$$\frac{r^e_2}{(1-p^e_2)} = \max\left\{\overline{d}_2,\ \frac{r^e_2}{(1-p^e_2)},\ \frac{\overline{d}_{21} + r^e_1}{1-p^e_1},\ \frac{r^e_2}{(\frac{\lambda}{\gamma}p^e_2 + 1 - p^e_2)},\ \frac{\overline{d}_{21} + r^e_1}{\frac{\lambda}{\gamma}p^e_1 + 1 - p^e_1}\right\} \qquad \text{(A.25)}$$

Consequently, we have the following:

$$\begin{aligned}\overline{d}_s &\geq \frac{r_2^e}{1-p_2^e} \\ \overline{d}_2 &= \frac{r_2^e}{1-p_2^e} \\ \overline{d}_{21} &= \frac{r_2^e\,(1-p_1^e)}{(1-p_2^e)} - r_1^e\end{aligned} \tag{A.26}$$

A.2.2 Proof of Theorem 1.iii

We will prove that there exists a non-negative upper bound $\overline{d}_2$ for $D(2,0)(\mathbf{x},s)$ for all $(\mathbf{x},s) \in \mathcal{S}$. Let system A be in state (x_1, x_2+1, s) and system B in state (x_1, x_2, s). Under some conditions, we will show that the difference in the total expected discounted cost between system A and B is less than $\overline{d}_2$. Note that we let $\overline{d}_2 = \frac{r_2^e}{1-p_2^e}$ in the previous section. We couple system A and system B such that system B follows the optimal strategy, and system A imitates system B whenever possible to couple with it. We consider the following cases:

CASE-1. $x_1 + x_2 + 1 < B$ AND $s < S^{max}$: System A can imitate all actions of system B. Further, there occurs a normal-discharge event of the additional recurring patient in system A with probability μ_2. That is:

$$\begin{aligned}&V_{n+1}^{\pi'}(x_1, x_2+1, s) - V_{n+1}^{\pi^*}(x_1, x_2, s) \\ &\leq \lambda\overline{d}_2 + s\gamma\overline{d}_2 + \mu_2\left(p_2 V_n(x_1, x_2, s+1) + (1-p_2)V_n(x_1, x_2, s) - V_n(x_1, x_2, s)\right) \\ &+ (x_1\mu_1 + x_2\mu_2)\,\overline{d}_2 \\ &+ \left((S^{max} - s)\gamma + B\mu_1 - x_1\mu_1 - (x_2+1)\mu_2\right)\left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2, s)\right) \\ &\leq \left(\lambda + S^{max}\gamma + B\mu_1\right)\overline{d}_2 + \mu_2 p_2\overline{d}_s - \mu_2\overline{d}_2 \\ &\leq \overline{d}_2\end{aligned} \tag{A.27}$$

which is true when $\overline{d}_s \leq \dfrac{r_2^e}{p_2\,(1-p_2^e)}$.

CASE-2. $x_1 + x_2 + 1 < B$ AND $s = S^{max}$: Discharged patients leave the system permanently in both systems.

$$\begin{aligned}&V_{n+1}^{\pi'}(x_1, x_2+1, s) - V_{n+1}^{\pi^*}(x_1, x_2, s) \\ &\leq \lambda\overline{d}_2 + s\gamma\overline{d}_2 + \mu_2\left(V_n(x_1, x_2, s) - V_n(x_1, x_2, s)\right) + \left(x_1\mu_1 \ + x_2\mu_2\overline{d}_2\right)\end{aligned} \tag{A.28}$$

$$+\left((S^{max}-s)\gamma+B\mu_1-x_1\mu_1-(x_2+1)\mu_2\right)\left(V_n(x_1,x_2+1,s)-V_n(x_1,x_2,s)\right)$$
$$\leq\left(\lambda+S^{max}\gamma+B\mu_1\right)\overline{d}_2-\mu_2\overline{d}_2$$
$$\leq\overline{d}_2$$

Case-3. $x_1+x_2+1=B$ and $s<S^{max}$: Actions of system B may change depending on the arriving patient's type. If system B admits an arriving patient, system A can only reject her or admit her with an early-discharge. Consider the following cases:

Subcase-3.a. System B admits both first-time and recurring patients. Let system A admit an arriving patient by discharging the additional recurring patient early. That is:

$$\begin{aligned}
&V_{n+1}^{\pi'}(x_1,x_2+1,s)-V_{n+1}^{\pi^*}(x_1,x_2,s) && \text{(A.29)}\\
&\leq\lambda\left(r_2^e+p_2^eV_n(x_1+1,x_2,s+1)+(1-p_2^e)V_n(x_1+1,x_2,s)-V_n(x_1+1,x_2,s)\right)\\
&+s\gamma\left(r_2^e+p_2^eV_n(x_1,x_2+1,s)+(1-p_2^e)V_n(x_1,x_2+1,s-1)-V_n(x_1,x_2+1,s_1-1)\right)\\
&+\mu_2\left(p_2^eV_n(x_1,x_2,s+1)+(1-p_2^e)V_n(x_1,x_2,s)-V_n(x_1,x_2,s)\right)+(x_1\mu_1+x_2\mu_2)\overline{d}_2\\
&+\left((S^{max}-s)\gamma+B\mu_1-x_1\mu_1-(x_2+1)\mu_2\right)\left(V_n(x_1,x_2+1,s)-V_n(x_1,x_2,s)\right)\\
&\leq\lambda\left(r_2^e+p_2^e\overline{d}_s\right)+s\gamma\left(r_2^e+p_2^e\overline{d}_s\right)+\mu_2p_2\overline{d}_s-\mu_2\overline{d}_2+\left((S^{max}-s)\gamma+B\mu_1\right)\overline{d}_2\\
&=\left(\lambda+S^{max}\gamma+B\mu_1\right)\overline{d}_2-\left(\lambda+s\gamma+\mu_2\right)\overline{d}_2+\left(\lambda p_2^e+s\gamma p_2^e+\mu_2p_2\right)\overline{d}_s+\left(\lambda+s\gamma\right)r_2^e\\
&\leq\overline{d}_2
\end{aligned}$$

which holds when

$$\overline{d}_s\leq\frac{r_2^e}{1-p_2^e}\left(\frac{\lambda+s\gamma+\mu_2-(\lambda+s\gamma)(1-p_2^e)}{\lambda p_2^e+s\gamma p_2^e+\mu_2p_2}\right) \qquad \text{(A.30)}$$

Note that $\left(\frac{\lambda+s\gamma+\mu_2-(\lambda+s\gamma)(1-p_2^e)}{\lambda p_2^e+s\gamma p_2^e+\mu_2p_2}\right)=\left(\frac{(\lambda+s\gamma)p_2^e+\mu_2}{(\lambda+s\gamma)p_2^e+\mu_2p_2}\right)\geq 1.$

Subcase-3.b. System B admits a first-time patient but rejects a recurring patient. Let system A admit the first-time patient by discharging the additional recurring patient early and reject the recurring patient. Following the same arguments in the previous case, we finally get the following to satisfy Theorem 1.iii:

$$V_{n+1}^{\pi'}(x_1,x_2+1,s)-V_{n+1}^{\pi^*}(x_1,x_2,s)$$
$$\leq\lambda\left(r_2^e+p_2^eV_n(x_1+1,x_2,s+1)+(1-p_2^e)V_n(x_1+1,x_2,s)-V_n(x_1+1,x_2,s)\right)$$

$$+ s\gamma \left(r_2 + V_n(x_1, x_2+1, s) - V_n(x_1, x_2+1, s)\right) \tag{A.31}$$

$$+ \mu_2 \left(p_2^e V_n(x_1, x_2, s+1) + (1-p_2^e) V_n(x_1, x_2, s) - V_n(x_1, x_2, s)\right) + (x_1\mu_1 + x_2\mu_2)\, \overline{d}_2$$

$$+ \left((S^{max} - s)\gamma + B\mu_1 - x_1\mu_1 - (x_2+1)\mu_2\right) \left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2, s)\right)$$

$$\leq \lambda \left(r_2^e + p_2^e \overline{d}_s\right) + s\gamma r_2 + \mu_2 p_2 \overline{d}_s - \mu_2 \overline{d}_2 + \left((S^{max} - s)\gamma + B\mu_1\right) \overline{d}_2$$

$$= \left(\lambda + S^{max}\gamma + B\mu_1\right) \overline{d}_2 - \left(\lambda + s\gamma + \mu_2\right) \overline{d}_2 + \left(\lambda p_2^e + \mu_2 p_2\right) \overline{d}_s + \lambda r_2^e + s\gamma r_2$$

$$\leq \overline{d}_2$$

which holds when

$$\overline{d}_s \leq \frac{r_2^e}{1-p_2^e} \left(\frac{\lambda + s\gamma + \mu_2 - \lambda\left(1-p_2^e\right)}{\lambda p_2^e + \mu_2 p_2}\right) - \frac{s\gamma r_2}{\lambda p_2^e + \mu_2 p_2} \tag{A.32}$$

$$= \frac{r_2^e}{1-p_2^e} \left(\frac{\lambda + s\gamma + \mu_2 - (\lambda + s\gamma)\left(1-p_2^e\right)}{\lambda p_2^e + \mu_2 p_2}\right) + s\gamma \frac{\left(1-p_2^e\right)}{\lambda p_2^e + \mu_2 p_2} \left(\frac{r_2^e}{1-p_2^e}\right) - s\gamma \left(\frac{r_2}{\lambda p_2^e + \mu_2 p_2}\right)$$

$$= \frac{r_2^e}{1-p_2^e} \left(\frac{\lambda + s\gamma + \mu_2 - (\lambda + s\gamma)\left(1-p_2^e\right)}{\lambda p_2^e + \mu_2 p_2}\right) + s\gamma \left(\frac{r_2^e}{\lambda p_2^e + \mu_2 p_2}\right) - s\gamma \left(\frac{r_2}{\lambda p_2^e + \mu_2 p_2}\right)$$

$$= \frac{r_2^e}{1-p_2^e} \left(\frac{\lambda + s\gamma + \mu_2 - (\lambda + s\gamma)\left(1-p_2^e\right)}{\lambda p_2^e + \mu_2 p_2}\right) - s\gamma \left(\frac{r_2 - r_2^e}{\lambda p_2^e + \mu_2 p_2}\right)$$

Case-4. $x_1 + x_2 + 1 = B$ and $s = S^{max}$: Only difference is that discharged patients leave the system permanently. Similarly, optimal actions of system B may change depending on the type of the arriving patient. We consider these situations as in the following:

Subcase-4.a. System B admits patients from both types. We let system A admit them by discharging the additional recurring patient early.

$$V_{n+1}^{\pi'}(x_1, x_2+1, s) - V_{n+1}^{\pi^*}(x_1, x_2, s)$$

$$\leq \lambda \left(r_2^e + V_n(x_1+1, x_2, s) - V_n(x_1+1, x_2, s)\right)$$

$$+ s\gamma \left(r_2^e + p_2^e V_n(x_1, x_2+1, s) + (1-p_2^e) V_n(x_1, x_2+1, s-1) - V_n(x_1, x_2+1, s_1-1)\right)$$

$$+ \mu_2 \left(V_n(x_1, x_2, s) - V_n(x_1, x_2, s)\right) + (x_1\mu_1 + x_2\mu_2)\, \overline{d}_2 \tag{A.33}$$

$$+ \left((S^{max} - s)\gamma + B\mu_1 - x_1\mu_1 - (x_2+1)\mu_2\right) \left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2, s)\right)$$

$$\leq \left(\lambda + S^{max}\gamma + B\mu_1\right) \overline{d}_2 - \left(\lambda + s\gamma + \mu_2\right) \overline{d}_2 + \lambda r_2^e + s\gamma (r_2^e + p_2^e \overline{d}_s)$$

$$\leq \overline{d}_2$$

which holds when

$$\overline{d}_s \leq \frac{r_2^e}{1-p_2^e} \left(\frac{\lambda + s\gamma + \mu_2 - (\lambda + s\gamma)\left(1-p_2^e\right)}{s\gamma p_2^e}\right) \tag{A.34}$$

Subcase-4.b. System B admits a first-time patient but rejects a recurring patient. Let system A admit the first-time patient by discharging the additional recurring patient early and reject the recurring patient. Consequently, we have:

$$\begin{aligned}
&V^{\pi'}_{n+1}(x_1, x_2+1, s) - V^{\pi^*}_{n+1}(x_1, x_2, s)\\
&\leq \lambda \left(r^e_2 + V_n(x_1+1, x_2, s) - V_n(x_1+1, x_2, s)\right)\\
&+ s\gamma \left(r_2 + V_n(x_1, x_2+1, s) - r_2 - V_n(x_1, x_2, s_1)\right)\\
&+ \mu_2 \left(V_n(x_1, x_2, s) - V_n(x_1, x_2, s)\right) + (x_1\mu_1 + x_2\mu_2)\,\overline{d}_2\\
&+ \left((S^{max} - s)\gamma + B\mu_1 - x_1\mu_1 - (x_2+1)\mu_2\right)\left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2, s)\right)\\
&\leq (\lambda + S^{max}\gamma + B\mu_1)\,\overline{d}_2 - (\lambda + \mu_2)\,\overline{d}_2 + \lambda r^e_2\\
&\leq \overline{d}_2
\end{aligned} \tag{A.35}$$

which holds since $\dfrac{\lambda r^e_2}{\lambda + \mu_2} \leq \overline{d}_2$, since $\overline{d}_2 = \dfrac{r^e_2}{1 - p^e_2}$ by (A.26).

To summarize, we have the following:

$$\begin{aligned}
\overline{d}_s \leq \min\Bigg\{ &\frac{r^e_2}{p_2(1-p^e_2)}, \frac{r^e_2}{(1-p^e_2)}\left(\frac{(\lambda + s\gamma)\,p^e_2 + \mu_2}{(\lambda + s\gamma)\,p^e_2 + \mu_2 p_2}\right),\\
&\frac{r^e_2}{(1-p^e_2)}\left(\frac{(\lambda + s\gamma)\,p^e_2 + \mu_2}{\lambda p^e_2 + \mu_2 p_2}\right) - s\gamma\left(\frac{r_2 - r^e_2}{\lambda p^e_2 + \mu_2 p_2}\right),\\
&\frac{r^e_2}{(1-p^e_2)}\left(\frac{(\lambda + s\gamma)\,p^e_2 + \mu_2}{\mu_2 p_2}\right)\Bigg\}
\end{aligned} \tag{A.36}$$

Note that

$$\begin{aligned}
&\frac{r^e_2\,(\lambda p^e_2 + \mu_2)}{p^e_2(1-p^e_2)} - (1 - \lambda - B\mu_1)\left(\frac{r_2 - r^e_2}{\lambda p^e_2 + \mu_2 p_2}\right)\\
&\leq \min\Bigg\{\frac{r^e_2}{p_2(1-p^e_2)}, \frac{r^e_2}{(1-p^e_2)}\left(\frac{(\lambda + s\gamma)\,p^e_2 + \mu_2}{(\lambda + s\gamma)\,p^e_2 + \mu_2 p_2}\right),\\
&\frac{r^e_2}{(1-p^e_2)}\left(\frac{(\lambda + s\gamma)\,p^e_2 + \mu_2}{\lambda p^e_2 + \mu_2 p_2}\right) - s\gamma\left(\frac{r_2 - r^e_2}{\lambda p^e_2 + \mu_2 p_2}\right),\\
&\frac{r^e_2}{(1-p^e_2)}\left(\frac{(\lambda + s\gamma)\,p^e_2 + \mu_2}{\mu_2 p_2}\right)\Bigg\}
\end{aligned} \tag{A.37}$$

Therefore, we set

$$\overline{d}_s \leq \frac{r^e_2\,(\lambda p^e_2 + \mu_2)}{p^e_2(1-p^e_2)} - (1 - \lambda - B\mu_1)\left(\frac{r_2 - r^e_2}{\lambda p^e_2 + \mu_2 p_2}\right) \tag{A.38}$$

where $S^{max}\gamma = 1 - \lambda - B\mu_1$ by uniformization.

A.2.3 Proof of Theorem 1.iv

Recall that we have the following from Equations A.26:

$$\overline{d}_{21} = \frac{r_2^e\,(1-p_1^e)}{(1-p_2^e)} - r_1^e$$

We want to show that $V(x_1, x_2+1, s) - V(x_1+1, x_2, s) \leq \overline{d}_{21}$. With probability $(\mu_1 - \mu_2)$, the additional first-time patient in system B is normally discharged, with probability μ_2 additional patients in both systems are normally discharged, with probability $x_1\mu_1 + x_2\mu_2$ remaining patients in both systems are normally discharged. The probability of a fictitious normal discharge in both systems is $(B\mu_1 - (x_1+1)\,\mu_1 - x_2\mu_2)$.

Case-1. $x_1 + 1 + x_2 < B$ and $s < S^{max}$: System B admits first-time patients. For a new recurring patient, system B may either admit or reject her, where in both cases System A can imitate it

$$\begin{aligned}
&V_{n+1}^{\pi'}(x_1, x_2+1, s) - V_{n+1}^{\pi^*}(x_1+1, x_2, s) \\
&\leq \lambda\overline{d}_{21} + s\gamma\overline{d}_{21} + (x_1\mu_1 + x_2\mu_2)\,\overline{d}_{21} \\
&+ (\mu_1 - \mu_2)\,(V_n(x_1, x_2+1, s) - p_1V_n(x_1, x_2, s+1) - (1-p_1)\,V_n(x_1, x_2, s)) \\
&+ \mu_2\big(p_2V_n(x_1, x_2, s+1) + (1-p_2)V_n(x_1, x_2, s) \\
&\qquad\qquad - p_1V_n(x_1, x_2, s+1) - (1-p_1)V_n(x_1, x_2, s)\big) \qquad \text{(A.39)} \\
&+ ((S^{max} - s)\,\gamma + B\mu_1 - (x_1+1)\,\mu_1 - x_2\mu_2)\,(V_n(x_1, x_2+1, s) - V_n(x_1+1, x_2, s)) \\
&\leq (\lambda + S^{max}\gamma + B\mu_1)\,\overline{d}_{21} + (\mu_1 - \mu_2)\,\overline{d}_2 - (\mu_1 - \mu_2)\,p_1\,(V_n(x_1, x_2, s+1) - V_n(x_1, x_2, s)) \\
&+ \mu_2\,(p_2 - p_1)\,(V_n(x_1, x_2, s+1) - V_n(x_1, x_2, s)) - \mu_1\overline{d}_{21} \\
&\leq (\lambda + S^{max}\gamma + B\mu_1)\,\overline{d}_{21} + (\mu_1 - \mu_2)\,\overline{d}_2 + (\mu_2p_2 - \mu_1p_1)\,\overline{d}_s - \mu_1\overline{d}_{21} \\
&\leq \overline{d}_{21}
\end{aligned}$$

which holds when $(\mu_2p_2 - \mu_1p_1)\,\overline{d}_s \leq \mu_1\overline{d}_{21} - (\mu_1 - \mu_2)\,\overline{d}_2$. By (A.26), we have the following:

$$(\mu_2p_2 - \mu_1p_1)\,\overline{d}_s \leq \mu_1\left(\frac{r_2^e\,(1-p_1^e)}{(1-p_2^e)} - r_1^e\right) - (\mu_1 - \mu_2)\left(\frac{r_2^e}{(1-p_2^e)}\right)$$

then:

$$\overline{d}_s \le \left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{\mu_2-\mu_1 p_1^e}{\mu_2 p_2-\mu_1 p_1}\right)-\left(\frac{\mu_1 r_1^e}{\mu_2 p_2-\mu_1 p_1}\right) \tag{A.40}$$

Case-2. $x_1+1+x_2<B$ and $s=S^{max}$: Discharged patients in both systems leave the system permanently.

$$\begin{aligned}
&V_{n+1}^{\pi'}(x_1,x_2+1,s)-V_{n+1}^{\pi^*}(x_1+1,x_2,s)\\
&\le \lambda\overline{d}_{21}+s\gamma\overline{d}_{21}+(x_1\mu_1+x_2\mu_2)\,\overline{d}_{21} \qquad\qquad \text{(A.41)}\\
&+(\mu_1-\mu_2)\left(V_n(x_1,x_2+1,s)-V_n(x_1,x_2,s)\right)+\mu_2\left(V(x_1,x_2,s)-V(x_1,x_2,s)\right)\\
&+\left((S^{max}-s)\gamma+B\mu_1-(x_1+1)\mu_1-x_2\mu_2\right)\left(V_n(x_1,x_2+1,s)-V_n(x_1+1,x_2,s)\right)\\
&\le (\lambda+S^{max}\gamma+B\mu_1)\,\overline{d}_{21}+(\mu_1-\mu_2)\,\overline{d}_2-\mu_1\overline{d}_{21}\\
&\le \overline{d}_{21}
\end{aligned}$$

which holds when $(\mu_1-\mu_2)\,\overline{d}_2 \le \mu_1\overline{d}_{21}$ and is true by the following, which is Condition 2:

$$0 \le \left(\frac{r_2^e}{1-p_2^e}\right)(\mu_2-\mu_1 p_1^e)-\mu_1 r_1^e \tag{A.42}$$

Case-3. $x_1+1+x_2=B$ and $s<S^{max}$: If system B rejects a new patient, system A can imitate it and we have Case-1. If system B admits a patient by discharging one of the existing patients early (not the additional one), then system A can also imitate it and we have again Case-1. Hence, the two systems differ when system B discharges the additional patient early upon arrival of a new patient.

Subcase-3.a. System B admits new patients by discharging the additional first-time patient early. Let system A admit new patients of both types by discharging the additional recurring patient early.

$$\begin{aligned}
&V_{n+1}^{\pi'}(x_1,x_2+1,s)-V_{n+1}^{\pi^*}(x_1+1,x_2,s)\\
&\le \lambda\big(r_2^e+p_2^e V_n(x_1+1,x_2,s+1)+(1-p_2^e)V_n(x_1+1,x_2,s)\\
&\qquad\qquad -r_1^e-p_1^e V_n(x_1+1,x_2,s+1)-(1-p_1^e)V_n(x_1+1,x_2,s)\big)\\
&+s\gamma\big(r_2^e+p_2^e V_n(x_1,x_2+1,s)+(1-p_2^e)V_n(x_1,x_2+1,s-1)\\
&\qquad\qquad -r_1^e-p_1^e V_n(x_1,x_2+1,s)-(1-p_1^e)V_n(x_1,x_2+1,s-1)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ (\mu_1 - \mu_2) \left(V_n (x_1, x_2 + 1, s) - p_1 V_n(x_1, x_2, s + 1) - (1 - p_1) V_n(x_1, x_2, s)\right) \\
&+ (x_1\mu_1 + x_2\mu_2) \overline{d}_{21} + \mu_2\big(p_2 V_n (x_1, x_2, s + 1) + (1 - p_2) V_n (x_1, x_2, s) \\
&\qquad\qquad - p_1 V_n (x_1, x_2, s + 1) - (1 - p_1) V_n (x_1, x_2, s) \big) \qquad (A.43) \\
&+ ((S^{max} - s) \gamma + B\mu_1 - (x_1 + 1) \mu_1 - x_2\mu_2) \left(V_n (x_1, x_2 + 1, s) - V_n (x_1 + 1, x_2, s)\right) \\
&\leq \lambda \left((r_2^e - r_1^e) + (p_2^e - p_1^e) \overline{d}_s\right) + s\gamma \left((r_2^e - r_1^e) + (p_2^e - p_1^e) \overline{d}_s\right) \\
&+ (\mu_1 - \mu_2) \overline{d}_2 - (\mu_1 - \mu_2) p_1 \left(V_n (x_1, x_2, s + 1) - V_n (x_1, x_2, s)\right) \\
&+ \mu_2 (p_2 - p_1) \left(V_n (x_1, x_2, s + 1) - V_n (x_1, x_2, s)\right) \\
&+ ((S^{max} - s) \gamma + B\mu_1) \overline{d}_{21} - \mu_1\overline{d}_{21} \\
&\leq \lambda \left((r_2^e - r_1^e) + (p_2^e - p_1^e) \overline{d}_s\right) + s\gamma \left((r_2^e - r_1^e) + (p_2^e - p_1^e) \overline{d}_s\right) - \lambda\overline{d}_{21} - s\gamma\overline{d}_{21} \\
&+ (\mu_1 - \mu_2) \overline{d}_2 + (\mu_2 p_2 - \mu_1 p_1) \overline{d}_s + (\lambda + S^{max}\gamma + B\mu_1) \overline{d}_{21} - \mu_1\overline{d}_{21} \\
&\leq \overline{d}_{21}
\end{aligned}
$$

which is true when $((\lambda + s\gamma) (p_2^e - p_1^e) + (\mu_2 p_2 - \mu_1 p_1)) \overline{d}_s \leq (\lambda + s\gamma + \mu_1) \overline{d}_{21} - (\mu_1 - \mu_2) \overline{d}_2 - (\lambda - s\gamma) (r_2^e - r_1^e)$. By (A.26), we have the following:

$$
\begin{aligned}
((\lambda + s\gamma) (p_2^e - p_1^e) + (\mu_2 p_2 - \mu_1 p_1)) \overline{d}_s \leq (\lambda + s\gamma + \mu_1) \left(\frac{r_2^e (1 - p_1^e)}{(1 - p_2^e)} - r_1^e\right) \\
- (\mu_1 - \mu_2) \left(\frac{r_2^e}{(1 - p_2^e)}\right) - (\lambda - s\gamma) (r_2^e - r_1^e)
\end{aligned}
$$

which is equivalent to the following: then:

$$
\begin{aligned}
\overline{d}_s \leq \left(\frac{r_2^e}{1 - p_2^e}\right) &\left(\frac{(\lambda + s\gamma) (1 - p_1^e) + \mu_2 - \mu_1 p_1^e}{(\lambda + s\gamma) (p_2^e - p_1^e) + (\mu_2 p_2 - \mu_1 p_1)}\right) \\
&- \left(\frac{(\lambda + s\gamma) r_2^e + \mu_1 r_1^e}{(\lambda + s\gamma) (p_2^e - p_1^e) + (\mu_2 p_2 - \mu_1 p_1)}\right)
\end{aligned} \qquad (A.44)
$$

Subcase-3.b. System B admits a new first-time patient by discharging the additional first-time patient early but rejects a recurring patient.

$$
\begin{aligned}
&V_{n+1}^{\pi'} (x_1, x_2 + 1, s) - V_{n+1}^{\pi^*} (x_1 + 1, x_2, s) \\
&\leq \lambda\big(r_2^e + p_2^e V_n(x_1 + 1, x_2, s + 1) + (1 - p_2^e) V_n(x_1 + 1, x_2, s) \\
&\qquad\qquad - r_1^e - p_1^e V_n(x_1 + 1, x_2, s + 1) - (1 - p_1^e) V_n(x_1 + 1, x_2, s)\big) \\
&+ s\gamma\overline{d}_{21} + (\mu_1 - \mu_2) \left(V_n (x_1, x_2 + 1, s) - p_1 V_n(x_1, x_2, s + 1) - (1 - p_1) V_n(x_1, x_2, s)\right) \\
&+ (x_1\mu_1 + x_2\mu_2) \overline{d}_{21} + \mu_2\big(p_2 V_n (x_1, x_2, s + 1) + (1 - p_2) V_n (x_1, x_2, s)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad - p_1 V_n\left(x_1, x_2, s+1\right) - (1-p_1) V_n\left(x_1, x_2, s\right)\big) \qquad \text{(A.45)}\\
&+ \left(\left(S^{max} - s\right)\gamma + B\mu_1 - \left(x_1+1\right)\mu_1 - x_2\mu_2\right)\left(V_n\left(x_1, x_2+1, s\right) - V_n\left(x_1+1, x_2, s\right)\right)\\
&\leq \lambda\left(\left(r_2^e - r_1^e\right) + \left(p_2^e - p_1^e\right)\overline{d}_s\right) + s\gamma\overline{d}_{21}\\
&+ \left(\mu_1 - \mu_2\right)\overline{d}_2 - \left(\mu_1 - \mu_2\right)p_1\left(V_n\left(x_1, x_2, s+1\right) - V_n\left(x_1, x_2, s\right)\right)\\
&+ \mu_2\left(p_2 - p_1\right)\left(V_n\left(x_1, x_2, s+1\right) - V_n\left(x_1, x_2, s\right)\right)\\
&+ \left(\left(S^{max} - s\right)\gamma + B\mu_1\right)\overline{d}_{21} - \mu_1\overline{d}_{21}\\
&\leq \lambda\left(\left(r_2^e - r_1^e\right) + \left(p_2^e - p_1^e\right)\overline{d}_s\right) - \lambda\overline{d}_{21}\\
&+ \left(\mu_1 - \mu_2\right)\overline{d}_2 + \left(\mu_2 p_2 - \mu_1 p_1\right)\overline{d}_s + \left(\lambda + S^{max}\gamma + B\mu_1\right)\overline{d}_{21} - \mu_1\overline{d}_{21}\\
&\leq \overline{d}_{21}
\end{aligned}
$$

which is true when the following holds:

$$
\begin{aligned}
\overline{d}_s \leq \left(\frac{r_2^e}{1-p_2^e}\right)&\left(\frac{\lambda\left(1-p_1^e\right) + \mu_2 - \mu_1 p_1^e}{\lambda\left(p_2^e - p_1^e\right) + \left(\mu_2 p_2 - \mu_1 p_1\right)}\right)\\
&- \left(\frac{\lambda r_2^e + \mu_1 r_1^e}{\lambda\left(p_2^e - p_1^e\right) + \left(\mu_2 p_2 - \mu_1 p_1\right)}\right)
\end{aligned} \qquad \text{(A.46)}
$$

Subcase-3.c. System B rejects a first-time patient and admits a recurring patient by discharging the additional first-time patient early.

$$
\begin{aligned}
&V_{n+1}^{\pi'}\left(x_1, x_2+1, s\right) - V_{n+1}^{\pi^*}\left(x_1+1, x_2, s\right)\\
&\leq \lambda\overline{d}_{21} + s\gamma\big(r_2^e + p_2^e V_n(x_1, x_2+1, s) + (1-p_2^e)V_n(x_1, x_2+1, s-1)\\
&\qquad\qquad\qquad - r_1^e - p_1^e V_n(x_1, x_2+1, s) - (1-p_1^e)V_n(x_1, x_2+1, s-1)\big)\\
&+ \left(\mu_1 - \mu_2\right)\left(V_n\left(x_1, x_2+1, s\right) - p_1 V_n(x_1, x_2, s+1) - \left(1-p_1\right)V_n(x_1, x_2, s)\right)\\
&+ \left(x_1\mu_1 + x_2\mu_2\right)\overline{d}_{21} + \mu_2\big(p_2 V_n\left(x_1, x_2, s+1\right) + (1-p_2)V_n\left(x_1, x_2, s\right)\\
&\qquad\qquad\qquad - p_1 V_n\left(x_1, x_2, s+1\right) - (1-p_1) V_n\left(x_1, x_2, s\right)\big) \qquad \text{(A.47)}\\
&+ \left(\left(S^{max} - s\right)\gamma + B\mu_1 - \left(x_1+1\right)\mu_1 - x_2\mu_2\right)\left(V_n\left(x_1, x_2+1, s\right) - V_n\left(x_1+1, x_2, s\right)\right)\\
&\leq \lambda\left(\left(r_2^e - r_1^e\right) + \left(p_2^e - p_1^e\right)\overline{d}_s\right) + s\gamma\left(\left(r_2^e - r_1^e\right) + \left(p_2^e - p_1^e\right)\overline{d}_s\right)\\
&+ \left(\mu_1 - \mu_2\right)\overline{d}_2 - \left(\mu_1 - \mu_2\right)p_1\left(V_n\left(x_1, x_2, s+1\right) - V_n\left(x_1, x_2, s\right)\right)\\
&+ \mu_2\left(p_2 - p_1\right)\left(V_n\left(x_1, x_2, s+1\right) - V_n\left(x_1, x_2, s\right)\right)\\
&+ \left(\left(S^{max} - s\right)\gamma + B\mu_1\right)\overline{d}_{21} - \mu_1\overline{d}_{21}\\
&\leq s\gamma\left(\left(r_2^e - r_1^e\right) + \left(p_2^e - p_1^e\right)\overline{d}_s\right) - s\gamma\overline{d}_{21}
\end{aligned}
$$

$$
\begin{aligned}
&+ (\mu_1 - \mu_2)\,\overline{d}_2 + (\mu_2 p_2 - \mu_1 p_1)\,\overline{d}_s + (\lambda + S^{max}\gamma + B\mu_1)\,\overline{d}_{21} - \mu_1 \overline{d}_{21} \\
&\leq \overline{d}_{21}
\end{aligned}
$$

which is true when the following holds:

$$
\begin{aligned}
\overline{d}_s \leq \left(\frac{r_2^e}{1-p_2^e}\right)&\left(\frac{s\gamma\,(1-p_1^e) + \mu_2 - \mu_1 p_1^e}{s\gamma\,(p_2^e - p_1^e) + (\mu_2 p_2 - \mu_1 p_1)}\right) \\
&- \left(\frac{s\gamma r_2^e + \mu_1 r_1^e}{s\gamma\,(p_2^e - p_1^e) + (\mu_2 p_2 - \mu_1 p_1)}\right)
\end{aligned}
\tag{A.48}
$$

Subcase-3.d. System B rejects patients from both types.

$$
\begin{aligned}
&V_{n+1}^{\pi'}(x_1, x_2+1, s) - V_{n+1}^{\pi^*}(x_1+1, x_2, s) \\
&\leq \lambda \overline{d}_{21} + s\gamma \overline{d}_{21} + (x_1\mu_1 + x_2\mu_2)\,\overline{d}_{21} \\
&+ (\mu_1 - \mu_2)\,(V_n(x_1, x_2+1, s) - p_1 V_n(x_1, x_2, s+1) - (1-p_1)\,V_n(x_1, x_2, s)) \\
&+ \mu_2 \big(p_2 V_n(x_1, x_2, s+1) + (1-p_2) V_n(x_1, x_2, s) \\
&\qquad\qquad - p_1 V_n(x_1, x_2, s+1) - (1-p_1) V_n(x_1, x_2, s)\big) \qquad \text{(A.49)} \\
&+ ((S^{max} - s)\,\gamma + B\mu_1 - (x_1+1)\,\mu_1 - x_2\mu_2)\,(V_n(x_1, x_2+1, s) - V_n(x_1+1, x_2, s)) \\
&\leq \lambda \overline{d}_{21} + s\gamma \overline{d}_{21} + (\mu_1 - \mu_2)\,\overline{d}_2 - (\mu_1 - \mu_2)\,p_1\,(V_n(x_1, x_2, s+1) - V_n(x_1, x_2, s)) \\
&+ \mu_2\,(p_2 - p_1)\,(V_n(x_1, x_2, s+1) - V_n(x_1, x_2, s)) \\
&+ ((S^{max} - s)\,\gamma + B\mu_1)\,\overline{d}_{21} - \mu_1 \overline{d}_{21} \\
&\leq (\mu_1 - \mu_2)\,\overline{d}_2 + (\mu_2 p_2 - \mu_1 p_1)\,\overline{d}_s + (\lambda + S^{max}\gamma + B\mu_1)\,\overline{d}_{21} - \mu_1 \overline{d}_{21} \\
&\leq \overline{d}_{21}
\end{aligned}
$$

which is true when the following holds:

$$
\overline{d}_s \leq \left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{\mu_2 - \mu_1 p_1^e}{\mu_2 p_2 - \mu_1 p_1}\right) - \left(\frac{\mu_1 r_1^e}{\mu_2 p_2 - \mu_1 p_1}\right) \tag{A.50}
$$

Case-4. $x_1 + 1 + x_2 = B$ and $s = S^{max}$: Different than Case-3, discharged patients leave the system permanently. We will consider all possible cases as in the previous case.

Subcase-4.a. System B admits patients of both types by discharging the additional first-time patient early. Let system A admit patients of both types by discharging the additional recurring patient early.

$$
V_{n+1}^{\pi'}(x_1, x_2+1, s) - V_{n+1}^{\pi^*}(x_1+1, x_2, s)
$$

$$
\begin{aligned}
&\leq \lambda\left(r_2^e + V_n\left(x_1+1,x_2,s\right) - r_1^e - V_n\left(x_1+1,x_2,s\right)\right)\\
&+\left(x_1\mu_1+x_2\mu_2\right)\overline{d}_{21} + s\gamma\big(r_2^e + p_2^e V_n(x_1,x_2+1,s) + (1-p_2^e)V_n(x_1,x_2+1,s-1)\\
&\qquad\qquad - r_1^e - p_1^e V_n(x_1,x_2+1,s) - (1-p_1^e)V_n(x_1,x_2+1,s-1)\big)\\
&+\left(\mu_1-\mu_2\right)\left(V_n\left(x_1,x_2+1,s\right) - V_n\left(x_1,x_2,s\right)\right) + \mu_2\left(V_n\left(x_1,x_2,s\right) - V_n\left(x_1,x_2,s\right)\right)\\
&+\left(\left(S^{max}-s\right)\gamma + B\mu_1 - \left(x_1+1\right)\mu_1 - x_2\mu_2\right)\left(V_n\left(x_1,x_2+1,s\right) - V_n\left(x_1+1,x_2,s\right)\right)\\
&\leq \lambda\left(r_2^e - r_1^e\right) + s\gamma\left(\left(r_2^e-r_1^e\right) + \left(p_2^e-p_1^e\right)\overline{d}_s\right) + \left(\mu_1-\mu_2\right)\overline{d}_2 \qquad (A.51)\\
&+\left(\left(S^{max}-s\right)\gamma + B\mu_1\right)\overline{d}_{21} - \mu_1\overline{d}_{21}\\
&\leq \lambda\left(r_2^e - r_1^e\right) + s\gamma\left(\left(r_2^e-r_1^e\right) + \left(p_2^e-p_1^e\right)\overline{d}_s\right) - \lambda\overline{d}_{21} - s\gamma\overline{d}_{21} + \left(\mu_1-\mu_2\right)\overline{d}_2\\
&+\left(\lambda + S^{max}\gamma + B\mu_1\right)\overline{d}_{21} - \mu_1\overline{d}_{21}\\
&\leq \overline{d}_{21}
\end{aligned}
$$

which is true when the following holds:

$$
\begin{aligned}
\overline{d}_s \leq \left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{\left(\lambda+s\gamma\right)\left(1-p_1^e\right)+\mu_2-\mu_1 p_1^e}{s\gamma\left(p_2^e-p_1^e\right)}\right)\\
-\left(\frac{\left(\lambda+s\gamma\right)r_2^e+\mu_1 r_1^e}{s\gamma\left(p_2^e-p_1^e\right)}\right)
\end{aligned} \qquad (A.52)
$$

Subcase-4.b. System B admits a first-time patient by discharging the additional first-time patient early but rejects a recurring patient.

$$
\begin{aligned}
&V_{n+1}^{\pi'}\left(x_1,x_2+1,s\right) - V_{n+1}^{\pi^*}\left(x_1+1,x_2,s\right)\\
&\leq \lambda\left(r_2^e + V_n\left(x_1+1,x_2,s\right) - r_1^e - V_n\left(x_1+1,x_2,s\right)\right) + \left(x_1\mu_1+x_2\mu_2\right)\overline{d}_{21} + s\gamma\overline{d}_{21}\\
&+\left(\mu_1-\mu_2\right)\left(V_n\left(x_1,x_2+1,s\right) - V_n\left(x_1,x_2,s\right)\right) + \mu_2\left(V_n\left(x_1,x_2,s\right) - V_n\left(x_1,x_2,s\right)\right)\\
&+\left(\left(S^{max}-s\right)\gamma + B\mu_1 - \left(x_1+1\right)\mu_1 - x_2\mu_2\right)\left(V_n\left(x_1,x_2+1,s\right) - V_n\left(x_1+1,x_2,s\right)\right)\\
&\leq \lambda\left(r_2^e - r_1^e\right) + \left(\lambda + S^{max}\gamma + B\mu_1\right)\overline{d}_{21} - \mu_1\overline{d}_{21} - \lambda\overline{d}_{21} + \left(\mu_1-\mu_2\right)\overline{d}_2 \qquad (A.53)\\
&\leq \overline{d}_{21}
\end{aligned}
$$

which is true by Condition 2:

$$
0 \leq \left(\frac{r_2^e}{1-p_2^e}\right)\left(\lambda\left(1-p_1^e\right)+\mu_2-\mu_1 p_1^e\right) - \left(\lambda r_2^e + \mu_1 r_1^e\right) \qquad (A.54)
$$

Subcase-4.c. System B rejects a first-time patient and admits a recurring patient by discharging the additional first-time patient early. We let system A reject the first-time patient and admit the recurring patient by discharging the additional recurring patient early.

$$
\begin{aligned}
&V_{n+1}^{\pi'}(x_1, x_2+1, s) - V_{n+1}^{\pi^*}(x_1+1, x_2, s) \\
&\leq \lambda \overline{d}_{21} + (x_1\mu_1 + x_2\mu_2)\,\overline{d}_{21} \\
&+ s\gamma\big(r_2^e + p_2^e V_n(x_1, x_2+1, s) + (1-p_2^e)V_n(x_1, x_2+1, s-1) \qquad\qquad (A.55)\\
&\qquad\qquad - r_1^e - p_1^e V_n(x_1, x_2+1, s) - (1-p_1^e)V_n(x_1, x_2+1, s-1)\big) \\
&+ (\mu_1 - \mu_2)\left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2, s)\right) + \mu_2\left(V_n(x_1, x_2, s) - V_n(x_1, x_2, s)\right) \\
&+ \left((S^{max} - s)\gamma + B\mu_1 - (x_1+1)\mu_1 - x_2\mu_2\right)\left(V_n(x_1, x_2+1, s) - V_n(x_1+1, x_2, s)\right) \\
&\leq (\lambda + S^{max}\gamma + B\mu_1)\,\overline{d}_{21} - s\gamma\overline{d}_{21} - \mu_1\overline{d}_{21} + s\gamma\left((r_2^e - r_1^e) + (p_2^e - p_1^e)\,\overline{d}_s\right) + (\mu_1 - \mu_2)\,\overline{d}_2 \\
&\leq \overline{d}_{21}
\end{aligned}
$$

which is true when the following holds:

$$
\overline{d}_s \leq \left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{s\gamma(1-p_1^e) + \mu_2 - \mu_1 p_1^e}{s\gamma(p_2^e - p_1^e)}\right) - \left(\frac{s\gamma r_2^e + \mu_1 r_1^e}{s\gamma(p_2^e - p_1^e)}\right) \tag{A.56}
$$

Subcase-4.d. System B rejects patients from both types.

$$
\begin{aligned}
&V_{n+1}^{\pi'}(x_1, x_2+1, s) - V_{n+1}^{\pi^*}(x_1+1, x_2, s) \\
&\leq \lambda \overline{d}_{21} + s\gamma\overline{d}_{21} + (x_1\mu_1 + x_2\mu_2)\,\overline{d}_{21} \\
&+ (\mu_1 - \mu_2)\left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2, s)\right) + \mu_2\left(V_n(x_1, x_2, s) - V_n(x_1, x_2, s)\right) \\
&+ \left((S^{max} - s)\gamma + B\mu_1 - (x_1+1)\mu_1 - x_2\mu_2\right)\left(V_n(x_1, x_2+1, s) - V_n(x_1+1, x_2, s)\right) \\
&\leq (\lambda + S^{max}\gamma + B\mu_1)\,\overline{d}_{21} - \mu_1\overline{d}_{21} + (\mu_1 - \mu_2)\,\overline{d}_2 \\
&\leq \overline{d}_{21}
\end{aligned}
$$

which is true by Condition 2:

$$
0 \leq \left(\frac{r_2^e}{1-p_2^e}\right)(\mu_2 - \mu_1 p_1^e) - \mu_1 r_1^e \tag{A.57}
$$

To summarize, we have:

$$
\overline{d}_s \leq \min\left\{ \left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{\mu_2 - \mu_1 p_1^e}{\mu_2 p_2 - \mu_1 p_1}\right) - \left(\frac{\mu_1 r_1^e}{\mu_2 p_2 - \mu_1 p_1}\right), \right. \tag{A.58}
$$

$$
\left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{(\lambda + s\gamma)(1-p_1^e) + \mu_2 - \mu_1 p_1^e}{(\lambda + s\gamma)(p_2^e - p_1^e) + (\mu_2 p_2 - \mu_1 p_1)}\right) \tag{A.59}
$$

$$-\left(\frac{(\lambda+s\gamma)\,r_2^e+\mu_1 r_1^e}{(\lambda+s\gamma)\,(p_2^e-p_1^e)+(\mu_2 p_2-\mu_1 p_1)}\right),$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{\lambda\,(1-p_1^e)+\mu_2-\mu_1 p_1^e}{\lambda\,(p_2^e-p_1^e)+(\mu_2 p_2-\mu_1 p_1)}\right) \tag{A.60}$$

$$-\left(\frac{\lambda r_2^e+\mu_1 r_1^e}{\lambda\,(p_2^e-p_1^e)+(\mu_2 p_2-\mu_1 p_1)}\right),$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{s\gamma\,(1-p_1^e)+\mu_2-\mu_1 p_1^e}{s\gamma\,(p_2^e-p_1^e)+(\mu_2 p_2-\mu_1 p_1)}\right) \tag{A.61}$$

$$-\left(\frac{s\gamma r_2^e+\mu_1 r_1^e}{s\gamma\,(p_2^e-p_1^e)+(\mu_2 p_2-\mu_1 p_1)}\right),$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{\mu_2-\mu_1 p_1^e}{\mu_2 p_2-\mu_1 p_1}\right)-\left(\frac{\mu_1 r_1^e}{\mu_2 p_2-\mu_1 p_1}\right), \tag{A.62}$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{(\lambda+s\gamma)\,(1-p_1^e)+\mu_2-\mu_1 p_1^e}{s\gamma\,(p_2^e-p_1^e)}\right) \tag{A.63}$$

$$-\left(\frac{(\lambda+s\gamma)\,r_2^e+\mu_1 r_1^e}{s\gamma\,(p_2^e-p_1^e)}\right),$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{s\gamma\,(1-p_1^e)+\mu_2-\mu_1 p_1^e}{s\gamma\,(p_2^e-p_1^e)}\right)-\left(\frac{s\gamma r_2^e+\mu_1 r_1^e}{s\gamma\,(p_2^e-p_1^e)}\right)\Bigg\} \tag{A.64}$$

Note that $\left(\frac{r_2^e}{1-p_2^e}\right)\frac{\left(\mu_2-\mu_1 p_1^e\right)}{(1-B\mu_1)\left(p_2^e-p_1^e\right)+(\mu_2 p_2-\mu_1 p_1)} - \frac{\left((1-B\mu_1)r_2^e+\mu_1 r_1^e\right)}{C_1}$, where $C_1=\min\left\{\gamma\,(p_2^e-p_1^e)\,,(\mu_2 p_2-\mu_1 p_1)\right\}$, is less than the terms in the minimum operator. Therefore, we set

$$\overline{d}_s \le \left(\frac{r_2^e}{1-p_2^e}\right)\frac{(\mu_2-\mu_1 p_1^e)}{(1-B\mu_1)\,(p_2^e-p_1^e)+(\mu_2 p_2-\mu_1 p_1)}-\frac{((1-B\mu_1)\,r_2^e+\mu_1 r_1^e)}{C_1} \tag{A.65}$$

Finally, by the proofs of Theorem i ii, iii and iv (i.e., by Equations A.16, A.26, A.38, A.65, respectively), we have the following expressions for $\overline{d}_s$:

$$\frac{r_2^e}{1-p_2^e} \le \overline{d}_s \tag{A.66}$$

$$\le \min\left\{\frac{(\mu_1+\lambda+\gamma)\,r_1-\gamma r_1^e}{p_1},\right. \tag{A.67}$$

$$\frac{r_2^e\,(\lambda p_2^e+\mu_2)}{p_2^e(1-p_2^e)}-(1-\lambda-B\mu_1)\left(\frac{r_2-r_2^e}{\lambda p_2^e+\mu_2 p_2}\right), \tag{A.68}$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\frac{(\mu_2-\mu_1 p_1^e)}{(1-B\mu_1)(p_2^e-p_1^e)+(\mu_2 p_2-\mu_1 p_1)} \tag{A.69}$$

$$\left. - \frac{((1-B\mu_1)\, r_2^e+\mu_1 r_1^e)}{C_1}\right\}$$

This completes the proof of Theorem 1.

A.3 Proof of Theorem 2

For the recurring patients to be a preferred type, we need to show that

$$V(x_1, x_2+1, s) - V(x_1, x_2, s+1) \leq r_2$$

for all $(\mathbf{x}, s) \in \mathcal{S}$. To do this, we need to prove the existence of finite upper bounds on the costs of having one more first-time patient in the ICU, having one more recurring patient in the ICU, having one more patient in the orbit and changing a first-time patient to a recurring patient, respectively. Let us assume that the following holds for $n \geq 0$ and prove that it also holds for $n+1$.

$$V_n(x_1, x_2+1, s) - V_n(x_1, x_2, s+1) \leq \overline{d}_{2s} \tag{A.70a}$$

$$V_n(x_1, x_2, s+1) - V_n(x_1, x_2, s) \leq \overline{d}_s \tag{A.70b}$$

$$V_n(x_1, x_2+1, s) - V_n(x_1, x_2, s) \leq \overline{d}_2 \tag{A.70c}$$

$$V_n(x_1, x_2+1, s) - V_n(x_1+1, x_2, s) \leq \overline{d}_{21} \tag{A.70d}$$

$$V_n(x_1+1, x_2, s) - V_n(x_1, x_2, s) \leq \overline{d}_1 \tag{A.70e}$$

where $\overline{d}_{2s}$, $\overline{d}_1$, $\overline{d}_2$, $\overline{d}_s$ and $\overline{d}_{21}$ are non-negative upper bounds on $D(2s)(\mathbf{x}, s)$, $D(s0)(\mathbf{x}, s)$, $D(20)(\mathbf{x}, s)$, $D(21)(\mathbf{x}, s)$ and $D(10)(\mathbf{x}, s)$, respectively. We will show that equations (A.70a), (A.70b), (A.70c), (A.70d), and (A.70e) hold for $n+1$. Without loss of generality, we let $\overline{d}_{2s} = r_2$.

A.3.1 Proof of Theorem 2.i

We define two two systems, system A and system B in a way such that system A starts in state (x_1, x_2+1, s) and system B starts in state $(x_1, x_2, s+1)$. From here on, we let system B follow the optimal policy and system A follow not necessarily the optimal policy. We will use the same arguments in the proofs of Theorem 2.ii, iii and iv. Note that by (A.70a), system B admits recurring patients whenever there is a free bed.

CASE-1. $x_1 + x_2 + 1 < B$ AND $s + 1 < S^{max}$: System A can imitate system B except for the arrival of the additional patient in the orbit of system B. Moreover, additional recurring patient in system A is normally discharged with probability μ_2.

$$\begin{aligned}
&V_{n+1}^{\pi'}(x_1, x_2+1, s) - V_{n+1}^{\pi^*}(x_1, x_2, s+1) \\
&\leq \lambda r_2 + s\gamma r_2 + \gamma\left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2+1, s)\right) + (x_1\mu_1 + x_2\mu_2) r_2 \\
&+ \mu_2\left(p_2 V_n(x_1, x_2, s+1) + (1-p_2)V_n(x_1, x_2, s) - V_n(x_1, x_2, s+1)\right) \qquad \text{(A.71)} \\
&+ \left((S^{max} - (s+1))\gamma + B\mu_1 - x_1\mu_1 - (x_2+1)\mu_2\right)\left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2, s+1)\right) \\
&\leq (\lambda + S^{max}\gamma + B\mu_1) r_2 - (\gamma + \mu_2) r_2 \\
&+ \mu_2\left(p_2 V_n(x_1, x_2, s+1) + (1-p_2)V_n(x_1, x_2, s) - V_n(x_1, x_2, s)\right) \\
&\leq (\lambda + S^{max}\gamma + B\mu_1) r_2 - (\gamma + \mu_2) r_2 + \mu_2 p_2 \overline{d}_s \\
&\leq r_2
\end{aligned}$$

which is true when the following holds:

$$\overline{d}_s \leq \frac{(\gamma + \mu_2) r_2}{\mu_2 p_2} \qquad \text{(A.72)}$$

Note that we used the $V(x_1, x_2, s+1) \geq V(x_1, x_2, s)$ in the second inequality, which is the result of Proposition 1. Proposition 1 still holds when early discharges are not allowed when there is a free bed.

CASE-2. $x_1 + x_2 + 1 < B$ AND $s + 1 = S^{max}$: The only difference from the previous case is that discharged patients in system B leave the system permanently. That is:

$$\begin{aligned}
&V_{n+1}^{\pi'}(x_1, x_2+1, s) - V_{n+1}^{\pi^*}(x_1, x_2, s+1) \\
&\leq \lambda r_2 + s\gamma r_2 - \gamma\left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2+1, s)\right) \\
&+ x_1\mu_1\left(p_1 V_n(x_1-1, x_2+1, s+1) + (1-p_1) V_n(x_1-1, x_2+1, s) - V_n(x_1-1, x_2, s+1)\right) \\
&+ x_2\mu_2\left(p_2 V_n(x_1, x_2, s+1) + (1-p_2) V_n(x_1, x_2, s) - V_n(x_1, x_2-1, s+1)\right) \\
&+ \mu_2\left(p_2 V_n(x_1, x_2, s+1) + (1-p_2)V_n(x_1, x_2, s) - V_n(x_1, x_2, s+1)\right) \qquad \text{(A.73)}
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\left(S^{max} - (s+1)\right)\gamma + B\mu_1 - x_1\mu_1 - (x_2+1)\mu_2\right)\left(V_n(x_1,x_2+1,s) - V_n(x_1,x_2,s+1)\right) \\
&\leq (\lambda + S^{max}\gamma + B\mu_1) r_2 - (\gamma+\mu_2) r_2 + \mu_2 p_2 \overline{d}_s + x_1\mu_1 p_1 \overline{d}_s + x_2\mu_2 p_2 \overline{d}_s \\
&\leq (\lambda + S^{max}\gamma + B\mu_1) r_2 - (\gamma+\mu_2) r_2 + B\mu_2 p_2 \overline{d}_s \\
&\leq r_2
\end{aligned}
$$

which holds when the following holds:

$$
\overline{d}_s \leq \frac{(\gamma+\mu_2) r_2}{B\mu_2 p_2} \tag{A.74}
$$

where we used Condition 1, that is, $\mu_1 p_1 \leq \mu_2 p_2$, and the fact that $x_1 + x_2 + 1 \leq B$.

Case-3. $x_1 + x_2 + 1 = B$ and $s+1 < S^{max}$: System B admits a recurring patient and system A rejects her by incurring a cost of r_2. System B admits or rejects a first-time patient. If system B rejects a first-time patient, then system A also rejects her and we have the results of Case-1. We focus on the case where system B admits first-time patients. System A admits firs-time patients by discharging additional recurring patient early. That is:

$$
\begin{aligned}
&V_{n+1}^{\pi'}(x_1,x_2+1,s) - V_{n+1}^{\pi^*}(x_1,x_2,s+1) \\
&\leq \lambda\left(r_2^e + p_2^e V_n(x_1+1,x_2,s+1) + (1-p_2^e) V_n(x_1+1,x_2,s) - V_n(x_1+1,x_2,s+1)\right) \\
&+ s\gamma\left(r_2 + V_n(x_1,x_2+1,s) - V_n(x_1,x_2+1,s)\right) + (x_1\mu_1 + x_2\mu_2) r_2 \\
&+ \gamma\left(V_n(x_1,x_2+1,s) - V_n(x_1,x_2+1,s)\right) \\
&+ \mu_2\left(p_2 V_n(x_1,x_2,s+1) + (1-p_2) V_n(x_1,x_2,s) - V_n(x_1,x_2,s+1)\right) \qquad \text{(A.75)} \\
&+ \left(\left(S^{max} - (s+1)\right)\gamma + B\mu_1 - x_1\mu_1 - (x_2+1)\mu_2\right)\left(V_n(x_1,x_2+1,s) - V_n(x_1,x_2,s+1)\right) \\
&\leq (\lambda + S^{max}\gamma + B\mu_1) r_2 - \lambda r_2 - (\gamma+\mu_2) r_2 + \lambda\left(r_2^e + p_2^e \overline{d}_s\right) + \mu_2 p_2 \overline{d}_s
\end{aligned}
$$

which is true when the following holds:

$$
\overline{d}_s \leq \frac{(\lambda+\gamma+\mu_2) r_2 - \lambda r_2^e}{\lambda p_2^e + \mu_2 p_2} \tag{A.76}
$$

Note that we used $V(x_1,x_2,s+1) \geq V(x_1,x_2,s)$ in the second inequality.

Case-4. $x_1 + x_2 + 1 = B$ and $s + 1 = S^{max}$: Discharged patients in system B leave the system permanently. Moreover, system B may admit or reject first-time patients. If system B reject first-time patients, we have the results of Case-2. We focus on the case where system B admits first-time patients.Let system A admit them by discharging the additional recurring patient early:

$$
\begin{aligned}
&V_{n+1}^{\pi'}\left(x_1,x_2+1,s\right)-V_{n+1}^{\pi^*}\left(x_1,x_2,s+1\right)\\
&\leq \lambda\left(r_2^e+p_2^e\overline{d}_s\right)+s\gamma\overline{d}_{21}+\gamma\left(V_n\left(x_1,x_2+1,s\right)-V_n\left(x_1,x_2+1,s\right)\right)\\
&+x_1\mu_1\left(p_1V_n\left(x_1-1,x_2+1,s+1\right)+\left(1-p_1\right)V_n\left(x_1-1,x_2+1,s\right)-V_n\left(x_1-1,x_2,s+1\right)\right)\\
&+x_2\mu_2\left(p_2V_n\left(x_1,x_2,s+1\right)+\left(1-p_2\right)V_n\left(x_1,x_2,s\right)-V_n\left(x_1,x_2-1,s+1\right)\right)\\
&+\mu_2\left(p_2V_n\left(x_1,x_2,s+1\right)+(1-p_2)V_n\left(x_1,x_2,s\right)-V_n\left(x_1,x_2,s+1\right)\right) \qquad \text{(A.77)}\\
&+\left(\left(S^{max}-(s+1)\right)\gamma+B\mu_1-x_1\mu_1-\left(x_2+1\right)\mu_2\right)\left(V_n\left(x_1,x_2+1,s\right)-V_n\left(x_1,x_2,s+1\right)\right)\\
&\leq\left(\lambda+S^{max}\gamma+B\mu_1\right)r_2-\left(\lambda+\gamma+\mu_2\right)r_2+\lambda\left(r_2^e+p_2^e\overline{d}_s\right)+B\mu_2p_2\overline{d}_s\\
&\leq r_2
\end{aligned}
$$

which is true when

$$\overline{d}_s\leq\frac{\left(\lambda+\gamma+\mu_2\right)r_2-\lambda r_2^e}{\lambda p_2^e+B\mu_2p_2} \tag{A.78}$$

Consequently, we have the following by (A.72), (A.74), (A.76) and (A.78):

$$
\begin{aligned}
\overline{d}_s\leq\min\Bigg\{&\frac{\left(\gamma+\mu_2\right)r_2}{\mu_2p_2},\ \frac{\left(\gamma+\mu_2\right)r_2}{B\mu_2p_2},\\
&\frac{\left(\lambda+\gamma+\mu_2\right)r_2-\lambda r_2^e}{\lambda p_2^e+\mu_2p_2},\ \frac{\left(\lambda+\gamma+\mu_2\right)r_2-\lambda r_2^e}{\lambda p_2^e+B\mu_2p_2}\Bigg\}
\end{aligned} \tag{A.79}
$$

Note that

$$
\begin{aligned}
\frac{\left(\gamma+\mu_2\right)r_2-\lambda r_2^e}{\lambda p_2^e+B\mu_2p_2}\leq\min\Bigg\{&\frac{\left(\gamma+\mu_2\right)r_2}{\mu_2p_2},\ \frac{\left(\gamma+\mu_2\right)r_2}{B\mu_2p_2},\\
&\frac{\left(\lambda+\gamma+\mu_2\right)r_2-\lambda r_2^e}{\lambda p_2^e+\mu_2p_2},\ \frac{\left(\lambda+\gamma+\mu_2\right)r_2-\lambda r_2^e}{\lambda p_2^e+B\mu_2p_2}\Bigg\}
\end{aligned} \tag{A.80}
$$

Therefore, we may define the following:

$$\overline{d}_s\leq\frac{\left(\gamma+\mu_2\right)r_2-\lambda r_2^e}{\lambda p_2^e+B\mu_2p_2} \tag{A.81}$$

A.3.2 Proof of Theorem 2.ii

System A is in state $(x_1, x_2, s+1)$ and system B is in state (x_1, x_2, s). We will consider the following cases:

CASE-1. $x_1 + x_2 < B$ AND $s + 1 < S^{max}$: System A can imitate decisions of system B upon a patient arrival. Further, additional patient in the orbit of system A arrives with probability γ and system A admits her. That is:

$$\begin{aligned}
&V^{\pi'}_{n+1}(x_1, x_2, s+1) - V^{\pi^*}_{n+1}(x_1, x_2, s) \\
&\leq \lambda\overline{d}_s + s\gamma\overline{d}_s + \gamma\left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2, s)\right) + x_1\mu_1\overline{d}_s + x_2\mu_2\overline{d}_s \\
&+ \left((S^{max} - s - 1)\gamma + B\mu_1 - x_1\mu_1 - x_2\mu_2\right)\left(V_n(x_1, x_2, s+1) - V_n(x_1, x_2, s)\right) \\
&\leq (\lambda + S^{max}\gamma + B\mu_1)\overline{d}_s - \gamma\overline{d}_s + \gamma\overline{d}_2 \qquad \text{(A.82)} \\
&\leq \overline{d}_s
\end{aligned}$$

which holds when $\overline{d}_2 \leq \overline{d}_s$.

CASE-2. $x_1 + x_2 < B$ AND $s + 1 = S^{max}$: Discharged patients in system A leave the system permanently.

$$\begin{aligned}
&V^{\pi'}_{n+1}(x_1, x_2, s+1) - V^{\pi^*}_{n+1}(x_1, x_2, s) \\
&\leq \lambda\overline{d}_s + s\gamma\overline{d}_s + \gamma\left(V_n(x_1, x_2+1, s) - V_n(x_1, x_2, s)\right) \qquad \text{(A.83)} \\
&+ x_1\mu_1\left(V_n(x_1-1, x_2, s+1) - p_1V_n(x_1-1, x_2, s+1) - (1-p_1)V_n(x_1-1, x_2, s)\right) \\
&+ x_2\mu_2\left(V_n(x_1, x_2-1, s+1) - p_2V_n(x_1, x_2-1, s+1) - (1-p_2)V_n(x_1, x_2-1, s)\right) \\
&+ \left((S^{max} - s - 1)\gamma + B\mu_1 - x_1\mu_1 - x_2\mu_2\right)\left(V_n(x_1, x_2, s+1) - V_n(x_1, x_2, s)\right) \\
&\leq (\lambda + S^{max}\gamma + B\mu_1)\overline{d}_s - \gamma\overline{d}_s + \gamma\overline{d}_2 \\
&\leq \overline{d}_s
\end{aligned}$$

which is the result of Case-1.

CASE-3. $x_1 + x_2 = B$ AND $s + 1 < S^{max}$: Upon new patient arrivals, system A can imitate decisions of system B. Let the additional patient in the orbit of system A arrive to the ICU and let system A admit her by early discharging one of the existing patients. The following subcases arise:

SUBCASE-3.A. $x_2 = B$: System A early discharges one the recurring patients early and we have the following:

$$
\begin{aligned}
& V_{n+1}^{\pi'}(x_1, x_2, s+1) - V_{n+1}^{\pi^*}(x_1, x_2, s) \\
& \leq \lambda \overline{d}_s + s\gamma \overline{d}_s + \gamma \left(r_2^e + p_2^e V_n(x_1, x_2, s+1) + (1-p_2^e) V_n(x_1, x_2, s) - V_n(x_1, x_2, s)\right) \\
& + x_2\mu_2 \overline{d}_s + \left((S^{max} - s - 1)\gamma + B\mu_1 - x_2\mu_2\right) \overline{d}_s \qquad \text{(A.84)} \\
& \leq (\lambda + S^{max}\gamma + B\mu_1)\overline{d}_s - \gamma \overline{d}_s + \gamma \left(r_2^e + p_2^e \overline{d}_s\right) \\
& \leq \overline{d}_s
\end{aligned}
$$

which is true when the following holds:

$$
\frac{r_2^e}{1-p_2^e} \leq \overline{d}_s \qquad \text{(A.85)}
$$

SUBCASE-3.B. $x_1 = B$: System A discharges one the first-time patients early and we have the following:

$$
\begin{aligned}
& V_{n+1}^{\pi'}(x_1, x_2, s+1) - V_{n+1}^{\pi^*}(x_1, x_2, s) \\
& \leq \lambda \overline{d}_s + s\gamma \overline{d}_s + x_1\mu_1 \overline{d}_s + \left((S^{max} - s - 1)\gamma + B\mu_1 - x_1\mu_1\right) \overline{d}_s \qquad \text{(A.86)} \\
& + \gamma \left(r_1^e + p_1^e V_n(x_1 - 1, x_2 + 1, s+1) + (1-p_1^e) V_n(x_1 - 1, x_2 + 1, s) - V_n(x_1, x_2, s)\right) \\
& \leq (\lambda + S^{max}\gamma + B\mu_1)\overline{d}_s - \gamma \overline{d}_s + \gamma \left(r_1^e + p_1^e \overline{d}_s + \overline{d}_{21}\right) \\
& \leq \overline{d}_s
\end{aligned}
$$

which is true when the following holds:

$$
\overline{d}_{21} \leq \overline{d}_s (1-p_1^e) - r_1^e \qquad \text{(A.87)}
$$

CASE-4. $x_1 + x_2 = B$ AND $s+1 = S^{max}$: We get the same results as in Equation A.21 and A.22. The corresponding conditions are as follows:

$$
r_2^e \leq \overline{d}_s \left(\frac{\lambda}{\gamma} p_2^e + 1 - p_2^e\right) \qquad \text{(A.88)}
$$

$$
\overline{d}_{21} \leq \overline{d}_s \left(\frac{\lambda p_1^e}{\gamma} + 1 - p_2^e\right) - r_1^e \qquad \text{(A.89)}
$$

By (A.26), we have

$$
\begin{aligned}
\overline{d}_s &\geq \frac{r_2^e}{1-p_2^e} \\
\overline{d}_2 &= \frac{r_2^e}{1-p_2^e} \qquad \text{(A.90)} \\
\overline{d}_{21} &= \frac{r_2^e (1-p_1^e)}{(1-p_2^e)} - r_1^e
\end{aligned}
$$

A.3.3 Proof of Theorem 2.iii

We will prove that there exists a non-negative upper bound $\overline{d}_2$ for $D(2,0)(\mathbf{x},s)$ for all $(\mathbf{x},s)\in\mathcal{S}$. Let system A be in state (x_1,x_2+1,s) and system B in state (x_1,x_2,s). We let system B follow the optimal strategy and system A not necessarily an optimal policy.

Case-1. $x_1+x_2+1<B$ and $s<S^{max}$: We considered exactly the same case in (A.27) and the condition is:

$$\overline{d}_s \le \frac{r_2^e}{p_2\left(1-p_2^e\right)} \tag{A.91}$$

Case-2. $x_1+x_2+1<B$ and $s=S^{max}$: Please see, A.28.

Case-3. $x_1+x_2+1=B$ and $s<S^{max}$: Consider the following cases:

Subcase-3.a. System B admits both first-time and recurring patients. Let system A admit an arriving patient by discharging the additional recurring patient early. Then, we have the result as in (A.29):

$$\overline{d}_s \le \frac{r_2^e}{1-p_2^e}\left(\frac{\mu_2+(\lambda+s\gamma)\,p_2^e}{\lambda p_2^e+s\gamma p_2^e+\mu_2 p_2}\right) \tag{A.92}$$

Subcase-3.b. System B rejects first-time patients and admits recurring patients:

$$\begin{aligned}
&V_{n+1}^{\pi'}(x_1,x_2+1,s)-V_{n+1}^{\pi^*}(x_1,x_2,s)\\
&\le \lambda\overline{d}_2+s\gamma\left(r_2^e+p_2^e V_n\left(x_1,x_2+1,s\right)+\left(1-p_2^e\right)V_n\left(x_1,x_2+1,s-1\right)-V_n\left(x_1,x_2+1,s-1\right)\right)\\
&+\mu_2\left(p_2V_n(x_1,x_2,s+1)+(1-p_2)V_n(x_1,x_2,s)-V_n(x_1,x_2,s)\right)+\left(x_1\mu_1+x_2\mu_2\right)\overline{d}_2\\
&+\left((S^{max}-s)\gamma+B\mu_1-x_1\mu_1-(x_2+1)\mu_2\right)\left(V_n(x_1,x_2+1,s)-V_n(x_1,x_2,s)\right)\\
&\le\left(\lambda+S^{max}\gamma+B\mu_1\right)\overline{d}_2-\left(s\gamma+\mu_2\right)\overline{d}_2+s\gamma\left(r_2^e+p_2^e\overline{d}_s\right)+\mu_2p_2\overline{d}_s\\
&\le\overline{d}_2
\end{aligned}$$

which is true when

$$\overline{d}_s \le \frac{r_2^e}{1-p_2^e}\left(\frac{\mu_2+s\gamma p_2^e}{s\gamma p_2^e+\mu_2 p_2}\right) \tag{A.93}$$

Case-4. $x_1+x_2+1=B$ and $s=S^{max}$: Discharged patients leave the system permanently.

SUBCASE-4.A. System B admits patients from both types. We let system A admit them by discharging the additional recurring patient early. Then, we have the result of (A.33), that is:

$$\bar{d}_s \leq \frac{r_2^e}{1-p_2^e}\left(\frac{\mu_2+(\lambda+s\gamma)\,p_2^e}{s\gamma p_2^e}\right) \tag{A.94}$$

SUBCASE-4.B. System B rejects first-time patients and admits recurring patients. Let system A admit recurring patients by discharging the additional recurring patient early and reject first-time patients.

$$\begin{aligned}
&V_{n+1}^{\pi'}(x_1,x_2+1,s)-V_{n+1}^{\pi^*}(x_1,x_2,s)\\
&\leq \lambda\bar{d}_2+\mu_2\left(V_n(x_1,x_2,s)-V_n(x_1,x_2,s)\right)+\left(x_1\mu_1+x_2\mu_2\right)\bar{d}_2 \qquad\qquad \text{(A.95)}\\
&+s\gamma\left(r_2^e+p_2^eV_n\left(x_1,x_2+1,s\right)+\left(1-p_2^e\right)V_n\left(x_1,x_2+1,s-1\right)-V_n\left(x_1,x_2+1,s-1\right)\right)\\
&+\left((S^{max}-s)\gamma+B\mu_1-x_1\mu_1-(x_2+1)\mu_2\right)\left(V_n(x_1,x_2+1,s)-V_n(x_1,x_2,s)\right)\\
&\leq\left(\lambda+S^{max}\gamma+B\mu_1\right)\bar{d}_2-\left(s\gamma+\mu_2\right)\bar{d}_2+s\gamma\left(r_2^e+p_2^e\bar{d}_s\right)\\
&\leq\bar{d}_2
\end{aligned}$$

which is true when the following holds:

$$\bar{d}_s \leq \frac{r_2^e}{1-p_2^e}\left(\frac{\mu_2+s\gamma p_2^e}{s\gamma p_2^e}\right) \tag{A.96}$$

We have the following:

$$\begin{aligned}
\bar{d}_s \leq \min\Bigg\{&\frac{r_2^e}{p_2(1-p_2^e)},\ \frac{r_2^e}{(1-p_2^e)}\left(\frac{(\lambda+s\gamma)\,p_2^e+\mu_2}{(\lambda+s\gamma)\,p_2^e+\mu_2p_2}\right),\\
&\frac{r_2^e}{1-p_2^e}\left(\frac{\mu_2+s\gamma p_2^e}{s\gamma p_2^e+\mu_2p_2}\right),\ \frac{r_2^e}{1-p_2^e}\left(\frac{\mu_2+(\lambda+s\gamma)\,p_2^e}{s\gamma p_2^e}\right) \qquad \text{(A.97)}\\
&\frac{r_2^e}{1-p_2^e}\left(\frac{\mu_2+s\gamma p_2^e}{s\gamma p_2^e}\right)\Bigg\}
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{r_2^e\left(\mu_2+(1-B\mu_1)\right)}{p_2^e\left(1-p_2^e\right)}\leq\min\Bigg\{&\frac{r_2^e}{p_2(1-p_2^e)},\ \frac{r_2^e}{(1-p_2^e)}\left(\frac{(\lambda+s\gamma)\,p_2^e+\mu_2}{(\lambda+s\gamma)\,p_2^e+\mu_2p_2}\right),\\
&\frac{r_2^e}{1-p_2^e}\left(\frac{\mu_2+s\gamma p_2^e}{s\gamma p_2^e+\mu_2p_2}\right),\ \frac{r_2^e}{1-p_2^e}\left(\frac{\mu_2+(\lambda+s\gamma)\,p_2^e}{s\gamma p_2^e}\right)
\end{aligned}$$

$$\frac{r_2^e}{1-p_2^e}\left(\frac{\mu_2+s\gamma p_2^e}{s\gamma p_2^e}\right)\Bigg\}$$

Therefore, we set

$$\overline{d}_s \leq \frac{r_2^e\left(\mu_2+\left(1-B\mu_1\right)p_2^e\right)}{p_2^e\left(1-p_2^e\right)} \tag{A.98}$$

A.3.4 Proof of Theorem 2.iv

We have the same results as in the Proof of 1.iv. That is:

$$\left(\lambda r_2^e+\mu_1 r_1^e\right) \leq \left(\frac{r_2^e}{1-p_2^e}\right)\left(\lambda\left(1-p_1^e\right)+\mu_2-\mu_1 p_1^e\right)$$

$$\overline{d}_s \leq \left(\frac{r_2^e}{1-p_2^e}\right)\frac{\left(\mu_2-\mu_1 p_1^e\right)}{\left(1-B\mu_1\right)\left(p_2^e-p_1^e\right)+\left(\mu_2 p_2-\mu_1 p_1\right)} - \frac{\left(\left(1-B\mu_1\right)r_2^e+\mu_1 r_1^e\right)}{C_1} \tag{A.99}$$

where $C_1 = \min\left\{\gamma\left(p_2^e-p_1^e\right),\left(\mu_2 p_2-\mu_1 p_1\right)\right\}$.

To summarize, we have the following by (A.81), (A.90), (A.98) and (A.99):

$$\frac{r_2^e}{1-p_2^e} \leq \overline{d}_s \tag{A.100}$$

$$\leq \min\left\{\frac{\left(\lambda+\mu_2\right)r_2-\lambda r_1^e}{\lambda r_2^e+B\mu_2 p_2},\right. \tag{A.101}$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{\mu_2+\left(1-B\mu_1\right)p_2^e}{p_2^e}\right), \tag{A.102}$$

$$\left(\frac{r_2^e}{1-p_2^e}\right)\left(\frac{\mu_2-\mu_1 p_1^e}{\left(1-B\mu_1\right)\left(p_2^e-p_1^e\right)+\left(\mu_2 p_2-\mu_1 p_1\right)}\right) \tag{A.103}$$

$$\left. - \left(\frac{\left(1-B\mu_1\right)r_2^e+\mu_1 r_1^e}{C_1}\right)\right\}$$

This completes the proof of Theorem 2.

Appendix B

B.1 Proof of Lemma 1

Consider the following NLP which is NLP (6.3) with the additional constraint $y_{i3} = 0$ for all $i \in \mathcal{I}$.

$$\min_{\substack{w_{il}, y_{il}, X_i \\ i\in\mathcal{I}, l\in\{1,2,3\}}} \quad \sum_{i\in\mathcal{I}} \Big(\left(r_{i1}w_{i3} + r^e_{i1}p^0_{i1}w_{i2}\right)\lambda_i + \left(r_{i2}y_{i3} + r^e_{i2}p^0_{i2}y_{i2}\right)\gamma_i X_i \Big) \tag{B.1a}$$

$$\text{s.t.} \quad X_i = \frac{\left(p_{i1}w_{i1} + p^e_{i1}w_{i2}\right)p^0_{i1}\lambda_i}{\gamma_i\left((1-p^0_{i2}p_{i2})y_{i1} + (1-p^0_{i2}p^e_{i2})y_{i2}\right)} \geq 0, \quad \forall i \in \mathcal{I}, \tag{B.1b}$$

$$\sum_{i\in\mathcal{I}} \left(\frac{(1-w_{i3})\,\lambda_i}{\mu^c_{i1}} + \frac{p^0_{i1}w_{i1}\lambda_i}{\mu^{sc}_{i1}} + \frac{(1-y_{i3})\gamma_i X_i}{\mu^c_{i2}} + \frac{p^0_{i2}y_{i1}\gamma_i X_i}{\mu^{sc}_{i2}} \right) \leq B, \tag{B.1c}$$

$$\sum_{l=1}^{3} w_{il} = 1, \quad \sum_{l=1}^{3} y_{il} = 1, \quad \forall i \in \mathcal{I}, \tag{B.1d}$$

$$y_{i3} = 0, \quad \forall i \in \mathcal{I}, \tag{B.1e}$$

$$w_{il} \geq 0, \quad y_{il} \geq 0, \quad \forall i \in \mathcal{I},\ l \in \{1,2,3\}. \tag{B.1f}$$

By (B.1d), (B.1e), and (B.1f), X_i is bounded for all $i \in \mathcal{I}$. Therefore, the decision variables of the NLP (B.1) have a bounded and closed (and thus compact) domain. Because the constraints of the NLP (B.1) are continuous functions of the decision variables, there exists an optimal solution to the NLP (B.1).

On the one hand, any feasible solution to NLP (B.1) is also a feasible solution to NLP (6.3) with the same objective function value. On the other hand, we will show that under Condition 4, for any feasible solution to NLP (6.3), there exists a feasible solution to NLP (B.1) with a less than or equal objective function value. This implies that an optimal solution to NLP (B.1) is also an optimal solution to NLP (6.3) under Condition 4, which will give us the desired result.

Let $(\mathbf{w}, \mathbf{y}, \mathbf{X})$ be a feasible solution to NLP (6.3) with the objective function value z. Let $(\mathbf{w}', \mathbf{y}', \mathbf{X}')$ be such that $\mathbf{w}' := \mathbf{w}$, $y'_{i1} := y_{i1}$, $y'_{i2} := y_{i2} + y_{i3}$, $y'_{i3} := 0$, and X'_i

is defined by the constraint (B.1b) for all $i \in \mathcal{I}$. We will prove that $(\mathbf{w}', \mathbf{y}', \mathbf{X}')$ is a feasible solution to NLP (B.1) with objective function value z' such that $z' \leq z$. By construction, $(\mathbf{w}', \mathbf{y}', \mathbf{X}')$ satisfies the constraints (B.1b), (B.1d), (B.1e), and (B.1f). Therefore, we will next prove that $(\mathbf{w}', \mathbf{y}', \mathbf{X}')$ satisfies the capacity constraint (B.1c). We let $\delta_i := X_i - X_i'$ for all $i \in \mathcal{I}$ and notice that $\delta_i \geq 0$ because $y_{i2}' \geq y_{i2}$. Specifically,

$$\begin{aligned}
\delta_i &= X_i - X_i' \\
&= \frac{\left(p_{i1} w_{i1} + p_{i1}^e w_{i2}\right) p_{i1}^0 \lambda_i}{\gamma_i \left((1 - p_{i2}^0 p_{i2}) y_{i1} + (1 - p_{i2}^0 p_{i2}^e) y_{i2}\right)} - \frac{\left(p_{i1} w_{i1}' + p_{i1}^e w_{i2}'\right) p_{i1}^0 \lambda_i}{\gamma_i \left((1 - p_{i2}^0 p_{i2}) y_{i1}' + (1 - p_{i2}^0 p_{i2}^e) y_{i2}'\right)} \\
&= \frac{\left(p_{i1} w_{i1} + p_{i1}^e w_{i2}\right) p_{i1}^0 \lambda_i}{\gamma_i \left((1 - p_{i2}^0 p_{i2}) y_{i1} + (1 - p_{i2}^0 p_{i2}^e) y_{i2}\right)} \\
&\qquad - \frac{\left(p_{i1} w_{i1} + p_{i1}^e w_{i2}\right) p_{i1}^0 \lambda_i}{\gamma_i \left((1 - p_{i2}^0 p_{i2}) y_{i1} + (1 - p_{i2}^0 p_{i2}^e) \left(y_{i2} + y_{i3}\right)\right)} \geq 0.
\end{aligned} \tag{B.2}$$

Then the left-hand-side (LHS) of (B.1c) under $(\mathbf{w}', \mathbf{y}', \mathbf{X}')$ is equal to

$$\begin{aligned}
&\sum_{i \in \mathcal{I}} \left(\frac{\left(1 - w_{i3}'\right) \lambda_i}{\mu_{i1}^c} + \frac{p_{i1}^0 w_{i1}' \lambda_i}{\mu_{i1}^{sc}} + \frac{\gamma_i X_i'}{\mu_{i2}^c} + \frac{p_{i2}^0 y_{i1}' \gamma_i X_i'}{\mu_{i2}^{sc}} \right) \\
&\quad = \sum_{i \in \mathcal{I}} \left(\frac{\left(1 - w_{i3}\right) \lambda_i}{\mu_{i1}^c} + \frac{p_{i1}^0 w_{i1} \lambda_i}{\mu_{i1}^{sc}} + \frac{\gamma_i \left(X_i - \delta_i\right)}{\mu_{i2}^c} + \frac{p_{i2}^0 y_{i1} \gamma_i \left(X_i - \delta_i\right)}{\mu_{i2}^{sc}} \right)
\end{aligned} \tag{B.3}$$

Because $(\mathbf{w}, \mathbf{y}, \mathbf{X})$ also satisfies (B.1c), if

$$\frac{\gamma_i \left(X_i - \delta_i\right)}{\mu_{i2}^c} + \frac{p_{i2}^0 y_{i1} \gamma_i \left(X_i - \delta_i\right)}{\mu_{i2}^{sc}} \leq \frac{\gamma_i \left(1 - y_{i3}\right) X}{\mu_{i2}^c} + \frac{p_{i2}^0 y_{i1} \gamma_i X_i}{\mu_{i2}^{sc}}, \qquad \forall i \in \mathcal{I}, \tag{B.4}$$

then the sum in (B.3) is less than or equal to B, which implies that $(\mathbf{w}', \mathbf{y}', \mathbf{X}')$ satisfies the constraint (B.1c). For a fixed $i \in \mathcal{I}$, the inequality in (B.4) is equivalent to

$$\begin{aligned}
& -\frac{\delta_i}{\mu_{i2}^c} - \frac{p_{i2}^0 y_{i1} \delta_i}{\mu_{i2}^{sc}} \leq -\frac{y_{i3} X_i}{\mu_{i2}^c} \\
\iff \quad & \frac{\delta_i}{\mu_{i2}^c} + \frac{p_{i2}^0 y_{i1} \delta_i}{\mu_{i2}^{sc}} \geq \frac{y_{i3} X_i}{\mu_{i2}^c} \\
\iff \quad & \delta_i \left(\frac{1}{\mu_{i2}^c} + \frac{p_{i2}^0 y_{i1}}{\mu_{i2}^{sc}} \right) \geq \frac{y_{i3} X_i}{\mu_{i2}^c}
\end{aligned} \tag{B.5}$$

Let us define

$$C_1 := \left(p_{i1} w_{i1} + p_{i1}^e w_{i2}\right) p_{i1}^0 \lambda_i$$

$$C_2 := (1 - p^0_{i1} p_{i2}) y_{i1} + (1 - p^0_{i1} p^e_{i2}) y_{i2}$$

$$C_3 := (1 - p^0_{i2} p_{i2}) y_{i1} + (1 - p^0_{i2} p^e_{i2}) (y_{i2} + y_{i3})$$

Then, by (6.3b) and (B.2), we have

$$X_i = \frac{C_1}{\gamma_i C_2}, \qquad \delta_i = \frac{C_1}{\gamma_i}\left(\frac{1}{C_2} - \frac{1}{C_3}\right) \qquad \forall i \in \mathcal{I},$$

respectively. Therefore, the condition (B.5) becomes equivalent to

$$\begin{aligned}
C_1\left(\frac{1}{C_2} - \frac{1}{C_3}\right)\left(\frac{1}{\mu^c_{i2}} + \frac{p^0_{i2} y_{i1}}{\mu^{sc}_{i2}}\right) &\geq \frac{y_{i3}}{\mu^c_{i2}} \frac{C_1}{C_2} \\
\left(\frac{C_3 - C_2}{C_2 C_3}\right)\left(\frac{\mu^{sc}_{i2} + \mu^c_{i2} p^0_{i2} y_{i1}}{\mu^{sc}_{i2}\mu^c_{i2}}\right) &\geq \frac{y_{i3}}{\mu^c_{i2}} \frac{1}{C_2} \\
\left(\frac{C_3 - C_2}{C_3}\right)\left(\frac{\mu^{sc}_{i2} + \mu^c_{i2} p^0_{i2} y_{i1}}{\mu^{sc}_{i2}}\right) &\geq y_{i3}, \\
\left(\frac{(1 - p^0_{i2} p^e_{i2}) y_{i3}}{(1 - p^0_{i2} p_{i2}) y_{i1} + (1 - p^0_{i2} p^e_{i2}) (y_{i2} + y_{i3})}\right)\left(\frac{\mu^{sc}_{i2} + \mu^c_{i2} p^0_{i2} y_{i1}}{\mu^{sc}_{i2}}\right) &\geq y_{i3} \\
\left(\frac{(1 - p^0_{i2} p^e_{i2})}{(1 - p^0_{i2} p_{i2}) y_{i1} + (1 - p^0_{i2} p^e_{i2}) (y_{i2} + y_{i3})}\right)\left(\frac{\mu^{sc}_{i2} + \mu^c_{i2} p^0_{i2} y_{i1}}{\mu^{sc}_{i2}}\right) &\geq 1 \\
(1 - p^0_{i2} p^e_{i2}) \mu^{sc}_{i2} + (1 - p^0_{i2} p^e_{i2}) \mu^c_{i2} p^0_{i2} y_{i1} &\geq (1 - p^0_{i2} p_{i2}) y_{i1} \mu^{sc}_{i2} \\
&\quad + (1 - p^0_{i2} p^e_{i2}) \mu^{sc}_{i2} (y_{i2} + y_{i3}) \\
(1 - p^0_{i2} p^e_{i2}) \mu^{sc}_{i2} + (1 - p^0_{i2} p^e_{i2}) \mu^c_{i2} p^0_{i2} y_{i1} &\geq (1 - p^0_{i2} p_{i2}) y_{i1} \mu^{sc}_{i2} \\
&\quad + (1 - p^0_{i2} p^e_{i2}) \mu^{sc}_{i2} (1 - y_{i1}) \\
(1 - p^0_{i2} p^e_{i2}) \mu^c_{i2} p^0_{i2} y_{i1} &\geq (1 - p^0_{i2} p_{i2}) y_{i1} \mu^{sc}_{i2} \\
&\quad - (1 - p^0_{i2} p^e_{i2}) \mu^{sc}_{i2} y_{i1} \\
(1 - p^0_{i2} p^e_{i2}) \mu^c_{i2} p^0_{i2} &\geq \mu^{sc}_{i2} (1 - p^0_{i2} p_{i2} - 1 + p^0_{i2} p^e_{i2}) \\
(1 - p^0_{i2} p^e_{i2}) \mu^c_{i2} p^0_{i2} &\geq \mu^{sc}_{i2} p^0_{i2} (p^e_{i2} - p_{i2}) \\
\frac{\mu^c_{i2}}{\mu^{sc}_{i2}} &\geq \frac{p^e_{i2} - p_{i2}}{1 - p^0_{i2} p^e_{i2}}
\end{aligned}$$

where in the second line, y_{i3} on both side cancels out, in the fourth line $(y_{i2} + y_{i3})$ is replaced with $(1 - y_{i1})$, in the fifth line $(1 - p^0_{i2} p^e_{i2}) \mu^{sc}_{i2}$ on both side cancel out. The resulting inequality holds under Condition 4. Therefore, under Condition 4, $(\mathbf{w}', \mathbf{y}', \mathbf{X}')$ is a feasible solution to NLP (B.1).

Finally,

$$z' = \sum_{i \in \mathcal{I}} \left(r_{i1} w'_{i3} + r^e_{i1} p^0_{i1} w'_{i2}\right) \lambda_i + r^e_{i2} p^0_{i2} y'_{i2} \gamma_i X'_i$$

$$
\begin{aligned}
&= \sum_{i\in\mathcal{I}} \left(r_{i1}w_{i3} + r^e_{i1}p^0_{i1}w_{i2}\right)\lambda_i + r^e_{i2}p^0_{i2}(y_{i2}+y_{i3})\gamma_i(X_i-\delta_i) \\
&= \sum_{i\in\mathcal{I}} \left(r_{i1}w_{i3} + r^e_{i1}p^0_{i1}w_{i2}\right)\lambda_i \\
&\qquad\qquad + \left(r_{i2}y_{i3} + r^e_{i2}p^0_{i2}y_{i2}\right)\gamma_i X_i + (r^e_{i2}p^0_{i2} - r_{i2})y_{i3}\gamma_i X_i - r^e_{i2}p^0_{i2}(y_{i2}+y_{i3})\gamma_i\delta_i \\
&= z - \sum_{i\in\mathcal{I}}(r_{i2} - r^e_{i2}p^0_{i2})y_{i3}\gamma_i X_i + r^e_{i2}p^0_{i2}(y_{i2}+y_{i3})\gamma_i\delta_i \\
&\leq z,
\end{aligned}
$$

where the inequality follows from the fact that $r_{i2} \geq r^e_{i2}$ (see the second to last paragraph of Chapter 3) and $\delta_i \geq 0$ for all $i \in \mathcal{I}$.

B.2 Proof of Lemma 2

Suppose that Conditions 4 and 5 hold and the system is overloaded, that is,

$$
\sum_{i\in\mathcal{I}} \lambda_i \left(\frac{1}{\mu^c_{i1}} + \frac{p^0_{i1}}{\mu^{sc}_{i1}} + \frac{p_{i1}p^0_{i1}}{1-p^0_{i2}p_{i2}}\left(\frac{1}{\mu^c_{i2}} + \frac{p^0_{i2}}{\mu^{sc}_{i2}}\right)\right) > B.
$$

By Lemma 1, there exists an optimal solution to NLP (6.3), denoted by $(\mathbf{w},\mathbf{y},\mathbf{X})$, such that $y_{i3} = 0$ for all $i \in \mathcal{I}$. Let z denote the optimal NLP (6.3) objective function value. We will show that the capacity constraint cannot be non-binding under $(\mathbf{w},\mathbf{y},\mathbf{X})$.

Suppose that the capacity constraint is not binding under $(\mathbf{w},\mathbf{y},\mathbf{X})$, that is, suppose that

$$
\sum_{i\in\mathcal{I}} \left(\frac{(1-w_{i3})\lambda_i}{\mu^c_{i1}} + \frac{p^0_{i1}w_{i1}\lambda_i}{\mu^{sc}_{i1}} + \frac{\gamma_i X_i}{\mu^c_{i2}} + \frac{p^0_{i2}y_{i1}\gamma_i X_i}{\mu^{sc}_{i2}}\right) < B. \tag{B.6}
$$

Because the system is overloaded and the capacity constraint is not binding, some patients must be either rejected or early discharged. Let $\mathcal{I}_0 \subset \mathcal{I}$ denote the set of patient types that are rejected or early discharged. Then, it must be the case that $\mathcal{I}_0 \neq \emptyset$, $\min\{w_{i1}, y_{i1}\} < 1$ for all $i \in \mathcal{I}_0$, and $w_{i1} = y_{i1} = 1$ for all $i \in \mathcal{I}\setminus\mathcal{I}_0$.

Next, we will construct a new feasible solution to NLP (6.3), denoted by $(\mathbf{w}',\mathbf{y}',\mathbf{X}')$, whose objective function value, denoted by z', is such that $z' < z$. Therefore, we will show that $(\mathbf{w},\mathbf{y},\mathbf{X})$ cannot be an optimal solution, which, in turn, will imply that the capacity constraint must be binding under $(\mathbf{w},\mathbf{y},\mathbf{X})$.

We let $w'_{il} = w_{il}$ and $y'_{il} = y_{il}$ (and thus $X'_i = X_i$ by (6.3b)) for all $i \in \mathcal{I} \setminus \mathcal{I}_0$ and $l \in \{1, 2, 3\}$. Let

$$\mathcal{I}_1 := \{i \in \mathcal{I}_0 : w_{i1} < 1, w_{i3} > 0\},$$
$$\mathcal{I}_2 := \{i \in \mathcal{I}_0 : w_{i1} < 1, w_{i3} = 0\},$$
$$\mathcal{I}_3 := \{i \in \mathcal{I}_0 : w_{i1} = 1\}.$$

Then, $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ is a disjoint partition of $\mathcal{I}_0$. Let us consider an arbitrary $i \in \mathcal{I}_0$.

- If $i \in \mathcal{I}_1$, we let $w'_{i1} = w_{i1} + \epsilon_i$, $w'_{i2} = w_{i2}$, $w'_{i3} = w_{i3} - \epsilon_i$ where $\epsilon_i \in (0, w_{i3}]$, and $y'_{il} = y_{il}$ for all $l \in \{1, 2, 3\}$.

- If $i \in \mathcal{I}_2$, we let $w'_{i1} = w_{i1} + \epsilon_i$, $w'_{i2} = w_{i2} - \epsilon_i$, $w'_{i3} = w_{i3}$ where $\epsilon_i \in (0, w_{i2}]$, and $y'_{il} = y_{il}$ for all $l \in \{1, 2, 3\}$.

- If $i \in \mathcal{I}_3$, we let $w'_{il} = w_{il}$ for all $l \in \{1, 2, 3\}$, $y'_{i1} = y_{i1} + \epsilon_i$, $y'_{i2} = y_{i2} - \epsilon_i$, and $y'_{i3} = y_{i3} = 0$ where $\epsilon_i \in (0, y_{i2}]$.

Finally, X'_i is uniquely defined by (6.3b) for all $i \in \mathcal{I}_0$. Observe that $(\mathbf{w}', \mathbf{y}', \mathbf{X}')$ satisfies the constraints (6.3b), (6.3d), and (6.3e) by construction.

Next, we will show that there exists a strictly positive vector $(\epsilon_i, i \in \mathcal{I}_0)$ under which $(\mathbf{w}', \mathbf{y}', \mathbf{X}')$ satisfies the constraint (6.3c) and thus it is a feasible solution to NLP (6.3). By construction and because $p_{ik} \le p^e_{ik}$ for all $i \in \mathcal{I}$ and $k \in \{1, 2\}$, we have

$$X'_i \le \frac{p^e_{i1} p^0_{i1} \lambda_i}{\gamma_i \left(1 - p^0_{i2} p^e_{i2}\right)}, \qquad \forall i \in \mathcal{I}, \tag{B.7}$$

and

$$X'_i - X_i = \frac{p_{i1} p^0_{i1} \lambda_i \epsilon_i}{\gamma_i \left((1 - p^0_{i2} p_{i2}) y_{i1} + (1 - p^0_{i2} p^e_{i2}) y_{i2}\right)} \le \frac{p_{i1} p^0_{i1} \lambda_i}{\gamma_i (1 - p^0_{i2} p^e_{i2})} \epsilon_i, \quad \forall i \in \mathcal{I}_1, \tag{B.8a}$$

$$X'_i - X_i = \frac{(p_{i1} - p^e_{i1}) p^0_{i1} \lambda_i \epsilon_i}{\gamma_i \left((1 - p^0_{i2} p_{i2}) y_{i1} + (1 - p^0_{i2} p^e_{i2}) y_{i2}\right)} \le 0, \quad \forall i \in \mathcal{I}_2, \tag{B.8b}$$

$$X'_i - X_i = \frac{\left(p_{i1} w_{i1} + p^e_{i1} w_{i2}\right) p^0_{i1} \lambda_i}{\gamma_i \left((1 - p^0_{i2} p_{i2})(y_{i1} + \epsilon_i) + (1 - p^0_{i2} p^e_{i2})(y_{i2} - \epsilon_i)\right)}$$

$$- \frac{(p_{i1}w_{i1} + p^e_{i1}w_{i2})\, p^0_{i1}\lambda_i}{\gamma_i\left((1-p^0_{i2}p_{i2})y_{i1} + (1-p^0_{i2}p^e_{i2})y_{i2}\right)} \le 0, \quad \forall i \in \mathcal{I}_3. \quad \text{(B.8c)}$$

Let

$$\begin{aligned}
C_1 &:= \sum_{i\in\mathcal{I}\setminus\mathcal{I}_0} \left(\frac{(1-w_{i3})\,\lambda_i}{\mu^c_{i1}} + \frac{p^0_{i1}w_{i1}\lambda_i}{\mu^{sc}_{i1}} + \frac{\gamma_i X_i}{\mu^c_{i2}} + \frac{p^0_{i2}y_{i1}\gamma_i X_i}{\mu^{sc}_{i2}} \right),\\
C_2 &:= \sum_{i\in\mathcal{I}_0} \left(\frac{(1-w_{i3})\,\lambda_i}{\mu^c_{i1}} + \frac{p^0_{i1}w_{i1}\lambda_i}{\mu^{sc}_{i1}} + \frac{\gamma_i X_i}{\mu^c_{i2}} + \frac{p^0_{i2}y_{i1}\gamma_i X_i}{\mu^{sc}_{i2}} \right),\\
C_3 &:= \sum_{i\in\mathcal{I}\setminus\mathcal{I}_0} \Big(\big(r_{i1}w_{i3} + r^e_{i1}p^0_{i1}w_{i2}\big)\,\lambda_i + \big(r_{i2}y_{i3} + r^e_{i2}p^0_{i2}y_{i2}\big)\,\gamma_i X_i \Big),\\
C_4 &:= \sum_{i\in\mathcal{I}_0} \Big(\big(r_{i1}w_{i3} + r^e_{i1}p^0_{i1}w_{i2}\big)\,\lambda_i + \big(r_{i2}y_{i3} + r^e_{i2}p^0_{i2}y_{i2}\big)\,\gamma_i X_i \Big).
\end{aligned}$$

Then, $C_1 + C_2 < B$ by (B.6) and $C_3 + C_4 = z$. The LHS of (6.3c) under $(\mathbf{w}', \mathbf{y}', \mathbf{X}')$ is

$$\begin{aligned}
&C_1 + \sum_{i\in\mathcal{I}_0} \left(\frac{(1-w'_{i3})\,\lambda_i}{\mu^c_{i1}} + \frac{p^0_{i1}w'_{i1}\lambda_i}{\mu^{sc}_{i1}} + \frac{\gamma_i X'_i}{\mu^c_{i2}} + \frac{p^0_{i2}y'_{i1}\gamma_i X'_i}{\mu^{sc}_{i2}} \right)\\
&\quad\le C_1 + C_2 + \sum_{i\in\mathcal{I}_0} \left(\frac{\lambda_i\epsilon_i}{\mu^c_{i1}} + \frac{p^0_{i1}\lambda_i\epsilon_i}{\mu^{sc}_{i1}} + \left(\frac{1}{\mu^c_{i2}} + \frac{p^0_{i2}y_{i1}}{\mu^{sc}_{i2}}\right)\gamma_i(X'_i - X_i) + \frac{p^0_{i2}\gamma_i X'_i\epsilon_i}{\mu^{sc}_{i2}} \right)\\
&\quad\le C_1 + C_2 + \sum_{i\in\mathcal{I}_0} \left(\frac{\lambda_i}{\mu^c_{i1}} + \frac{p^0_{i1}\lambda_i}{\mu^{sc}_{i1}} + \left(\frac{1}{\mu^c_{i2}} + \frac{p^0_{i2}}{\mu^{sc}_{i2}}\right)\frac{p_{i1}p^0_{i1}\lambda_i}{1-p^0_{i2}p^e_{i2}} + \frac{p^e_{i1}p^0_{i1}p^0_{i2}\lambda_i}{\left(1-p^0_{i2}p^e_{i2}\right)\mu^{sc}_{i2}} \right)\epsilon_i,
\end{aligned} \quad \text{(B.9)}$$

where the second inequality follows from (B.7) and (B.8). Because $C_1 + C_2 < B$, there exists a sufficiently small and strictly positive vector $(\epsilon_i, i \in \mathcal{I}_0)$ such that the term in the right-hand-side of (B.9) is less than or equal to B. Therefore, under the aforementioned vector $(\epsilon_i, i \in \mathcal{I}_0)$, $(\mathbf{w}', \mathbf{y}', \mathbf{X}')$ satisfies the constraint (6.3c) and thus it is a feasible solution to NLP (6.3).

Next, let us consider the objective function value of $(\mathbf{w}', \mathbf{y}', \mathbf{X}')$. We have

$$\begin{aligned}
z' - z &= C_3 + \sum_{i\in\mathcal{I}_0} \Big(\big(r_{i1}w'_{i3} + r^e_{i1}p^0_{i1}w'_{i2}\big)\,\lambda_i + \big(r_{i2}y'_{i3} + r^e_{i2}p^0_{i2}y'_{i2}\big)\,\gamma_i X'_i \Big) - z\\
&= \sum_{j=1}^{3}\sum_{i\in\mathcal{I}_j} \Big(\big(r_{i1}w'_{i3} + r^e_{i1}p^0_{i1}w'_{i2}\big)\,\lambda_i + \big(r_{i2}y'_{i3} + r^e_{i2}p^0_{i2}y'_{i2}\big)\,\gamma_i X'_i \Big) - C_4\\
&= \sum_{i\in\mathcal{I}_1} \left(-r_{i1}\lambda_i + r^e_{i2}p^0_{i2}y_{i2}\frac{p_{i1}p^0_{i1}\lambda_i}{(1-p^0_{i2}p_{i2})y_{i1} + (1-p^0_{i2}p^e_{i2})y_{i2}} \right)\epsilon_i
\end{aligned} \quad \text{(B.10)}$$

$$+\sum_{i\in\mathcal{I}_2}\left(-r^e_{i1}p^0_{i1}\lambda_i + r^e_{i2}p^0_{i2}y_{i2}\frac{(p_{i1}-p^e_{i1})p^0_{i1}\lambda_i}{(1-p^0_{i2}p_{i2})y_{i1}+(1-p^0_{i2}p^e_{i2})y_{i2}}\right)\epsilon_i \qquad \text{(B.11)}$$

$$+\sum_{i\in\mathcal{I}_3}\left(r^e_{i2}p^0_{i2}(y_{i2}-\epsilon_i)\frac{(p_{i1}w_{i1}+p^e_{i1}w_{i2})\,p^0_{i1}\lambda_i}{(1-p^0_{i2}p_{i2})(y_{i1}+\epsilon_i)+(1-p^0_{i2}p^e_{i2})(y_{i2}-\epsilon_i)}\right.$$

$$\left. - r^e_{i2}p^0_{i2}y_{i2}\frac{(p_{i1}w_{i1}+p^e_{i1}w_{i2})\,p^0_{i1}\lambda_i}{(1-p^0_{i2}p_{i2})y_{i1}+(1-p^0_{i2}p^e_{i2})y_{i2}}\right), \qquad \text{(B.12)}$$

$$< 0. \qquad \text{(B.13)}$$

Because $(\epsilon_i, i\in\mathcal{I}_0)$ is a strictly positive vector and $p_{ik}\le p^e_{ik}$ for all $i\in\mathcal{I}$ and $k\in\{1,2\}$, both the sum in (B.11) and the sum in (B.12) are strictly less than 0. Furthermore, the sum in (B.10) is less than or equal to 0 by Condition 5. Consequently, the strict inequality in (B.13) follows.

The strict inequality in (B.13) implies that $(\mathbf{w},\mathbf{y},\mathbf{X})$ is not an optimal NLP (6.3) solution, which is a contradiction. Therefore, the capacity constraint must be binding under $(\mathbf{w},\mathbf{y},\mathbf{X})$ when the Conditions 4 and 5 hold.

Appendix C

Table C.1: Percentage Improvement over Reject Policy - One-Patient-Type ICU, Hospital A

			Hospital A, Bed Capacity $B = 51$							
			Demand							
			$\rho = 0.9$	$\rho = 1$	$\rho = 1.25$	$\rho = 1.5$	$\rho = 0.9$	$\rho = 1$	$\rho = 1.25$	$\rho = 1.5$
			Cost Case 1				Cost Case 2			
		Policy	Mean LOS of Readmitted Patients is the same as First-time Patients' LOS							
$\frac{1}{\gamma}$	2 Days	Fluid NLP	-	-	45.9	31.3	-	-	44.3	36.9
		Early Disch	45.8	33.6	7.8	-4.5	-55.7	-91.3	-158.3	-162.2
		OneBed Policy	10.4	11.0	11.4	11.4	29.3	30.3	31.0	31.2
		Fluid Policy	54.0	44.5	30.1	20.1	29.2	30.3	31.0	31.2
	4 Days	Fluid NLP	-	-	45.8	31.3	-	-	44.0	36.7
		Early Disch	47.3	34.6	8.3	-3.9	-48.9	-87.8	-157.0	-161.3
		OneBed Policy	13.9	13.4	12.9	12.2	30.3	30.3	30.7	31.0
		Fluid Policy	55.0	45.4	29.5	20.3	30.4	30.2	30.7	31.1
	6 Days	Fluid NLP	-	-	45.8	31.2	-	-	44.1	36.5
		Early Disch	47.6	34.7	8.8	-3.6	-48.9	-86.6	-156.1	-161.7
		OneBed Policy	10.5	10.7	11.3	11.3	29.7	29.9	30.7	30.8
		Fluid Policy	55.4	45.5	29.7	20.3	29.7	30.0	30.7	30.7
			Mean LOS of Readmitted Patients is 25% higher than First-time Patients' LOS							
	2 Days	Fluid NLP	-	-	44.5	29.1	-	-	44.5	36.8
		Early Disch	45.0	31.4	4.0	-7.8	-61.4	-105.1	-175.1	-172.6
		OneBed Policy	11.4	11.5	11.7	11.6	29.4	30.0	31.2	31.1
		Fluid Policy	53.4	46.0	28.1	18.2	29.4	30.1	31.2	31.1
	4 Days	Fluid NLP	-	-	44.4	29.2	-	-	44.3	37.0
		Early Disch	46.6	32.3	4.8	-7.0	-55.6	-100.6	-173.5	-169.9
		OneBed Policy	12.0	11.7	11.8	11.7	30.2	30.0	31.0	31.2
		Fluid Policy	54.5	43.9	28.4	18.5	30.1	30.0	30.9	31.2
	6 Days	Fluid NLP	-	-	44.2	29.2	-	-	43.8	37.0
		Early Disch	46.8	32.8	4.6	-6.8	-55.0	-98.9	-175.6	-169.1
		OneBed Policy	11.3	11.3	11.4	11.7	29.5	29.9	30.4	31.3
		Fluid Policy	54.7	44.0	28.4	18.6	29.5	29.9	30.4	31.3
			Mean LOS of Readmitted Patients is 50% higher than First-time Patients' LOS							
	2 Days	Fluid NLP	-	-	42.7	27.1	-	-	44.2	36.8
		Early Disch	44.6	28.9	0.2	-11.0	-66.9	-119.2	-191.3	-179.9
		OneBed Policy	12.3	11.9	11.9	11.8	29.9	29.9	30.7	31.2
		Fluid Policy	54.1	44.5	25.6	16.2	29.9	29.8	30.7	31.1
	4 Days	Fluid NLP	-	-	42.7	27.1	-	-	44.3	37.0
		Early Disch	46.0	30.6	1.1	-10.3	-62.1	-113.5	-189.9	-177.8
		OneBed Policy	12.7	12.3	12.0	11.9	29.9	30.2	30.8	31.2
		Fluid Policy	54.9	45.6	26.1	16.5	29.9	30.2	30.9	31.3
	6 Days	Fluid NLP	-	-	42.7	27.0	-	-	44.3	36.6
		Early Disch	46.6	31.1	1.2	-10.1	-60.1	-112.1	-189.7	-178.7
		OneBed Policy	13.2	12.5	11.9	11.7	29.9	29.9	30.8	30.9
		Fluid Policy	54.5	45.8	26.2	16.5	29.8	29.8	30.8	30.9

Table C.2: Percentage Improvement over Reject Policy - One-Patient-Type ICU, Hospital B

			Hospital B, Bed Capacity $B = 14$							
			Demand							
			$\rho = 0.9$	$\rho = 1$	$\rho = 1.25$	$\rho = 1.5$	$\rho = 0.9$	$\rho = 1$	$\rho = 1.25$	$\rho = 1.5$
			Cost Case 1				Cost Case 2			
		Policy	Mean LOS of Readmitted Patients is the same as First-time Patients' LOS							
$\frac{1}{\gamma}$	2 Days	Fluid NLP	-	-	49.2	34.6	-	-	57.8	49.6
		Early Disch	17.7	10.8	-7.6	-15.1	-88.7	-101.8	-127.4	-131.1
		OneBed Policy	14.4	14.1	14.4	14.6	37.5	36.9	37.6	38.0
		Fluid Policy	31.4	26.4	14.1	10.8	37.5	36.9	37.6	38.0
	4 Days	Fluid NLP	-	-	49.2	34.6	-	-	57.8	49.8
		Early Disch	22.2	15.6	-3.8	-11.3	-76.9	-88.9	-117.2	-121.1
		OneBed Policy	14.0	14.1	14.4	14.6	36.9	37.0	37.5	38.1
		Fluid Policy	34.2	29.1	15.9	12.5	36.9	37.1	37.5	38.1
	6 Days	Fluid NLP	-	-	49.3	34.6	-	-	57.9	49.7
		Early Disch	24.5	17.5	-2.0	-10.4	-71.4	-84.6	-112.2	-118.8
		OneBed Policy	14.0	14.0	14.5	14.6	36.9	36.8	37.6	37.9
		Fluid Policy	32.8	30.1	16.9	13.2	36.9	36.7	37.6	37.9
			Mean LOS of Readmitted Patients is 25% higher than First-time Patients' LOS							
	2 Days	Fluid NLP	-	-	46.2	32.4	-	-	57.9	49.5
		Early Disch	14.3	7.7	-11.4	-18.9	-101.3	-112.1	-137.9	-140.7
		OneBed Policy	14.1	14.3	14.4	14.5	37.1	37.4	37.8	37.9
		Fluid Policy	29.1	24.0	13.9	10.3	37.2	37.4	37.8	37.9
	4 Days	Fluid NLP	-	-	46.2	32.4	-	-	57.8	49.5
		Early Disch	19.8	12.6	-7.7	-15.2	-87.6	-99.7	-128.0	-131.5
		OneBed Policy	14.0	14.1	14.2	14.4	37.0	37.2	37.3	37.7
		Fluid Policy	32.2	27.0	13.1	9.8	37.0	37.2	37.3	37.7
	6 Days	Fluid NLP	-	-	46.3	32.3	-	-	58.1	49.4
		Early Disch	21.9	14.5	-5.5	-13.8	-81.8	-95.5	-121.7	-128.3
		OneBed Policy	14.0	14.0	14.5	14.4	37.0	36.9	37.8	37.6
		Fluid Policy	33.5	28.1	14.4	10.6	37.0	36.8	37.8	37.6
			Mean LOS of Readmitted Patients is 50% higher than First-time Patients' LOS							
	2 Days	Fluid NLP	-	-	43.9	30.8	-	-	57.8	49.6
		Early Disch	11.4	4.5	-15.4	-21.7	-111.6	-123.2	-148.3	-148.1
		OneBed Policy	14.0	14.2	14.3	14.4	37.1	37.3	37.4	37.7
		Fluid Policy	28.7	24.1	11.2	8.4	37.0	37.3	37.4	37.8
	4 Days	Fluid NLP	-	-	43.9	30.8	-	-	57.9	49.5
		Early Disch	16.9	9.0	-11.0	-18.6	-97.4	-112.0	-137.1	-139.1
		OneBed Policy	13.9	14.1	14.3	14.4	36.8	37.0	37.4	37.6
		Fluid Policy	30.4	24.7	10.2	10.0	36.8	37.0	37.4	37.6
	6 Days	Fluid NLP	-	-	44.1	30.9	-	-	58.2	49.7
		Early Disch	19.3	11.5	-8.8	-17.1	-90.9	-106.4	-131.3	-134.2
		OneBed Policy	14.0	13.8	14.5	14.4	36.8	36.7	37.8	37.8
		Fluid Policy	31.7	26.0	11.3	8.7	36.8	36.7	37.8	37.8