VORTEX TRANSFER IN CHARGED-NEUTRAL SUPERFLUID MIXTURES

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By

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August, 2013

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Bose-Einstein Condensation (BEC) was introduced by Einstein 1925. It took 70 years to confirm BEC by experiments. BEC creates a suitable environment to observe macroscopic-quantum behavior. Condensates consist of ultracold atoms allow physicists to create superfluids and also they allow to manipulate these quantum structures easily. One of the main tool needed to manipulate these structures is synthetic magnetic field. Under the light of these experimental achievements we studied the angular momentum transfer in the N-body systems. First of all, to develop physical intuition, we solved 2-body problem. This problem can be defined as: The system consist of two particles and confined in a ring. Particles interact with each other and charged one coupled to the magnetic field. We used two approaches to solve the system and compared these approaches in the small limit of inter-particle interaction. Finally, we studied N-body systems and vortex transfer in the two-component superfluid mixtures via Gross-Pitaevski equation and Bogoulibov equations. We observed that for various parameters neutral-neutral mixtures do not possess vortex transfer, yet charged-neutral mixtures coupled to the magnetic field experience vortex transfer.

Keywords: Superfluidity, charged-neutral mixture, vortices, vortex transfer.

ÖZET

YÜKLÜ-YÜKSÜZ ÜSTÜNAKIŞKAN KARIŞIMLARINDA GİRDAP AKTARIMI

Semih Kaya
Fizik, Yüksek Lisans
Tez Yöneticisi: Doç. Dr. M. Özgür Oktel
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Bose-Einstein yoğuşması (BEY) 1925 senesinde Einstein tarafndan ileri sürüldü. Bu fiziksel olayı doğrulamak deneycilerin 70 yılını aldı. BEY makroskopik-kuantum davranışını gözlemlemek için uygun bir ortam yaratmaktadr. Aşırı soğuk atomlardan oluşan yoğuşuklar fizikçilere üstünakışkan yaratma ve bu kuantum yapılarını rahatlıkla kullanma imkanı vermiştir. Bu yapıları kullanabilmek için ihtiyaç duyulan ana araçlardan biri yapay manyetik alandır. Bu deneysel başarıların ışığı altında N-parçacıklı sistemlerde açısal momentum aktarımını çalıştık. İlk olarak, fiziksel sezgimizi geliştirmek için 2-parçacık problemini çözdük. Bu problemi şu şekilde tanımlayabiliriz: Sistem iki parçacıktan oluşmuştur ve bir halka üzerindedir. Parçacıklar birbirleriyle etkileşim halindedir ve yüklü olan manyetik alanın etkisi altındadır. Bu problemi çözmek için iki yaklaşım kullandık ve bu yaklaşımları etkileşmenin küçük limitinde karşılaştırdık. Son olarak, Gross-Pitaevski ve Bogoulibov denklemleri aracılığıyla N-parçacklı sistemleri ve iki bileşenli üstünakışkan karışımlarında girdap aktarımını çalıştık. Çeşitli parametreler için yüksüz-yüksüz karışımların girdap aktarımı içermediğini, fakat manyetik alan altındaki yüklü-yüksüz karışımların girdap aktarımı sergilediklerini gözlemledik.

Anahtar sözcükler: Üstünakışkanlık, yüklü-yüksüz karışımlar, girdaplar,vorteksler, girdap aktarımı.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

In this thesis, we investigate a scenerio where Gross-Pitaevski equation is quantitatively and qualitatively wrong. We present vortex transfer conditions between two components of charged-neutral superfluid mixtures where the charged one has vortex. We study this problem for different cases.

In the 2^{nd} chapter, to gain physical insight for vortex transfer, we first studied 2-body problem. We can state the problem as; two particles-one of them is charged and subjected to magnetic field as well as the other one is neutral-interacting via delta potential on a ring. We investigate the angular momentum transfer from charged particle to the neutral one in the ground state. We solve the problem perturbatively and numerically. Then, we compare the results of the two approaches.

In chapter 3, we build necessary background to understand and solve N-body transfer problem. We mention 3 main subjects; Gross-Pitaevski Equation, The Bogoliubov equations and vortices in BEC and applied them to a very basic model to grasp the topics.

In the 4^{th} chapter, we solve the Gross-Pitaevski equation and The Bogoliubov equations in many different systems with use of numerical techniques and try to observe vortex transfer in these systems. We make comparison with another

paper to verify our numerical approach.

Finally, we review our results for 2-body and N-body problems and comment on the obstacles that we tackled.

To make this introduction more complete, we mention the fundamental concepts associated with our study.

1.1 BEC in Ultracold Atoms

Bose Einstein Condensation also known as BEC and ultracold atoms are very important concepts due to their applications and their potentials to produce new ideas in physics.

In 1924, De-Broglie came up with the idea of matter waves [1]. BEC concept in gases was introduced by Einstein in 1925 after reading S. Bose's paper on photon statistics. Einstein immediately combined the two ideas of De-Broglie and Bose. [2]

BEC is an interesting quantum behaviour which is exhibited by N-body systems which obey Bose-Einstein statistics while approaching zero temperature. [3] We can define BEC as; a state where most of the particles in the N-body system occupy the ground state near the zero temperature. We can state BEC in a more mathematical manner; for 3D ideal Bose gas, [4]

$$\rho\lambda^3 > 2.612$$

where ρ is density of gas and λ is De-Broglie wavelength of particles. As a final comment in simple manner, when De-Broglie wavelength of particles are comparable with interatomic spacing between them, BEC starts to occur.

A decade after from Einstein's paper, the physics community was interested in the mysterious phase transition of ^{4}He and was trying to understand in physical concepts what was happening when ^{4}He was cooled down to the specific temperature which called critical temperature. While scientists from the low temperature

physics community were struggling with the problem and trying to find different angles to attack, Fritz London carried the problem onto a different level and brought forward an argument.

Fritz London, a German physicist and known as the founder of novel theory of superconductivity which is constructed on the concept of the macroscopic wave function-the idea which later has been used to provide verification of quantum behaviour at the macroscopic scale- was aware of the Einstein's ideas. ^{4}He is known as boson and London connected the concepts of BEC and Superconductivity to explain superfluidity of ^{4}He. But his ideas lacked mathematical foundations and there were no quantitative predictions. London's innovative arguments had to wait to be put on a mathematical basis until Bogoliubov's 1947 research on the microscopic model of weakly interacting Bose gas was complete. Thus, ^{4}He seems to be suitable playground for experimentalists to observe BEC-related quantum phenomenon. But, due to strong interactions between particles, the fraction of the condensate is only 10%, in contrast with the novel concept of BEC. This handicap sets limitations on control of the experiment. At this point, ultracold atoms enter the stage. [2, 4]

Ultracold atoms which are alkali metals and have integral spin, hold the title of '*ultracold*' due to their critical temperatures between 1nK and 1μK, unlike ^{4}H whose critical temperature is at the order of 1K. The main advantage of ultracold atoms is that, at near zero temperature 99% of the atoms occupy the ground state. That condition gives experimentalists the oppurtunity to examine BEC over a broad extent of scenerios, by this means scientists gain great control on experiments. Regardless of the fact that lowest state is highly populated by the particles, interactions between atoms play an active role in the system. Thus, standard ideal Bose gas model does not work here, This paradigm needs to be modified on the basis of mean field approach. [4–6]

Creating a succesful BEC requires two main elements; cooling down to near zero temperatures and trapping atoms. Reaching very low temperatures and trapping atoms is a very difficult, but highly rewarding quest for physicists. To overcome this challenge, scientists had to modernize their experimental techniques

for cooling and trapping alkali atoms. These efforts yielded a Nobel Prize in physics in 1997. Cloude Cohen Tannoudji, William D. Phillips and Steven Chou was awarded with Nobel Prize, 'For development of methods to cool and trap atoms with laser light'. In 1995, physicists managed to create BEC at JILA and MIT, Carl E. Wieman, Eric A. Cornell and Wolfgang Ketterle won the Nobel Prize in Physics, 'For the achivement of Bose-Einstein condensations in dilute gases of alkali atoms, and for early fundemental studies of the properties of the condensates' in 2001. [7]

After BEC was achieved, superfluids created by using BEC in ultracold atoms are scrutinized by physicists. Subjects of superconductivity can be investigated by using charged Superfluids, thus observing and understanding vortices in Superfluids became a main target for scientists. Rotation of the superfluid is enough to create a vortex, yet there are obstacles in exploring the subjects like quantum Hall effect using rotating charged superfluids. Experimental techniques have limitations for obtaininig high rotation velocity which is crucially needed for quantum Hall effect. Experimentalist tried to find creative methods to overcome this difficulty and Synthetic magnetic field turned out to be one of the possible answers to their problem. [8]

1.2 Superfluid Mixtures And Synthetic Magnetic Fields

Superfluids consisting of two or more different components are called Superfluid mixtures. Since superfluids are made up of alkali atoms, electrical neutrality of atoms seems to be a problem while examining quantum Hall effect via charged superfluids. This difficulty has been overcome by establishing a connection with Lorentz force and Coriolis force with use of mechanical rotation, yet rotational velocity is not sufficient to replicate high magnetic fields required for quantum Hall effect. Synthetic or artificial magnetic field technology is developed to overcome this hardship. [8]

Creating synthetic magnetic fields can be achieved by use of two counter-propagating laser beams. Y.-J. Lin *et al*. [8] succeeded in producing the Synthetic magnetic fields by means of lasers in ultracold atoms of ^{87}Rb. Electro-magnetic field of lasers spatially stimulate alteration between internal atomic states. Thus, laser fields create dressed states. For the center of mass motion, spatial nature of dressed states alters the momentum operator $\boldsymbol{p} = -i\hbar\nabla$ to $\boldsymbol{p} - \boldsymbol{A}$ in n^{th} internal state. In this way, a unit charge's motion in magnetic field can be mimiced by the atomic motion in n^{th} dressed state. The technology allows us to generate high magnetic fields necessitated to enter quantum Hall regime. [8, 9]

Chapter 2

Angular Momentum Transfer in the Two-Body Problem

2.1 Perturbative Approach

In this section, our main objective is to calculate angular momentum transfer from charged particle subjected to the magnetic field to neutral particle via interaction by means of perturbation theory.

We can describe the problem as; two particle confined in a ring, interacting via delta function potential and charged one is coupled to the magnetic field. In order to apply perturbation theory, we need to solve unperturbed Hamiltonian. To solve the interaction-free system, we can identify this problem as; two distinguishable particle problem. Because of distinguishability, we can easily construct interaction-free wave function by solving one-body problem of each particle. After calculating one-body wave functions, we multiply them to obtain the interaction-free wave function.

The solution of the one neutral particle on a ring is straightforward and its wave function given by the expression

$$\psi_n(\phi) = \frac{1}{\sqrt{2\pi R}} e^{in\phi}. \tag{2.1}$$

Now, we must deal with the charged particle on a ring subjected to the magnetic field. This problem, also known as Aharonov-Bohm Effect, has been discussed in many texts. In this problem, we are assuming that the magnetic field is generated by a solenoid.

Standard Hamiltonian for the system is

$$H = \frac{1}{2m}(\frac{\hbar}{i}\nabla - q\vec{A})^2 = \frac{1}{2m}[-\hbar^2\vec{\nabla}^2 + q^2A^2 - \frac{2\hbar}{i}\vec{A}\cdot\vec{\nabla}]. \tag{2.2}$$

We calculated vector potential of solenoid as [10]

$$\vec{A} = \frac{\Phi}{2\pi r}\hat{\phi}. \tag{2.3}$$

Φ is magnetic flux of the solenoid and r is displacement from the center. Now we have to solve eigenvalue problem. Our Schrödinger equation looks like [11]

$$\frac{d^2}{d\phi^2}\psi - 2\beta i\frac{d\psi}{d\phi} + \gamma\psi = 0 \tag{2.4}$$

where

$$\beta = \tfrac{q\Phi}{2\pi\hbar},\ \widetilde{E} = \tfrac{2mR^2E}{\hbar^2} \text{ and } \gamma = \widetilde{E} - \beta^2.$$

β, γ and $\widetilde{E}$ are dimensionless quantities. R is defined as trajectory radius of ring and E as energy. By solving the Schrodinger equation we can obtain eigenstates of the system as

$$\psi_m(\phi) = \frac{1}{\sqrt{2\pi R}} e^{im\phi}. \tag{2.5}$$

m is an integer and can be positive, negative or zero.
Energy eigenstates are

$$E_n = \frac{\hbar^2}{2mR^2}(m-\beta)^2. \tag{2.6}$$

We obtained the eigenfunction and the eigenenergies of the one particle problem. But we are interested in the two particle system. The two particle Hamiltonian is

$$H = -\frac{\hbar^2}{2m_1R^2}\Big[\frac{\partial}{\partial\phi_1} - \beta\Big]^2 - \frac{\hbar^2}{2m_2R^2}\Big[\frac{\partial^2}{\partial{\phi_2}^2}\Big]. \tag{2.7}$$

We take the mass of charged particle as m_1 and neutral particle's mass as m_2. To obtain the eigenfunction of the system, we don't have to solve this Hamiltonian from beginning. We can constuct eigenfunction by simply multiplying single particle eigenfunctions due to two particles' distinguishability.

The two particle eigenfunction is

$$\Psi_{n_1,n_2}(\phi_1,\phi_2) = \frac{1}{2\pi R} e^{i(n_1\phi_1+n_2\phi_2)}. \tag{2.8}$$

We can also write the total energy by summing single particle energies to yields

$$E_{n_1,n_2} = \frac{\hbar^2}{2m_1R^2}[(n_1-\beta)^2 + \frac{m_1}{m_2}{n_2}^2]. \tag{2.9}$$

β contains magnetic flux parameter which belongs to solenoid. Thus, we are free to choose β values. We assume β is in the interval; $-\frac{1}{2} < \beta < \frac{1}{2}$, for these values of β, possible wave function with lowest energy is Ψ_{00}.

Thus far, we have made calculations that exclude the interaction between two particles. Because we need to know interaction-free system's eigenfuntions to apply perturbation theory to transfer problem. Next, we include the interparticle interaction. We assume interaction is in the form of delta function. We define interacting Hamiltonian as

$$H' = U_o\delta(\phi_1-\phi_2) \tag{2.10}$$

where U_0 is in energy units. We can calculate the first order correction to the ground state energy [12]

$$\begin{aligned}
\Delta^{(1)} &= \langle\Psi_{00}|H'|\Psi_{00}\rangle && (2.11)\\
&= \int_0^{2\pi}\int_0^{2\pi} \frac{R^2}{(2\pi R)^2} U_0\delta(\phi_1-\phi_2)d\phi_1 d\phi_2 && (2.12)\\
\Delta^{(1)} &= \frac{U_0}{2\pi}. && (2.13)
\end{aligned}$$

The first order correction to the ground stateis [13]

$$\Psi_{00}^{(1)} = \sum_{n_1}\sum_{n_2} \frac{\langle\Psi_{n_1n_2}^{(0)}|H'|\Psi_{00}^{(0)}\rangle}{E_{00}^{(0)} - E_{n_1n_2}^{(0)}}\Psi_{00}^{(0)} \tag{2.14}$$

where $n_1 \neq 0$ and $n_2 \neq 0$.

Let us calculate matrix element $\langle \Psi^{(0)}_{n_1 n_2} | H' | \Psi^{(0)}_{00} \rangle$

$$= \int_0^{2\pi} \frac{R}{2\pi R} d\phi_1 \int_0^{2\pi} \frac{R}{2\pi R} e^{-i(n_1\phi_1 + n_2\phi_2)} U_0 \delta(\phi_1 - \phi_2) d\phi_2 \quad , \tag{2.15}$$

$$\tag{2.16}$$

taking advantage of the delta function yields to

$$\langle \Psi^{(0)}_{n_1 n_2} | H' | \Psi^{(0)}_{00} \rangle = \frac{U_0}{4\pi^2} \int_0^{2\pi} e^{-i\phi_1(n_1+n_2)} d\phi_1 \tag{2.17}$$

$$= \frac{U_o}{2\pi} \delta_{n_1, -n_2}. \tag{2.18}$$

Hence,

$$\Psi^{(1)}_{00} = \frac{U_0}{2\pi} \frac{2m_1 R^2}{\hbar^2} \sum_{n_1 \neq 0} \frac{1}{\beta^2 - [(n_1 - \beta)^2 + \frac{m_1}{m_2} {n_1}^2]} \Psi^{(0)}_{n_1, -n_1}. \tag{2.19}$$

The ground state eigenfunction with first order correction is

$$\Psi_{G.S.} = \Psi^{(0)}_{00} + \Psi^{(1)}_{00} \tag{2.20}$$

. Also, the ground state eigenket with first order correction is

$$|\Psi_{G.S.}\rangle = |0,0\rangle^{(0)} + |0,0\rangle^{(1)} \tag{2.21}$$

where

$$|0,0\rangle^{(1)} = \frac{U_0 m_1 R^2}{\pi \hbar^2} \sum_{n_1 \neq 0} \frac{1}{\beta^2 - [(n_1 - \beta)^2 + \frac{m_1}{m_2} {n_1}^2]} |n_1, -n_1\rangle^{(0)} \tag{2.22}$$

. We define the angular momentum operator for the second particle as

$$L_{z_2} = \frac{\hbar}{i} \frac{\partial}{\partial \phi_2} \tag{2.23}$$

. We're looking for the expectation value of the second particle's angular momentum

$$\langle L_{z_2} \rangle = \langle \Psi_{G.S.} | L_{z_2} | \Psi_{G.S.} \rangle, \tag{2.24}$$

$$\langle L_{z_2}\rangle = ({}^{(0)}\langle 0,0| + {}^{(1)}\langle 0,0|)L_{z_2}(|0,0\rangle^{(0)} + |0,0\rangle^{(1)}), \tag{2.25}$$

$$\langle L_{z_2}\rangle = -\hbar\left(\frac{U_0 m_1 R^2}{\pi\hbar^2}\right)^2 \sum_{n_1\neq 0} \frac{n_1}{\left[\beta^2 - [(n_1-\beta)^2 + \frac{m_1}{m_2}{n_1}^2]\right]^2}. \tag{2.26}$$

We have almost calculated the expectation value of angular momentum. Now, we have to deal with summation. In order to simplify, we make some arrengements over summation. For instance, there is no need to hold "1" index under n any more.

$$\sum_{n_1\neq 0} \cdots \rightarrow \sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{n}{\left[\beta^2 - [(n-\beta)^2 + \frac{m_1}{m_2}n^2]\right]^2} \tag{2.27}$$

$$= \sum_{n=1}^{\infty} \frac{n}{\left[2m\beta - n^2(\frac{m_1}{m_2}+1)\right]^2} - \frac{n}{\left[2m\beta + n^2(\frac{m_1}{m_2}+1)\right]^2} \tag{2.28}$$

Next we introduce variable $\eta = \frac{m_1}{m_2} + 1$, with some simple algebra summation reduces to

$$\sum_{n=1}^{\infty} \frac{8\beta\eta}{[4\beta^2 - n^2\eta^2]^2} \tag{2.29}$$

. With aid of "wolfram-alpha" we find

$$\sum_{n=1}^{\infty} \frac{8\beta\eta}{[4\beta^2 - n^2\eta^2]^2} = \frac{2\pi^2\beta^2 csc^2\left(\frac{2\pi\beta}{\eta}\right) + \pi\beta\eta cot\left(\frac{2\pi\beta}{\eta}\right) - \eta^2}{4\beta^3\eta} \tag{2.30}$$

. Finally, putting all pieces together, we obtain the angular momentum of the second particle as

$$\langle L_{z_2}\rangle = -\hbar(\frac{U_0 m_1 R^2}{\pi\hbar^2})^2 \frac{2\pi^2\beta^2 csc^2\left(\frac{2\pi\beta}{\eta}\right) + \pi\beta\eta cot\left(\frac{2\pi\beta}{\eta}\right) - \eta^2}{4\beta^3\eta}, \tag{2.31}$$

with dimensionless parameters

$$\langle L_{z_2}\rangle = -\hbar\widetilde{U_0}^2 \frac{2\pi^2\beta csc^2\left(\frac{2\pi\beta}{\eta}\right) + \pi\eta cot\left(\frac{2\pi\beta}{\eta}\right) - \eta^2}{16\pi^2\beta^2\eta} \tag{2.32}$$

where $\widetilde{U_0} = \frac{2mR^2U}{\hbar^2}$.For certain $\frac{m_1}{m_2}$ values, we can keep track of the behaviour of $\langle L_{z_2}\rangle$ due to β.

$$\frac{m_1}{m_2} = 0.5 \longrightarrow \eta = 1.5$$

$$\frac{m_1}{m_2} = 1 \longrightarrow \eta = 2$$

$$\frac{m_1}{m_2} = 2 \longrightarrow \eta = 3$$

In numerical part, we assumed $\eta = 2$.

2.2 Numerical Approach

In this approach, we solve the problem utilizing MATLAB. We constructed matrix representation of full Hamiltonian and extracted the necessary information from it[1].

We are looking to solve the eigenvalue problem;

$$-\frac{\hbar^2}{2mR^2}\Big[\frac{\partial}{\partial\phi_1}-\beta\Big]^2-\frac{\hbar^2}{2mR^2}\Big[\frac{\partial^2}{\partial\phi_2^2}\Big]\Psi(\phi_1,\phi_2)-U_0\delta(\phi_1-\phi_2)\Psi(\phi_1,\phi_2)=E\Psi(\phi_1,\phi_2) \tag{2.33}$$

The energy eigenfunction of H_0 is;

$$\langle\phi_1,\phi_2|n_1,n_2\rangle = \psi_{n_1,n_2}(\phi_1,\phi_2) = \frac{1}{2\pi R}e^{i(n_1\phi_1+n_2\phi_2)} \tag{2.34}$$

here R is the radius of ring, n_1 and n_2 are charged and neutral particle's quantum numbers respectively.

Before making any attempt to solve the problem numerically, we must obtain the dimesionless form of Schrödinger equation and in order to do this, we divide both sides of the equation by $\frac{\hbar^2}{2mR^2}$.

Hence, we obtain the form

$$\widetilde{H}\widetilde{\Psi} = \widetilde{E}_{n_1,n_2}\widetilde{\Psi} \tag{2.35}$$

[1]see Perturbation section

where $R\Psi \rightarrow \widetilde{\Psi}$ and $\widetilde{U_0}$ is dimensionless strength of interaction as well as β is dimensionless parameter.To solve system numerically, first we are assuming complete set of states;

$$\sum_{n_1}\sum_{n_2}|n_1,n_2\rangle\langle n_1,n_2| = 1 \tag{2.36}$$

and observe that

$$\langle n_1,n_2|\widetilde{H'}|m_1,m_2\rangle = \frac{\widetilde{U}_0}{2\pi}\delta_{n_1+n_2,m_1+m_2}. \tag{2.37}$$

This expression gives us a clue about solution of the system. States feel interaction, if $n_1 + n_2 = m_1 + m_2$ condition holds. We use this property to construct appropriate matrices. m_1+m_2 gives us total angular momentum quantum number L. For a given total angular momentum L, we can determine matrix representation of physical quantities. If we choose $L = 0$, we can see $n_1 = -n_2$, or $L = 1$ then $n_1 = 1 - n_2$. Generally

$$n_1 = L - n_2. \tag{2.38}$$

If the second particle is in the m^{th} state then first particle is in $(L-m)^{th}$ state. Taking advantage of completeness we propose that our general state ket in the form

$$|\Psi\rangle = \sum_{-\infty}^{\infty} c_m|L-m,m\rangle \tag{2.39}$$

and also the general wave function is

$$\langle\phi_1,\phi_2|\Psi\rangle = \Psi(\phi_1,\phi_2) = \sum_{-\infty}^{\infty} c_m e^{i(L-m)\phi_1+im\phi_2}. \tag{2.40}$$

Before demonstrating the numerical approach, we can add a comment here. If we make some arrangements on this expression, we obtain

$$\langle\phi_1,\phi_2|\Psi\rangle = \Psi(\phi_1,\phi_2) = e^{iL\phi_1}\sum_{-\infty}^{\infty} c_m e^{im(\phi_2-\phi_1)}. \tag{2.41}$$

We can write it in a more simplified form by a making coordinate transformation where $\theta = \phi_2 - \phi_1$. Hence,

$$\Psi(\phi_1,\theta) = e^{iL\phi_1}f(\theta). \tag{2.42}$$

This form can be taken to calculate analytical solution of the problem. [14]
Energy eigenvalues of the unperturbed system are

$$E_{L-m,m} = (L-m-\beta)^2 + m^2. \tag{2.43}$$

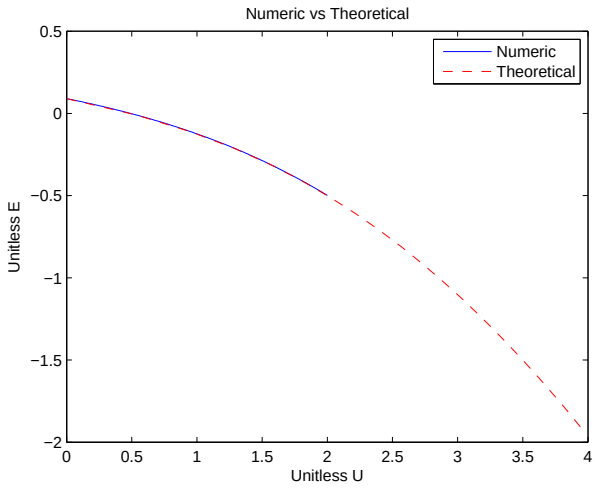

Figure 2.1: Ground-state Energy Vs. Interaction Potential, for L=0 and $\beta = 0.2$

Now, We can construct H matrix numerically by using our state ket. H_0 matrix is diagonal and H' matrix is unit matrix multiplied with $\frac{\widetilde{U_0}}{2\pi}$ constant. For given L values, We can find eigenvalues and eigenvectors of H by using simple *eigs* command in $MATLAB$.

For $L = 0$, We calculated ground-state energy numerically and compared with analytical expression [14] with varying $\widetilde{U_0}$. We can compare theoretical and numerical values in Figure 2.1 which are compatible with each other. The result meets our expectations. Increasing interaction strength reinforces bounding energy between particles and lowers energy.

Our main purpose is to calculate angular momentum transfer from charged particle coupled to the magnetic field to the neutral one via delta-interaction. To compute expectation value of neutral particle's angular momentum, all we have to do is to find eigenvector of Hamiltonian and construct the L_{z_2} matrix which is diagonal. Let a be an eigenvector of full Hamiltonian matrix. Then, in matrix notation expectation value of neutral particle can be calculated as

$$\langle L_{z_2} \rangle = a^\dagger L_{z_2} a.$$

We can compare perturbation and numerical angular momentum transfer results with varying β in Figure 2.2. We can also compare them with varying U_0 in

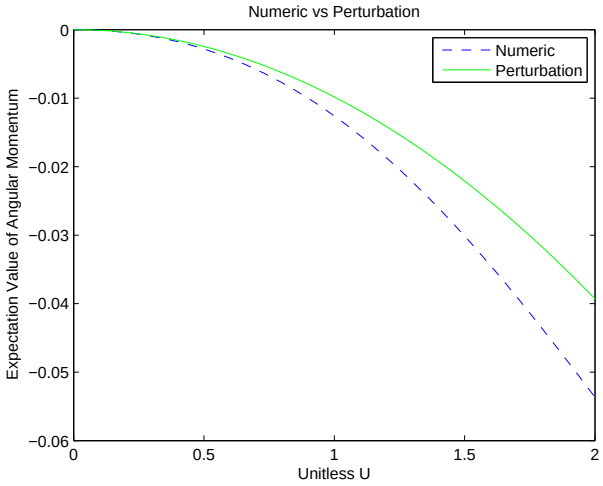

Figure 2.2: Neutral Particle's Angular Momentum Vs. $\widetilde{U_0}$ for L=0, β=0.2 in ground state

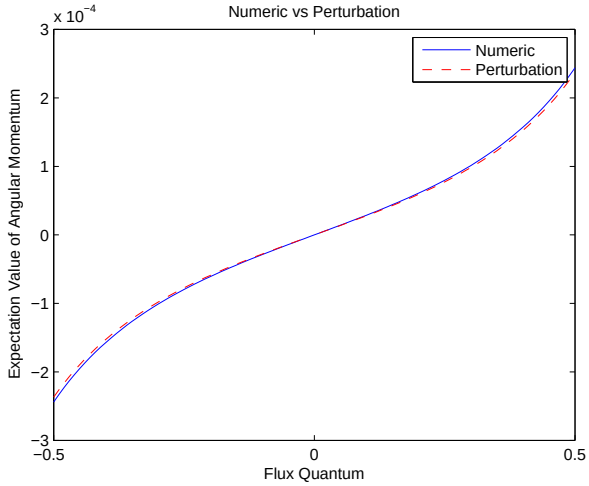

Figure 2.3: Neutral Particle's Angular Momentum Vs. β for L=0, $\widetilde{U_0}$=0.1 in ground state

Figure 2.3 where we see that numeric and perturbative results agree with each other for small $\widetilde{U_0}$ which is meaningful.

The numerical method gives us the freedom of examining the problem in all interaction regimes, in contrast with perturbative approach which is only applicable in the weak interaction regime.

Chapter 3

Theoretical Background for the N-Body Problem

3.1 Gross-Pitaevski Equation

Disclaimer: The notes in this section are far from originality and closely follow the subjects in the book: C. Pethick and H. Smith, *Bose − Einstein Condensation in Dilute Gases*. Cambridge University Press, 2000.

Bose Einstein Condensation including interactions can be described by Gross-Pitaevski equation known as non-linear Schrödinger equation. Gross-Pitaevski equation can explain many aspects of BEC and the existence of quantized vortices. [7] At the zero temperature, electronic degrees of freedom are frozen and atoms can be treated as particles. [4] Due to the ultracold nature of atoms, s-wave collisions have an important role in interactions. Therefore, we can model this interaction as contact interaction where $V(r_1 - r_2) = U_0\delta(r_1 - r_2)$ and $U_0 = 4\pi\hbar a/m$ which is characterized by scattering length a. We can interpret this type of interaction between all atoms as; apart from the trapping potential, an atom in the condensate that senses an extra potential dependent on the local atomic density due to mean field of other atoms. The positivity or negativity of a presents us whether atoms repel each other or attract each other. We can safely

state that if inter-particle spacing is much greater than scattering length a then we can investigate the system's properties with use of Gross-Pitaevski Equation at zero temperature. [15]

We follow the mean-field or Hartree approach to calculate energy of N-body system. At zero temperature, all bosons occupy ground state and they are in the same single particle state $\psi(r)$. *Hamiltonian* of the system is

$$\mathbf{H} = \sum_{n=1}^{N} \left[\frac{p_n^2}{2m} + V(r_n) \right]. \tag{3.1}$$

Thus, we can write the wave function of the N-Body system as

$$\Psi(r_1, r_2, ..., r_N) = \prod_{n=1}^{N} \psi(r_n). \tag{3.2}$$

The Normalization of $\psi(r)$ is

$$\int dr |\psi(r)|^2 = 1. \tag{3.3}$$

This wave function belongs to BEC created by ideal Bose gas. Thus, interactions are excluded. We have to include interactions to achieve the true form of wave function. We consider two-body interactions in the form of delta function where U_0 is the strength of the interaction potential.
The *Hamiltonian* that contains interactions can be written as

$$\mathbf{H} = \sum_{n=1}^{N} \left[\frac{p_n^2}{2m} + V(r_n) \right] + \frac{U_0}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} \delta(r_n - r_m) \tag{3.4}$$

where $V(r)$ stands for trapping potential.

In the Hartree Approximation, the energy of the system can be calculated as

$$E = N \int dr \left[\frac{\hbar^2}{2m} |\nabla \psi(r)|^2 + V(r) |\psi(r)|^2 + \frac{(N-1)}{2} U_0 |\psi(r)|^4 \right]. \tag{3.5}$$

In the Mean Field approach, we are assuming that all of the atoms are in condensed state, yet some of the atoms are not. Actually, the number of atoms in condensed state is lower than N due to interactions. The microscopic theory

of weakly interacting Bose gas can give us the real number of atoms in non-condensed state. For uniform Bose gas, the number of particles in the non-condensed state can be calculated by means of microscopic theory, fraction of atoms also known as *depletion of condensate* is in the order of $(na^3)^{1/2}$ where n is the atomic density. [15] To calculate depletion of the condensate roughly, we assume the average volume occupied by an atom, is a sphere and has the radius R.

$$n = \frac{N}{V} = \frac{N}{N\frac{4}{3}\pi R^3} = \frac{1}{\frac{4}{3}\pi R^3} \tag{3.6}$$

Therefore, depletion of condensate is in the order of $(a/R)^{3/2}$ which is $\sim 1\%$ in the experiments conducted with alkali atoms. [15] Thus, we can safely ignore the depletion for the sake of our calculations.

A heuristic calculation for the energy of uniform Bose gas whose trapping potential is zero in a finite volume V. By use of eqn. 3.3 we may find the wave function of single particle in ground state is $1/V^{1/2}$. Thus, we can easily see from eqn. 3.5 that kinetic energy vanishes and interaction energy between two atom is U_0/V. We are looking for the energy of the entire system. So, this is just a counting problem. The number of possible ways of choosing pairs of atoms can be shown as; [15]

$$W = \frac{N!}{2!(N-2)!} = \frac{N(N-1)}{2}. \tag{3.7}$$

If N$\gg$1, then we can approximate the number of possible ways as;

$$W \approx \frac{N^2}{2}. \tag{3.8}$$

Thus, the energy of the system can be simply expressed as;

$$E = \frac{N^2 U_0}{2V} = \frac{n^2 V}{2}. \tag{3.9}$$

where n is atomic density. Recalling from statistical sechanics, the chemical potential $\mu = \partial E/\partial N$. By using this relation, we obtain

$$\mu = U_0 n. \tag{3.10}$$

So far, physical quantities for uniform Bose gas have been calculated heuristically. To investigate non-uniform Bose gas, calculations must be made in a more

formal manner. We define a new wave function for condensate state which is more appropriate to use

$$\Psi(r) = N^{1/2}\psi(r). \tag{3.11}$$

With this, atomic density becomes

$$n(r) = |\Psi(r)|^2. \tag{3.12}$$

The energy of the system in terms of the newly defined wave function can be written as

$$E(\Psi) = \int dr\left[\frac{\hbar^2}{2m}|\nabla\Psi(r)|^2 + V(r)|\Psi(r)|^2 + \frac{U_0}{2}|\Psi(r)|^4\right]. \tag{3.13}$$

where N$\gg$1 approximation has been used.

We need to minimize the energy expression by varying $\Psi(r)$ and $\Psi^*(r)$ to obtain the most suitable form of Ψ imposed by constraint

$$N = \int dr|\Psi(r)|^2. \tag{3.14}$$

Minimization of the energy can be carried out by taking chemical potential μ as Lagrange multiplier which guarantees a fixed atom number. This procedure gives; [15]

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(r) + V(r)\Psi(r) + U_0|\Psi|^2\Psi(r) = \mu\Psi(r) \tag{3.15}$$

which is called time-independent Gross-Pitaevski Equation. This form, also known as non-linear Schrödinger equation, contains the non-linear term $U_0|\Psi(r)|^2$ generated by the mean field of other atoms apart from usual Schrödinger equation.

Simple application of Gross-Pitaevski equation to uniform bose gas reveals;

$$U_0|\Psi(r)|^2 = \mu = U_0 n \tag{3.16}$$

which confirms the eqn. 3.10

The properties of a Bose gas at thermodynamic equilibrium can be studied with the use of time-independent Gross-Pitaevski equation. To examine dynamical stucture of the gas, we need to address time-dependent Gross-Pitaevski equation which is given by [15]

$$i\hbar\frac{\partial\Psi(r,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi(r,t) + V(r)\Psi(r,t) + U_0|\Psi(r,t)|^2. \tag{3.17}$$

3.1.1 Thomas-Fermi Approximation

In condensates where interactions are repulsive and the particle number is of order 10^6 or higher [4], interaction energy dominates kinetic energy. This gives us the opportunity to obtain a very accurate simple solution by disregarding the kinetic term in the Gross-Pitaevski equation. Therefore, eqn. 3.15 reduces to

$$\left[V(r) + U_0|\Psi(r)|^2\right]\Psi(r) = \mu\Psi(r). \tag{3.18}$$

This admits the solution as

$$|\Psi(r)|^2 = \left[\mu - V(r)\right]/U_0. \tag{3.19}$$

If we put this approximation into words; The energy needed to add an atom anywhere in the condensate is constant. [15]

3.1.2 Coherence Length

We have studied condensates confined to a volume V. It is now appropriate to mention a parameter which enables us to infer information about behaviour of condensate in a confined volume. Here, the term 'confined volume' refers to the volume with solid boundaries. At the boundary, the wave function is zero and somewhere in the middle of the volume, the wave function reaches the value that the wave function of uniform Bose gas has in equilibrium. The distance between boundary and the point where gas reaches its bulk value can be evaluated by Gross-Pitaevski equation and this distance is called 'coherence length' or 'healing length' which is given by [15]

$$\xi(r) = \left(\frac{\hbar^2}{2mn(r)U_0}\right)^{1/2}. \tag{3.20}$$

The kinetic energy term in Gross-Pitaevski equation can be ignored when healing length is shorter than typical lengths in the system. [16]

3.1.3 Two-Component Condensates

We have investigated single component BEC. We can widen our perspective to discuss two-component BEC which includes two macroscopically-occupied quantum states. These two components can be isotopes of the same atoms or just different atoms.

To obtain Gross-Pitaevski equations for two-component condensate, we use the same arguments that have been used to derive a single component Gross-Pitaevski equation. The many-body wave function for two-component condensate which excludes interaction is [15]

$$\Psi(r_1, ..., r_{N_1}; r_1^{'}, ..., r_{N_2}^{'}) = \prod_{n=1}^{N_1} \psi_1(r_n) \prod_{m=1}^{N_2} \psi_2(r_m^{'}) \tag{3.21}$$

where r_n and r_m denotes coordinates of different components and ψ_1 and ψ_2 are ordinary normalized single-particle wave functions. Once again we introduce appropriate wave functions where $\Psi_1 = N_1^{1/2}$ and $\Psi_2 = N_2^{1/2}$. The energy of the system referring to the single component is [15]

$$\begin{aligned} E = \int dr \Big[& \frac{\hbar^2}{2m_1} |\nabla \Psi_1|^2 + V_1(r)|\Psi_1|^2 + \frac{\hbar^2}{2m_2} |\nabla \Psi_2|^2 \\ & + V_2(r)|\Psi_2|^2 + \frac{1}{2} U_{11} |\Psi_1|^4 + \frac{1}{2} U_{22} |\Psi_2|^4 + U_{12} |\Psi_1|^2 |\Psi_2|^2 \Big] \end{aligned} \tag{3.22}$$

where we assumed $N_1 \gg 1$ and $N_2 \gg 1$. m_1 and m_2 represents the mass of two different atoms and V_i stands for external potentials which may be dependent on internal structures of atoms. U_{ij} is the strength of interaction which is associated with scattering length a_{ij} and $U_{12} = U_{21}$. To minimize energy by using μ_1 and μ_2 as Lagrange multipliers yields to two coupled time-independent Gross-Pitaevski equations as [15]

$$\begin{aligned} -\frac{\hbar^2}{2m_1} \nabla^2 \Psi_1 + V_1(r)\Psi_1 + U_{11}|\Psi_1|^2 \Psi_1 + U_{12}|\Psi_2|^2 \Psi_1 &= \mu_1 \Psi_1, \\ -\frac{\hbar^2}{2m_2} \nabla^2 \Psi_2 + V_2(r)\Psi_2 + U_{22}|\Psi_2|^2 \Psi_2 + U_{12}|\Psi_1|^2 \Psi_2 &= \mu_2 \Psi_2. \end{aligned} \tag{3.23}$$

3.2 The Bogoliubov Equations

Investigation of BEC under time-dependence provides invaluable information about the physical properties of the condensate such as elementary excitations and ballistic expansion of the cloud. [15] The thermodynamics of the system can be studied with use of the excitation spectrum. In addition, elementary excitations are needed to realize dynamical instability of the system. In this section, we will offer a method to calculate these excitations.

We can calculate the excitation spectrum starting from Gross-Pitaevski equation. The time-dependent condensate wave function of unperturbed state may be written as; $\Psi(r,t) = \psi(r)e^{-i\mu t/\hbar}$ where $\psi(r)$ is time-independent solution of the condensate wave function. To describe spectrum of elementary excitations, we consider the deviation of the condensate wave function from equilibrium is small and the contribution of weak perturbations of the wave function is given by the *variational form* [17]

$$\Psi(r,t) = e^{-i\mu t/\hbar}\big[\psi(r) + u(r)e^{-i\omega t} + \upsilon^*(r)e^{i\omega t}\big] \tag{3.24}$$

where $u(r)$ and $v(r)$ need to be solved. We plug the *ansatz* into eqn. 3.17. By equating coefficients of exponential terms and only regarding the terms linear in $u(r)$ and $\upsilon(r)$, we end up with the two coupled equations of $u(r)$ and $\upsilon(r)$;

$$\bigg[-\frac{\hbar^2}{2m}\nabla^2 + V(r) + 2|\psi(r)|^2U_0 - \mu - \hbar\omega\bigg]u(r) - |\psi(r)|^2U_0\upsilon(r) = 0 \tag{3.25}$$

and

$$\bigg[-\frac{\hbar^2}{2m}\nabla^2 + V(r) + 2|\psi(r)|^2U_0 - \mu + \hbar\omega\bigg]\upsilon(r) - |\psi(r)|^2U_0u(r) = 0 \tag{3.26}$$

which are known as the *Bogoliubov equations*.As in usual, we use these equations to study the uniform Bose gas where the form of $u(r)$ and $\upsilon(r)$ are

$$u(r) = u_k\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{V^{1/2}} \tag{3.27}$$

and

$$\upsilon(r) = \upsilon_k\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{V^{1/2}}. \tag{3.28}$$

The Bogoliubov equations for the uniform system are;

$$\hbar\omega u_k = \Big(\frac{\hbar^2 k^2}{2m} + nU_0\Big)u_k - nU_0 v_k \tag{3.29}$$

and

$$-\hbar\omega v_k = \Big(\frac{\hbar^2 k^2}{2m} + nU_0\Big)v_k - nU_0 u_k \tag{3.30}$$

where chemical potential is nU_0. By equating the determinant to zero and setting $E_k^0 = \hbar^2 k^2/2m$, one can find that

$$\begin{aligned}(\hbar\omega)^2 &= (E_k^0 + nU_0)^2 - (nU_0)^2 \\ &= E_k^0(E_k^0 + 2nU_0)\end{aligned} \tag{3.31}$$

which is consistent with the results obtained with use of microscopic theory of Bose gas. [15] There are two roots for the excitation spectrum. The physically relevant one is positive root.
Thus;

$$E_k = \sqrt{E_k^0(E_k^0 + 2nU_0)}. \tag{3.32}$$

There can be imaginary excitation energies which depict dynamical instability of the system. These imaginary energies will be our guide when we investigate vortex transfer between two-components.

3.3 Quantization Of Circulation and the Single-Vortex Structure

As we mentioned before, investigation of rotation of superfluid is an important quest for physicists. Concepts studied in superfluid theory such as; phase of wave function, vortex structure, found place in research of rotational properties of atomic Bose-Einstein condensates. These acquisitions has been used to resolve rotating condensates under trapping potentials.

The Hydrodynamic equations which can be derived from Gross-Pitaevski equation [15] imply that velocity of condensate is gradient of phase of wave function.

$$\mathbf{v} = \frac{\hbar}{m}\nabla\phi \tag{3.33}$$

from vector calculus we can easily infer that curl of a gradient is zero.

$$\nabla \times \mathbf{v} = 0. \tag{3.34}$$

This form indicates irrotational velocity field. Yet, if wave function has a singularity in its phase that leads us to quantization of circulation. This interesting property can be investigated by taking line integral of velocity in a closed contour. We expect that to preserve single-valuedness of wave function, the phase must change in multiples of 2π when line integral is carried out in closed contour.
Hence from vector calculus

$$\Delta\phi = \int \nabla\phi \cdot d\mathbf{l}, \tag{3.35}$$

for a closed contour due to single-valuedness of wave function is

$$\Delta\phi = \oint \nabla\phi \cdot d\mathbf{l} = 2\pi n \tag{3.36}$$

where n is an integer. Therefore integral of velocity field in a closed contour which is called circulation yields to [15];

$$\oint \mathbf{v} \cdot d\mathbf{l} = n\frac{h}{m} \tag{3.37}$$

which implies that circulation is quantized as multiples of $\hbar/m$.

For example, by considering smooth azimuthal flow under a rotationally invariant trap about the z axis, some simple properties of the wave function of that state can be deduced. Because of single-valuedness, the phase of wave function in the form $e^{in\varphi}$ where φ is azimuthal angle. Only ϕ component of the velocity vector survives. Using polar coordinates velocity vector can be written as

$$\vec{v} = n\frac{h}{2\pi m r}\hat{\phi}. \tag{3.38}$$

Thus, line integral of velocity field yields to nh/m if the taken path encloses z axis, on the other hand if it is not, integral gives zero. If the circulation is non-zero, velocity field diverges on the z axis, Thus, kinetic energy goes to infinity. To overcome this ambiguity the value of wave function on the z axis must be zero. For a cylindrically symmetric trap and a state with non-zero circulation about z axis corresponds to each particle in trap carries $n\hbar$ azimuthal angular

momentum, thus total angular momentum about z axis is Nn$\hbar$. We can write the wave function in cylindrical coordinates is

$$\Psi(r,z) = \psi(r,z)e^{in\phi}. \tag{3.39}$$

Therefore the updated Gross-Pitaevski equation becomes

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial\psi}{\partial r}\Big) + \frac{\partial^2\psi}{\partial z^2}\Big] + \frac{\hbar^2}{2mr^2}n^2\psi + V(r,z)\psi + U_0\psi^3 = \mu\psi. \tag{3.40}$$

The ground state of wave function does not depend z, therefore z derivative does not give any contribution and we can model the system in 2-dimensions [15].

Chapter 4

N-Body Problem

In this section, we will demonstrate solutions to particular systems. First, we will solve Gross-Pitaevski equation for those systems and by means of these solutions, we will attempt to solve the *Bogoliubov* equations. For rotationally symmetric systems, by observing excitation energies we will conclude whether vortex transfer exist or not.

4.1 1-D Box

The second most basic system to solve for BEC is 1-d box with solid walls. The analytical solutions for BEC is very limited due to non-linear terms in Gross-Pitaevski equation. So, we have to use numerical techniques[1] to resolve condensate systems.

Our system is BEC with repulsive interactions confined in a box with length L. We need to know one more parameter to solve this problem and normally this parameter is particle number of the system. But, due to our numerical technique,

[1] see Appendix

we assume chemical potential of the system is given. The time-independent Gross-Pitaevski equation for one-component gas in 1-d box is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(r) + g|\Psi|^2\Psi(r) = \mu\Psi(r) \tag{4.1}$$

where g is strength of interaction and in $Energy \times length$ units.

Boundary Conditions: Due to hard walls, wave function of the condensate vanishes at the boundaries. Our numerical calculation must take into account these conditions where $\psi(0) = \psi(L) = 0$.

The standart procedure before making any numerical calculation is choosing suitable dimensionless parameters which are compatible with the natural parameters of the system under consideration.
Dimensionless parameters which are chosen for 1-d box are

$$\begin{aligned} \widetilde{x} &\rightarrow x/L, \\ \widetilde{\psi} &\rightarrow \psi\sqrt{L}, \\ \widetilde{g} &= \frac{2mL}{\hbar^2}g, \\ \widetilde{\mu} &= \frac{2mL^2}{\hbar^2}\mu \end{aligned} \tag{4.2}$$

with dimensionless parameters Gross-Pitaevski eqn. becomes

$$-\frac{d^2\widetilde{\psi}}{d\widetilde{x}^2} + \widetilde{g}|\widetilde{\psi}|^2\widetilde{\psi} = \widetilde{\mu}\widetilde{\psi} \tag{4.3}$$

with the normalization,

$$N = \int |\widetilde{\psi}(\widetilde{x})|^2 d\widetilde{x}. \tag{4.4}$$

In terms of new parameters healing length is

$$\xi = \frac{L}{\sqrt{\widetilde{\mu}}}. \tag{4.5}$$

By adjusting dimensionless parameter $\widetilde{\mu}$, the proportion of healing length to box's length can be determined. In our calculations, we usually take this proportion as; $\xi/L = 0.01$.
The Numerical solution of $\widetilde{\psi}$ is in figure 4.1 where $N \approx 9.7\text{x}10^4$. In figure 4.2

another wave function with different parameters is shown. We can make a comparison between these wave functions, we can see that first wave function reaches its bulk value in very short distance, yet other wave function reaches its bulk value in a longer distance. These results are compatible with their coherence lengths which second wave function has the longer one.

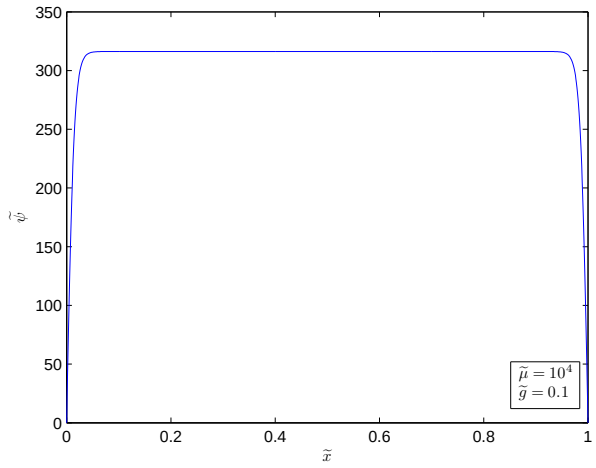

Figure 4.1: Wave function in 1-d Box where $\xi/L = 0.01$

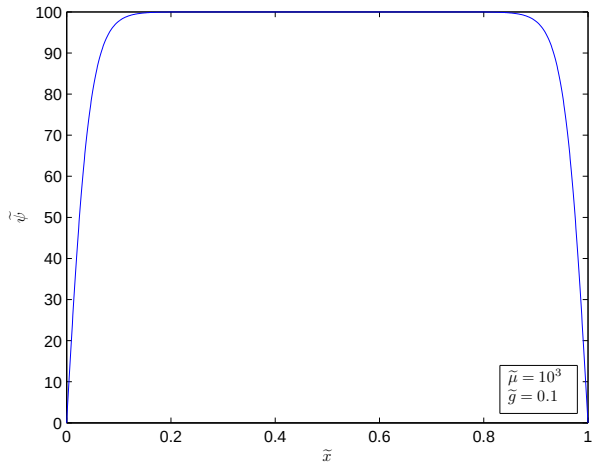

Figure 4.2: Wave function in 1-d Box where $\xi/L \approx 0.03$

The bulk value of wave function of uniform bose gas can be calculated as in term of dimensionless parameters; $\widetilde{\psi} = \sqrt{\widetilde{\mu}/\widetilde{g}}$, this gives the number 316.23. The bulk value of figure 4.1 reads 316.20. Two numbers agree with each other as expected.

The excitation spectrum of the system can be found with inserting the ansatz 3.24 into 4.1 and disregarding second and higher orders of u and υ, resulting equations are

$$\hbar\omega u = -\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + 2g|\psi|^2u - \mu u + g(\psi)^2\upsilon \tag{4.6}$$

and

$$-\hbar\omega\upsilon^* = -\frac{\hbar^2}{2m}\frac{d^2\upsilon^*}{dx^2} + 2g|\psi|^2\upsilon^* - \mu\upsilon^* + g(\psi^*)^2u^*. \tag{4.7}$$

Once again, we need to use dimensionless parameters to apply numerical methods. We assume u,υ and ψ are real and set $\epsilon = \hbar\omega$. By dividing the both sides of equations with $2mL^2/\hbar^2$, we obtain the dimensionless set that is found before. Addition to them new dimensionless parameters need to be defined as

$$\begin{aligned} \widetilde{u} &\rightarrow u\sqrt{L}, \\ \widetilde{\epsilon} &= \frac{2mL^2\epsilon}{\hbar^2}, \\ \widetilde{\upsilon} &\rightarrow \upsilon\sqrt{L}. \end{aligned} \tag{4.8}$$

Hence, dimensionless *Bogoliubov* equations can be written as

$$\widetilde{\epsilon}\widetilde{u} = -\frac{d^2\widetilde{u}}{dx^2} + 2\widetilde{g}|\psi|^2\widetilde{u} - \widetilde{\mu}\widetilde{u} + \widetilde{g}(\widetilde{\psi})^2\widetilde{\upsilon} \tag{4.9}$$

and

$$\widetilde{\epsilon}\widetilde{\upsilon} = \frac{d^2\widetilde{\upsilon}}{dx^2} - 2\widetilde{g}|\psi|^2\widetilde{\upsilon} + \widetilde{\mu}\widetilde{\upsilon} - \widetilde{g}(\widetilde{\psi})^2\widetilde{u}. \tag{4.10}$$

Two coupled equations with same eigenvalue, by discretizing derivatives we can solve this eigenvalue problem numerically. First four of excitation energies are shown in the table 4.2

0^{th}	-2.6312e-11 + 7.1957i -2.6312e-11 - 7.1957i
1^{st}	450.6552 -450.6552
2^{nd}	902.1604 -902.1604
3^{rd}	1354.9816 -1354.9816
4^{th}	1809.8202 -1809.8202

Table 4.1: Bogoliubov energies of 1-d box

Here, we need to emphasize a point about energies, there is zero mode energy which does not correspond to instability of the system in the Table 4.1. In the *Bogoliubov* equations, if we set $\widetilde{\epsilon} = 0$, it can be shown that equations may be satisfied by $u(r) = -\upsilon(r) \sim c\psi$ where c is arbitrary constant. When this condition is met, zero mode shows up [18], otherwise it is not observed.

4.2 Cylinder Geometry

Axially symmetric traps are important tools to study BEC especially with vortex structure. In this section, we will start with one-component gas without vortex and gas with vortex. Then, we will continue with BEC with vortex under magnetic field and also study the mixed condensates to reveal vortex transfer.

4.2.1 One Component Gas

Our system is BEC confined in cylinder with radius R. The time-independent Gross-Pitaevski equation of the system is

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\Big]\psi + g|\psi|^2\psi = \mu\psi \tag{4.11}$$

where g in $Energy \times Area$ units, due to 2-d geometry. Because BEC does not possess vortex, ϕ derivative does not give any contribution, moreover, ground state of the system does not depend on the z variable [15], so the z derivative also vanishes. Thus, problem reduces to two-dimension.
G.P. equation becomes

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}\Big]\psi+g|\psi|^2\psi=\mu\psi. \tag{4.12}$$

Wave function only depends on radial coordinate.

Boundary Conditions: $\psi(R)=0$ as expected because of solid walls. $\psi(0)$ needs to be finite, does not have to be zero. This condition can be satisfied by derivative of ψ with respect to r variable on z axis is zero.

We can define dimensionless parameters as usual

$$\begin{aligned} x &\rightarrow r/R, \\ \widetilde{\psi} &\rightarrow R\psi, \\ \widetilde{g} &= \frac{2m}{\hbar^2}g, \\ \widetilde{\mu} &= \frac{2mR^2}{\hbar^2}\mu. \end{aligned} \tag{4.13}$$

Dimensionless G.P. equation is

$$-\Big[\frac{1}{x}\frac{d}{dx}+\frac{d^2}{dx^2}\Big]\widetilde{\psi}+\widetilde{g}|\widetilde{\psi}|^2\widetilde{\psi}=\widetilde{\mu}\widetilde{\psi}. \tag{4.14}$$

Normalization may be written as

$$N=2\pi\int x|\widetilde{\psi}(x)|^2dx. \tag{4.15}$$

Numerically calculated wave function is shown in the figure 4.3; where ψ is plotted in the range $[0,R]$ and reaches its bulk value 100, as estimated. Calculated number of particles is $N=2.86\times10^4$.

One of the our tasks is to find excitation spectrum. We will follow the standard procedure by plugging ansatz 3.24 into time-dependent G.P. equation and ignoring second or higher orders of u and υ. The Bogoliubov equations are

$$\hbar\omega u=-\frac{\hbar^2}{2m}\nabla^2u+2g|\psi|^2u-\mu u+g(\psi)^2\upsilon \tag{4.16}$$

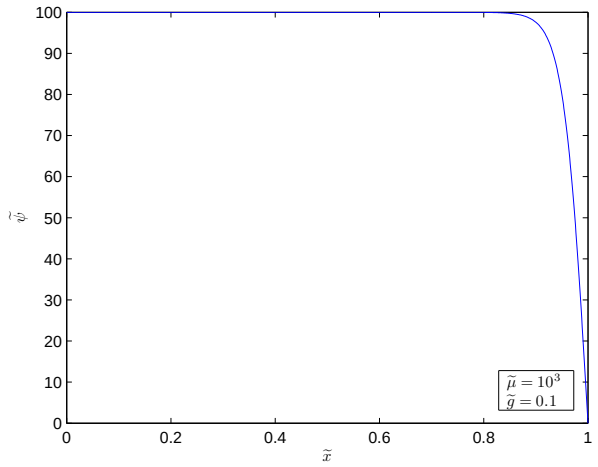

Figure 4.3: One-component Wave function in cylinder where $\xi/L \approx 0.03$

and

$$-\hbar\omega\upsilon = -\frac{\hbar^2}{2m}\nabla^2\upsilon + 2g|\psi|^2\upsilon - \mu\upsilon + g(\psi)^2 u \tag{4.17}$$

where ∇^2 is Laplacian operator in polar coordinates. $u(r,\phi)$ and $\upsilon(r,\phi)$ both depend on r and ϕ. Therefore, we make fourier expansion of them as

$$\begin{aligned} u(r,\phi) &= \sum_{m=-\infty}^{\infty} u_m(r)e^{im\phi}, \\ \upsilon(r,\phi) &= \sum_{n=-\infty}^{\infty} \upsilon(r)_n e^{-in\phi} \end{aligned} \tag{4.18}$$

by inserting these expansion into *Bogoliubov* equations and equating coefficients of same powers of $e^{i\phi}$, we obtain set of equations for u_m and υ_n components. These equations are

$$\hbar\omega u_m = -\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{d}{dr} + \frac{d^2}{dr^2} - \frac{m^2}{r^2}\Big]u_m + 2g|\psi|^2 u_m - \mu u_m + g(\psi)^2\upsilon_{-m} \tag{4.19}$$

and

$$-\hbar\omega\upsilon_n = -\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{d}{dr} + \frac{d^2}{dr^2} - \frac{n^2}{r^2}\Big]u_n + 2g|\psi|^2\upsilon_n - \mu\upsilon_n + g(\psi)^2 u_{-n} \tag{4.20}$$

where $m = -n$. By introducing the dimesionless forms as

$$\begin{aligned} \widetilde{u} &\rightarrow Ru, \\ \widetilde{\epsilon} &= \frac{2mR^2\epsilon}{\hbar^2}, \\ \widetilde{\upsilon} &\rightarrow R\upsilon. \end{aligned} \tag{4.21}$$

For clarity, we drop tilde symbols. The dimensionless form of *Bogoliubov* equations are

$$\epsilon u_m = -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{m^2}{x^2}\Big]u_m + 2g|\psi|^2 u_m - \mu u_m + g(\psi)^2 \upsilon_{-m} \tag{4.22}$$

and

$$-\epsilon \upsilon_{-m} = -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{m^2}{x^2}\Big]\upsilon_{-m} + 2g|\psi|^2 \upsilon_{-m} - \mu \upsilon_{-m} + g(\psi)^2 u_m \tag{4.23}$$

where $\epsilon = \hbar\omega\frac{2mR^2}{\hbar^2}$. Here m values corresponds to the m unit angular momentum of perturbative part of wave function. Calculated energies are shown in the table 4.2. The energies are paired and real as expected. The equation set with $m = 0$

1^{st}	176.0733
	-176.0733
2^{nd}	325.2482
	-325.2482
3^{rd}	478.0522
	-478.0522
4^{th}	637.3014
	-637.3014

Table 4.2: Bogoliubov energies of One-Component Gas in Cylinder where m=0

produces *zero mode* energy, but we discarded it.

4.2.2 Single-Vortex in Cylinder

Our system is BEC with single vortex structure confined in cylinder. Time-independent Gross-Pitaevski equation that describes this system is

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\Big]\Psi(r,\phi) + g|\Psi(r,\phi)|^2\Psi(r,\phi) = \mu\Psi(r,\phi) \tag{4.24}$$

As we discussed before z derivative vanishes. Yet, ϕ variable does not vanish this time. BEC with singly-quantized vortex has the wave function is given by

$$\Psi(r,\phi) = \psi(r)e^{i\phi} \tag{4.25}$$

Therefore, G.P. equation reduces the form

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}-\frac{1}{r^2}\Big]\psi(r)+g|\psi(r)|^2\psi(r)=\mu\psi(r). \tag{4.26}$$

Boundary Conditions: Due to vortex structure, $\psi(0)=0$ and $\psi(R)=0$ as usual. [15] We use same dimensionless parameters and normalization condition that we defined in previous section, equations 4.13 and 4.15. We may write dimensionless form of G.P. equation as

$$-\Big[\frac{1}{x}\frac{d}{dx}+\frac{d^2}{dx^2}-\frac{1}{x^2}\Big]\widetilde{\psi(x)}+\widetilde{g}|\widetilde{\psi(x)}|^2\widetilde{\psi(x)}=\widetilde{\mu}\widetilde{\psi(x)}. \tag{4.27}$$

Solution of G.P. equation is exhibited in the figure 4.4, the wave function starts from zero, ends up at its bulk value, then drops to zero to satisfy boundary conditions. Number of particles calculated as $N=2.84\times10^4$.

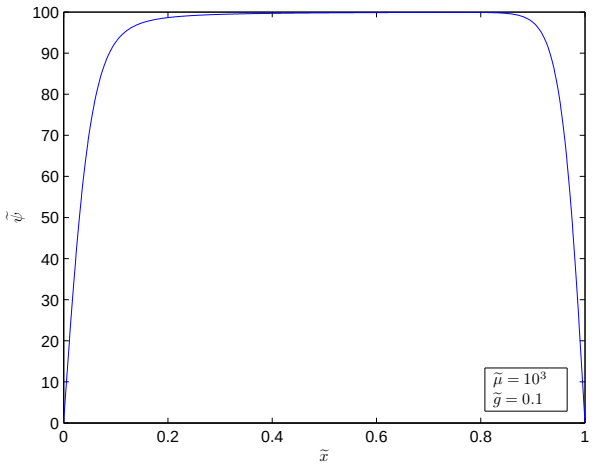

Figure 4.4: Singly quantized vortex in cylinder where $\xi/L\approx0.03$

The *Bogoliubov* equations are

$$\hbar\omega u=-\frac{\hbar^2}{2m}\nabla^2u+2g|\psi|^2u-\mu u+g(\psi)^2e^{2i\phi}\upsilon \tag{4.28}$$

and

$$-\hbar\omega\upsilon=-\frac{\hbar^2}{2m}\nabla^2\upsilon+2g|\psi|^2\upsilon-\mu\upsilon+g(\psi)^2e^{-2i\phi}u \tag{4.29}$$

where *ansatz* eqn. 3.24 inserted into time-dependent G.P. equation. By plugging fourier expansion of u and υ (4.18) into *Bogoliubov* equations and equating coefficents of $e^{i\phi}$, coupled equations of u_m and υ_n can be obtained. We define standart dimensionless parameters where we used in previous section. For brevity, we drop tilde symbols.

Coupled expansion components of u and υ are

$$\epsilon u_n = -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{n^2}{x^2}\Big]u_n + 2g|\psi|^2 u_n - \mu u_n + g(\psi)^2 \upsilon_{2-n} \tag{4.30}$$

and

$$-\epsilon \upsilon_{2-n} = -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{(2-n)^2}{x^2}\Big]\upsilon_{2-n} + 2g|\psi|^2 \upsilon_{2-n} - \mu \upsilon_{2-n} + g(\psi)^2 u_n \tag{4.31}$$

where $2 - m = n$ and $\epsilon = \hbar\omega\frac{2mR^2}{\hbar^2}$. Calculated excitation energies can be seen in table 4.3.

1^{st}	176.1804
	-176.1804
2^{nd}	325.6745
	-325.6745
3^{rd}	479.2158
	-479.2158
4^{th}	639.8025
	-639.8025

Table 4.3: Bogoliubov energies of Single Vortex in Cylinder where n=1

For $n = 1$ energies are paired and *zero mode* emerges, because equations are symmetric under the exchange $\epsilon \to -\epsilon$. For other values of n, instead of 1, equations are not symmetric. Thus, energies are not paired, as seen in the table 4.4.

4.2.3 Singly-Quantized Vortex Under Magnetic Field In Cylinder

As distinct from previous systems, we will examine charged condensate under external potential. Our system will be charged BEC with singly-quantized vortex

1^{st}	-2.4828
	83.6234
2^{nd}	-84.2891
	242.0179
3^{rd}	-253.3864
	393.0887
4^{th}	-410.9724
	549.7267

Table 4.4: Bogoliubov energies of Single Vortex in Cylinder where n=0

confined in cylinder geometry under magnetic field. We assume the magnetic field is generated by a selenoid. The magnetic field that applied to system modifies the momentum operator, by using the derived quantities discussed in *chapter* 2, The Gross-Pitaevski equation that depicts the system is

$$\frac{1}{2m}\Big[-\hbar^2\nabla^2+q^2|\overrightarrow{A}|^2+2i\hbar q\overrightarrow{A}\cdot\overrightarrow{\nabla}\Big]\Psi(r,\phi)+g|\Psi(r,\phi)|^2\Psi(r,\phi)=\mu\Psi(r,\phi) \quad (4.32)$$

where q is charge of particles. ∇^2 and $\overrightarrow{\nabla}$ operators are defined in cylindrical coordinates. Thus,

$$\overrightarrow{A}\cdot\overrightarrow{\nabla}=\frac{|\overrightarrow{B}|}{2}\frac{\partial}{\partial\phi} \quad (4.33)$$

where $\overrightarrow{A}$ is vector potential and $|\overrightarrow{A}|^2=\frac{\overrightarrow{|B|^2}}{2}r^2$, as well as $|\overrightarrow{B}|$ is strength of magnetic field. For brevity, we will write as; $\overrightarrow{A}^2=A^2$ and $\overrightarrow{|B|^2}=B^2$ The wave function that contains all the information about the system is

$$\Psi=\psi(r)e^{i\phi}. \quad (4.34)$$

As we stated before, wave function does not depend on z variable, so our system reduces to 2 dimension. G. P. equation may be written as

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}-\frac{1}{r^2}+\frac{q^2A^2}{2m}-\frac{\hbar q}{2m}B\Big]\psi(r)+g|\psi(r)|^2\psi(r)=\mu\psi(r). \quad (4.35)$$

By inserting the definition of A^2 into G.P. equation, obtained expression is

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}-\frac{1}{r^2}\Big]\psi(r)+\underline{\Big[\frac{q^2B^2r^2}{8m}-\frac{\hbar q}{2m}B\Big]}\psi(r)+g|\psi(r)|^2\psi(r)=\mu\psi(r). \quad (4.36)$$

If we compare the two G.P. equations, 4.26 and 4.36, the only difference is the underlined term in the equation 4.36. This term operates like an isotropic harmonic potential acting on the system.

Boundary Conditions: The wave function satisfies the same boundary conditions as in section 4.2.2.

Now, we can introduce dimensionless parameters, in addition to the set (4.14), we define a new dimensionless parameter that is a measure of the strength of magnetic field as

$$\gamma = \frac{q^2 B^2 R^4}{4\hbar^2}. \tag{4.37}$$

By taking into account dimensionless parameters, G.P. equation can be written as

$$-\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{1}{x^2}\Big]\widetilde{\psi(x)} + \gamma x^2 \widetilde{\psi(x)} - 2\sqrt{\gamma}\widetilde{\psi(x)} + \widetilde{g}|\widetilde{\psi(x)}|^2\widetilde{\psi(x)} = \widetilde{\mu}\widetilde{\psi(x)}. \tag{4.38}$$

Response of the wave function to the magnetic fields in different strengths are shown in the figures 4.5, 4.6 and 4.7. In figures 4.5 and 4.6, the wave functions

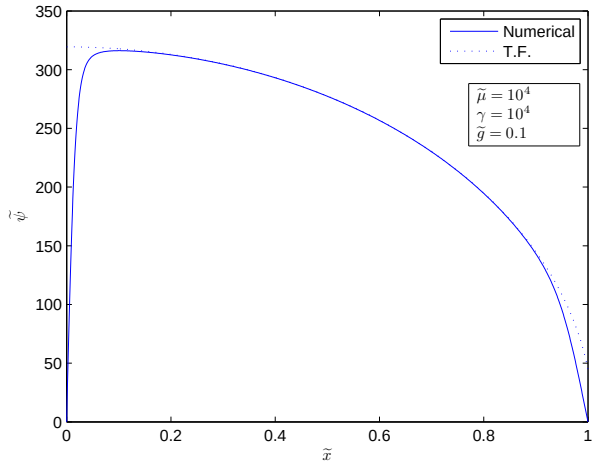

Figure 4.5: Singly quantized vortex under magnetic field where $N = 1.61 \times 10^5$

satisfies Thomas-Fermi Approximation after reaching their bulk values, due to their high particle number [4]. Figure 4.7 exhibits the form of wave function under very strong magnetic field.

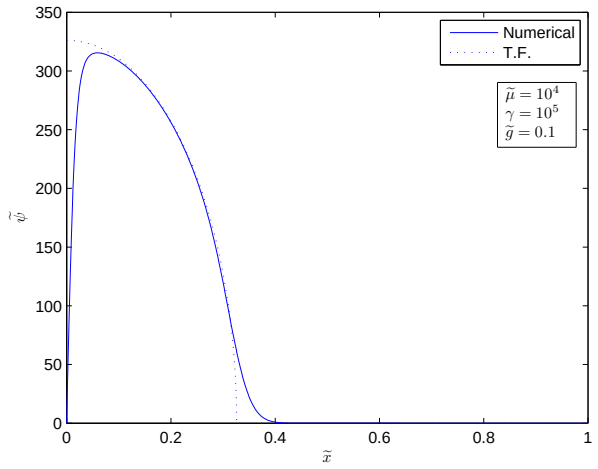

Figure 4.6: Singly quantized vortex under magnetic field where $N = 1.74 \times 10^4$

To investigate stability of the system, we need to calculate excitation spectrum. All energy eigenvalues that belongs to spectrum must be real to ensure dynamical stability [19]. We follow the same procedure to obtain *Bogoliubov* equations as we did in previous sections.
The Dimensionless *Bogoliubov* equations are

$$\begin{aligned} \epsilon u_m &= -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{m^2}{x^2}\Big]u_m + (\gamma x^2 - 2m\sqrt{\gamma} - \mu + 2g|\Psi|^2)u_m \\ &\quad + g(\Psi(x))^2 \upsilon_{2-m} \end{aligned} \tag{4.39}$$

and

$$\begin{aligned} -\epsilon \upsilon_{2-m} &= -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{(2-m)^2}{x^2}\Big]u_m + (\gamma x^2 - 2m\sqrt{\gamma} - \mu + 2g|\Psi|^2)\upsilon_{2-m} \\ &\quad + g(\Psi(x))^2 u_m \end{aligned} \tag{4.40}$$

Numerically calculated energies are shown in the tables, 4.5 and 4.6. We tested our system for various parameters and did not observe any imaginary eigenvalue even under extreme magnetic fields. Thus, we concluded that system is dynamically stable. This result is consistent with the findings in literature [20].

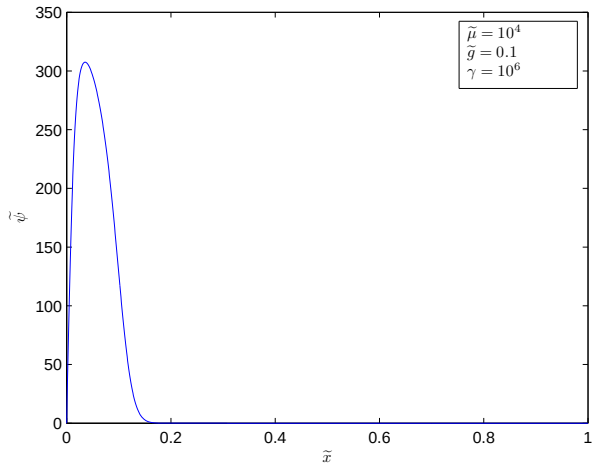

Figure 4.7: Singly quantized vortex under magnetic field where $N = 1.96 \times 10^3$

1^{st}	-14.1126
	182.8434
2^{nd}	-395.3633
	414.5050
3^{rd}	-752.2662
	787.6866
4^{th}	-1110.9396
	1138.8743

Table 4.5: Bogoliubov energies of Single Vortex under magnetic Field where $\gamma = 10^4$, m=0

4.3 Vortex Transfer

Our main goal is to observe vortex transfer between two-species Bose-Einstein condensate. At this point, observing instability-imaginary excitation energy-corresponds to vortex transfer between two species. We could not predict vortex transfer via Gross-Pitaevski equation, because phases of wave functions vanish in the mean field interaction terms. Thus, it can only be checked by solving excitation spectrum.

1^{st}	12.9933
	1282.6750
2^{nd}	-3675.0347
	4019.8617
3^{rd}	-7048.0715
	7323.9770
4^{th}	-10463.3924
	10622.5487

Table 4.6: Bogoliubov energies of Single Vortex under magnetic field where $\gamma = 10^6$, m=0

4.3.1 Singly-Quantized Vortex Interacting with Condensate

The system under investigation resides in a cylinder and consists of two condensate which one of them possess vortex. To solve this system, we need coupled Gross-Pitaevski equations (3.23) with usual wave functions defined below

$$\Psi_1(r) = \psi_2(r) \text{ and } \Psi_2(r,\phi) = \psi_1(r)e^{i\phi}. \tag{4.41}$$

By plugging wave functions into eqn. (3.23), the obtained equations where g_{12} taken as inter-species interaction and $m_1 = m_2$ are

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\Big]\psi_1 + g_{11}|\psi|_1^2\psi_1 + g_{12}|\psi_2|^2\psi_1 = \mu_1\psi_1 \tag{4.42}$$

and

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\Big]\psi_2 + g_{22}|\psi|_2^2\psi_2 + g_{12}|\psi_1|^2\psi_2 = \mu_2\psi_2. \tag{4.43}$$

Boundary Conditions:Wave function with vortex vanishes at z axis and cylinder wall, condensate wave function without rotation has a finite value at z axis and vanishes at cylinder wall. To start numerical process we define dimensionless

parameters as

$$\begin{aligned} x &\rightarrow r/R, \\ \widetilde{\psi_1} &\rightarrow R\psi_1, \\ \widetilde{\psi_2} &\rightarrow R\psi_2, \\ \widetilde{g_{11}} &= \frac{2m}{\hbar^2} g_{11}, \\ \widetilde{g_{22}} &= \frac{2m}{\hbar^2} g_{22}, \\ \widetilde{\mu_1} &= \frac{2mR^2}{\hbar^2} \mu_1, \\ \widetilde{\mu_2} &= \frac{2mR^2}{\hbar^2} \mu_2. \end{aligned} \tag{4.44}$$

Gross-Pitaevski equations in terms of dimensionless parameters given by

$$-\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{x^2}\Big]\widetilde{\psi_1} + \widetilde{g_{11}}|\widetilde{\psi_1}|^2\widetilde{\psi_1} + \widetilde{g_{12}}|\widetilde{\psi_2}|^2\widetilde{\psi_1} = \widetilde{\mu_1}\widetilde{\psi_1} \tag{4.45}$$

and

$$-\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{1}{x^2}\Big]\widetilde{\psi_2} + \widetilde{g_{22}}|\widetilde{\psi_2}|^2\widetilde{\psi_2} + \widetilde{g_{12}}|\widetilde{\psi_1}|^2\widetilde{\psi_2} = \widetilde{\mu_2}\widetilde{\psi_2}. \tag{4.46}$$

Solutions for different parameters can be seen in the figures 4.8 and 4.9. Figures agree with our expectations. For repulsive inter-species interaction, different type atoms try to avoid each other, at least near the z-axis. For attractive inter-species interaction, different type of atoms try to stay close each other.

By using ground-state wave function, *Bogoliubov* energies can be calculated. For two-species condensate, the ansatz that needs to be inserted in time-dependent Gross-Pitaevski equation may be written as

$$\Psi_1(r,\phi,t) = \big[\psi_1(r) + u_1(r,\phi)e^{-i\omega t} + \upsilon_1^*(r,\phi)e^{i\omega t}\big]e^{-i\mu t/\hbar} \tag{4.47}$$

and

$$\Psi_2(r,\phi,t) = \big[\psi_2(r) + u_2(r,\phi)e^{-i\omega t} + \upsilon_2^*(r,\phi)e^{i\omega t}\big]e^{-i\mu t/\hbar}. \tag{4.48}$$

Thus, we obtain four-coupled differential equations. By taking advantage of definitions of (4.21), (4.44) and using fourier expansion of perturbed wave functions where

$$\begin{aligned} u_1 &= \sum_n u_{1n}e^{in\phi} \qquad \upsilon_1 = \sum_m \upsilon_{1m}e^{-im\phi}, \\ u_2 &= \sum_k u_{2k}e^{ik\phi} \qquad \upsilon_2 = \sum_p \upsilon_{2p}e^{-ip\phi}. \end{aligned} \tag{4.49}$$

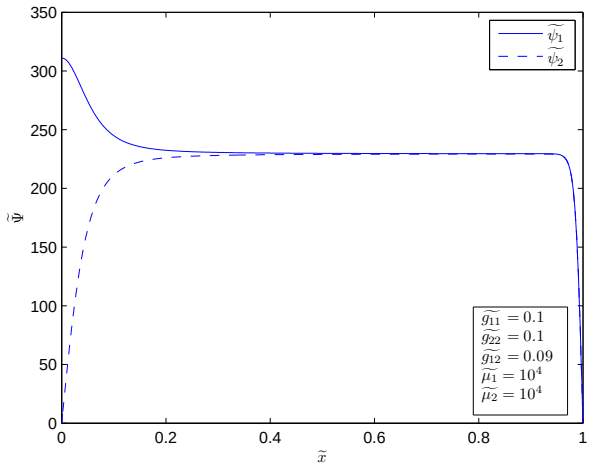

Figure 4.8: Wave functions of mixture with repulsive inter-species interaction where $N_1 = 1.62 \times 10^5$ and $N_2 = 1.59 \times 10^5$

The *Bogoliubov* equations follows as
the first one is

$$\epsilon u_{1n} = -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{n^2}{x^2} - \mu_1 + 2g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2\Big]u_{1n} + g_{11}(\psi_1)^2\upsilon_{1(-n)} + g_{12}\psi_1\psi_2\upsilon_{2(1-n)} + g_{12}\psi_1\psi_2^*u_{2(n+1)}, \tag{4.50}$$

the second one is

$$-\epsilon\upsilon_{1(-n)} = -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{(-n^2)}{x^2} - \mu_1 + 2g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2\Big]\upsilon_{1(-n)} + g_{11}(\psi_1^*)^2u_{1n} + g_{12}\psi_1^*\psi_2\upsilon_{2(1-n)} + g_{12}\psi_1^*\psi_2^*u_{2(n+1)}, \tag{4.51}$$

the third one is

$$\epsilon u_{2(n+1)} = -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{(n+1)^2}{x^2} - \mu_2 + 2g_{22}|\psi_2|^2 + g_{12}|\psi_1|^2\Big]u_{2(n+1)} + g_{22}(\psi_2)^2\upsilon_{2(1-n)} + g_{12}\psi_1^*\psi_2u_{1n} + g_{12}\psi_1\psi_2\upsilon_{1(-n)}, \tag{4.52}$$

the fourth one is

$$-\epsilon\upsilon_{2(1-n)} = -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{(1-n)^2}{x^2} - \mu_2 + 2g_{22}|\psi_2|^2 + g_{12}|\psi_1|^2\Big]\upsilon_{2(1-n)} + g_{22}(\psi_2^*)^2u_{2(n+1)} + g_{12}\psi_1\psi_2^*\upsilon_{1(-n)} + g_{12}\psi_1^*\psi_2^*u_{1-n} \tag{4.53}$$

where $m = -n$, $p = 1 - n$ and $k = n + 1$ and once again we drop tilde symbols. The excitation energies for different solutions of Gross-Pitaevski equation can be seen in the tables 4.7 and 4.8. We ran the code for various parameters, but we did not observe *vortex transfer* in the system.

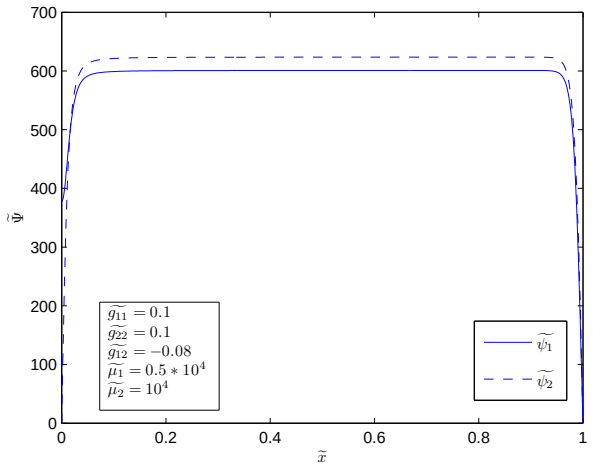

Figure 4.9: Wave functions of mixture with attractive inter-species interaction where $N_1 = 1.09 \times 10^6$ and $N_2 = 1.18 \times 10^6$

4.3.2 Charged-Neutral Superfluid Mixture Under Magnetic Field

The last system that we will investigate in this section has two condensate under magnetic field which one of them charged and has vortex. Also, two condensate interact with each other. We need to start with solving Gross-Pitaevski equation to study vortex-transfer. The wave functions that defined in equation 4.41 can describe the system and as in the previous section, system has two coupled G.P. equation as

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\Big]\psi_1 + g_{11}|\psi|_1^2\psi_1 + g_{12}|\psi_2|^2\psi_1 = \mu_1\psi_1 \tag{4.54}$$

and

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{q^2B^2r^2}{8m} - \frac{\hbar qB}{2m}\Big]\psi_2 + g_{22}|\psi|_2^2\psi_2 + g_{12}|\psi_1|^2\psi_2 = \mu_2\psi_2. \tag{4.55}$$

Boundary Conditions: System has same boundary conditions as we depicted in the previous section.

We will follow the standard procedure to initiate numerical process by using

1^{st}	-96.1883
	103.5266
	-213.6367
	232.7449
2^{nd}	-323.1548
	354.9493
	-428.6085
	436.4028
3^{rd}	-437.6475
	484.0542
	-558.01288
	623.4641
4^{th}	-689.3830
	775.5332
	-834.3192
	941.7919

Table 4.7: Excitation energies of two-component system where $g_{12} = 0.09$, n=2

defined dimensionless parameters (4.37) and (4.44). The dimensionless Gross-Pitaevski equation may be written as

$$-\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{x^2}\Big]\widetilde{\psi_1} + \widetilde{g_{11}}|\widetilde{\psi_1}|^2\widetilde{\psi_1} + \widetilde{g_{12}}|\widetilde{\psi_2}|^2\widetilde{\psi_1} = \widetilde{\mu_1}\widetilde{\psi_1} \tag{4.56}$$

and

$$-\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{1}{x^2} + \gamma x^2 - 2\sqrt{\gamma}\Big]\widetilde{\psi_2} + \widetilde{g_{22}}|\widetilde{\psi_2}|^2\widetilde{\psi_2} + \widetilde{g_{12}}|\widetilde{\psi_1}|^2\widetilde{\psi_2} = \widetilde{\mu_2}\widetilde{\psi_2}. \tag{4.57}$$

To observe *vortex transfer*, we ran our code for different parameters. One of the scenario that we tried is shown in the figure 4.10. Due to high magnetic field, the particles in the charged condensate gather near the z axis, because of this high concentration, density of the other condensate is lowered in this region. Results shown in the figure agree with our expectations. Our next step is to present *Bogoliubov* equations in dimensionless form.

By using variational form (4.47),(4.48) and fourier expansions of $u(r,\phi)$ and $\upsilon(r,\phi)$, four-coupled dimesionless *Bogoliubov* equations follows as the first one is

$$\begin{aligned}\epsilon u_{1m} = &-\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{m^2}{x^2} - \mu_1 + 2g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2\Big]u_{1m} \\ &+ g_{11}(\psi_1)^2\upsilon_{1(-m)} + g_{12}\psi_1\psi_2\upsilon_{2(1-m)} + g_{12}\psi_1\psi_2^*u_{2(m+1)},\end{aligned} \tag{4.58}$$

1^{st}	370.9914
	377.9047
	-819.381
	836.5595
2^{nd}	-1115.6995
	1123.2726
	-1219.7203
	1246.9724
3^{rd}	-1614.5755
	1651.7953
	-2009.3827
	2056.6861
4^{th}	-2405.9701
	2463.6903
	-2467.4343
	2486.4067

Table 4.8: Excitation energies of two-component system where $g_{12} = -0.08$, n=2

the second one is

$$-\epsilon \upsilon_{1(-m)} = -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{(-m^2)}{x^2} - \mu_1 + 2g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2\Big]\upsilon_{1(-m)} + g_{11}(\psi_1^*)^2 u_{1m} + g_{12}\psi_1^*\psi_2\upsilon_{2(1-m)} + g_{12}\psi_1^*\psi_2^* u_{2(m+1)}, \tag{4.59}$$

the third one is

$$\epsilon u_{2(m+1)} = -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{(m+1)^2}{x^2} - \mu_2 + 2g_{22}|\psi_2|^2 + g_{12}|\psi_1|^2\Big]u_{2(m+1)} + \Big[\gamma x^2 - 2\sqrt{\gamma}(m+1)\Big]u_{2(m+1)} + g_{22}(\psi_2)^2\upsilon_{2(1-m)} + g_{12}\psi_1^*\psi_2 u_{1m} + g_{12}\psi_1\psi_2\upsilon_{1(-m)}, \tag{4.60}$$

the fourth one is

$$-\epsilon \upsilon_{2(1-m)} = -\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2} - \frac{(1-m)^2}{x^2} - \mu_2 + 2g_{22}|\psi_2|^2 + g_{12}|\psi_1|^2\Big]\upsilon_{2(1-m)} + \Big[\gamma x^2 - 2\sqrt{\gamma}(1-m)\Big]\upsilon_{2(1-m)} + g_{22}(\psi_2^*)^2 u_{2(m+1)} + g_{12}\psi_1\psi_2^*\upsilon_{1(-m)} + g_{12}\psi_1^*\psi_2^* u_{1-m} \tag{4.61}$$

where $k = -m$, $p = 1 - m$ and $n = m + 1$. Excitation energies are exhibited in the table 4.9. Non-real eigenvalues are observed and we conclude that the system is experiencing $\underline{vortex\,transfer}$.

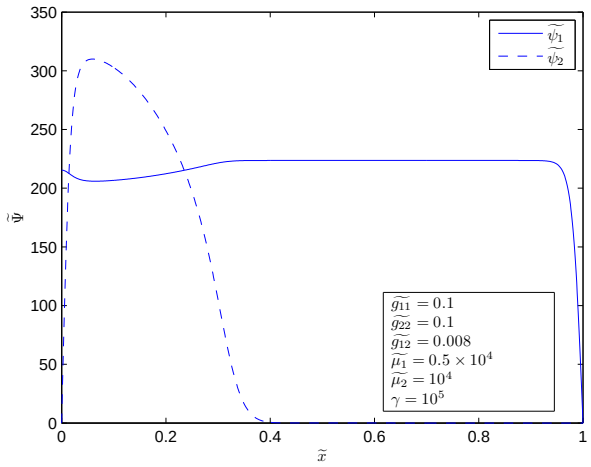

Figure 4.10: Wave functions of charged-neutral superfluid mixture

1^{st}	305.9299 -306.8793 - 0.8177i -306.8793 + 0.8177i -674.4201
2^{nd}	674.7199 842.7632 -1001.5476 1002.8715
3^{rd}	-1325.2200 1327.4081 -1649.4168 1651.1409
4^{th}	-2405.9701 2463.6903 -2467.4343 2486.4069

Table 4.9: Excitation energies of charged-neutral superfluid

Chapter 5

Conclusion

In our study, we have investigated properties of superfluids and vortex transfer in the superfluid mixtures. Before dive into N-body problem, we examine two-body problem to acquire physical insight for N-body problem.

We solved two-body problem by using two different methods: perturbation and numerical. The physical quantities were calculated via different methods and were compared with each other for small interaction parameter. We observed angular momentum transfer from charged particle to neutral particle. Also we presented the dependence of angular momentum transfer to magnetic field. We found that increase in magnetic field reinforces the transfer in the ground state.

We studied N-body problem under various scenarios. By means of numerical methods, we obtained the wave function of N-particles. We controlled the consistency of these solutions with suitable parameters. By exploiting solution of Gross-Pitaevski equation and $MATLAB$ we calculated the excitation spectrum of the systems to check stability. Then, we shifted our focus to vortex transfer between two components. We found vortex transfer between two components when vortex state coupled to the magnetic field. Vortex transfer revealed itself via non-real eigenvalue.

Our next study aims to produce phase diagram of vortex transferred system.

Also, the numerical method that is used to solve Gross-Pitaevski equation may be modified to number-conserved form. Because, in our codes we designate chemical potential and our codes find number of particles associated with specified chemical potential. Thus, this is a minor handicap to compare our results with experiments. We may include the modification of program into our further study.

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Appendix A

Numerical Methods

A.1 Relaxation Scheme

The best way to explain the method that we used in calculations is clarifying it with an example. We start with writing dimensionless 1-d G.P. equation in the form

$$-\frac{\partial \Psi}{\partial t} = -\left[\frac{\partial}{\partial x^2} + V - \mu\right]\Psi \tag{A.1}$$

where V is dimensionless trapping potential and μ is dimensionless chemical potential. We discretize the equations in both space and time as

$$\Psi_{j+1}^{n+1} = \Psi_j^n + \tau\Big[\frac{\Psi_{j+1}^n - 2\Psi_j^n + \Psi_{j-1}^n}{\Delta x} + V_j\Psi_j^n - \mu\Psi_j^n\Big]. \tag{A.2}$$

Here, upper index (n) stands for time discretization and lower index (j) corresponds to space discretization. Δx and τ are minimum intervals in space and time respectively. At the beginning we assign a trial value to Ψ which satisfies boundary conditions . If we apply this scheme iteratively, we end up with steady-state solution which corresponds to the ground state. In every iteration, we compare old and new values of Ψ regarding to specified *tolerance* value which enable us to control relaxation time. *Tolerance* value can be defined roughly as; difference between old and new values of Ψ after every iteration, in our codes it is in the order of 10^-8 and 10^-9. Taking 1000 space point is providing good accuracy. To

realize a successful relaxation, parameters must obey this criteria where [21]

$$\tau \leq \frac{(\Delta x)^2}{4}. \tag{A.3}$$

Otherwise this scheme becomes unstable.

A.2 Solving Eigen-Value Problem

To solve an eigenvalue problem with coupled differential equations, we need to discretize derivatives in space domain, in this way, problem reduces to (2N-4) (due to boundary conditions) set of algebraic equations.

Once again, we work on an example to clarify our arguments. If the system is one-component condensate in a box, then *Bogoliubov* equations of the system can be written as

$$\epsilon u = -\frac{d^2u}{dx^2} + 2g|\psi|^2 u - \mu u + g(\psi)^2 \upsilon \tag{A.4}$$

and

$$\epsilon \upsilon = \frac{d^2\upsilon}{dx^2} - 2g|\psi|^2 \upsilon + \mu \upsilon - g(\psi)^2 u. \tag{A.5}$$

After discretization,

$$\epsilon u(j) = -\frac{u(j+1) - 2u(j) + u(j-1)}{(\Delta x)^2} + 2g|\psi(j)|^2 u(j) - \mu u(j) + g(\psi(j))^2 \upsilon(j) \tag{A.6}$$

and

$$\epsilon \upsilon(j) = \frac{\upsilon(j+1) - 2\upsilon(j) + \upsilon(j-1)}{(\Delta x)^2} - 2g|\psi(j)|^2 \upsilon(j) + \mu \upsilon(j) - g(\psi(j))^2 u(j) \tag{A.7}$$

where Δx is the minimum space interval which depends on N. j indices describe the points in discretized space. Using this form we can write (2N-4) set of equations in the matrix form. First define the column matrix as

$$\begin{pmatrix} u(2) \\ \vdots \\ u(N-1) \\ v(2) \\ \vdots \\ v(N-1) \end{pmatrix} = \mathbf{v}$$

Thus, (2N-4) set of equations reduces to form;

$$\epsilon \mathbf{v} = \mathbf{A}\mathbf{v}. \tag{A.8}$$

All we have to do is constructing **A** matrix on the basis of set of equations. Eigenvalues of this matrix correspond to excitation energies. So, simple *eig* command in $MATLAB$ does the job.

Appendix B

Verification of Results

B.1 Gross-Pitaevski Equation for 2-d Harmonic Trap

To be sure of our results, we need to test our findings. In order to verify reliability of solution of Gross-Pitaevski equation, we solved G.P. equation in 2-d harmonic trap with cylinder geometry and crosschecked the results with the paper. [22] G.P. equation can be written with 2-d cylindrical harmonic potential as

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}\Big]\psi+\frac{1}{2}\omega^2 r^2\psi+g|\psi|^2\psi=\mu\psi. \tag{B.1}$$

We used the dimensionless parameters which are defined in the paper [22]. These are

$$\begin{aligned}\alpha &= \mu/\hbar\omega\\ c &= \frac{\hbar}{\omega a^2 m}\\ x &= r/a\\ \psi(x) &= \frac{a\sqrt{2mg}}{\hbar}\psi(r)\end{aligned} \tag{B.2}$$

where $a = \sqrt{\hbar/m\omega}$ and c carries the sign of g.
G.P. eqn. becomes

$$-\Big[\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2}\Big]\psi(x) + x^2\psi(x) + c|\psi(x)|^2\psi(x) - 2\alpha\psi(x) = 0. \tag{B.3}$$

Usual normalization for the system is

$$N = 2\pi \int r|\psi(r)|^2 dr. \tag{B.4}$$

Normalization with dimensionless parameters is given by

$$\int x|\psi(x)|^2 dx = \eta N = n \tag{B.5}$$

where n is reduced number of atoms. In the paper, $\eta = 10^{-4}$ [22]. Comparison between our calculated n values and results of the paper is a good test to investigate reliability of our numerical method.

α	n	n
2	2.9303	2.9253
2.2	3.7509	3.7464
2.4	4.6518	4.6476
2.6	5.6331	5.6292
3.0	7.8377	7.8341
3.4	10.3651	10.3617

Table B.1: α and Reduced number of atoms where c=1

First column is dimesionless chemical potential values. Second column exhibits the reduced number of atoms calculated by our numerical approach. Third column indicates the values from paper. [22] Our values have a good agreement with paper's results. Error is in the order of $0.2\% - 1\%$.

We can safely use our numerical method to calculate Gross-Pitaevski equation unless c is negative. One of the handicaps of numerical method that we followed is our program could not realize relaxation for attractive interactions.

B.2 Bogoliubov Equations in 1-d Box

We can make simple approximation to test our numerical method required to solve Bogoliubov equations. Let's start with equations that we derived in *chapter* 4. The Bogoliubov equations for 1-d box with one-component gas is

$$\widetilde{\epsilon u} = -\frac{\hbar^2}{2m}\frac{d^2\widetilde{u}}{dx^2} + 2\widetilde{g}|\psi|^2\widetilde{u} - \widetilde{\mu}\widetilde{u} + \widetilde{g}(\widetilde{\psi})^2\widetilde{v} \tag{B.6}$$

and

$$\widetilde{\epsilon v} = \frac{\hbar^2}{2m}\frac{d^2\widetilde{v}}{dx^2} - 2\widetilde{g}|\psi|^2\widetilde{v} + \widetilde{\mu}\widetilde{v} - \widetilde{g}(\widetilde{\psi})^2\widetilde{u}. \tag{B.7}$$

Here, for very short coherence length such as $\xi \approx 0.01$ we assume $\widetilde{\psi} = constant$ and define new parameters as

$$\begin{aligned} \alpha &= 2\widetilde{g}|\psi|^2 - \widetilde{\mu} \\ \beta &= \widetilde{g}(\psi)^2 \end{aligned} \tag{B.8}$$

and coupled equations reduce to

$$\widetilde{\epsilon u} = -\frac{\hbar^2}{2m}\frac{d^2\widetilde{u}}{dx^2} + \alpha\widetilde{u} + \beta\widetilde{v} \tag{B.9}$$

and

$$\widetilde{\epsilon v} = \frac{\hbar^2}{2m}\frac{d^2\widetilde{v}}{dx^2} - \alpha\widetilde{v} - \beta\widetilde{u}. \tag{B.10}$$

by arrenging equations for v we obtain as

$$\frac{d^4v}{dx^4} - 2\alpha\frac{d^2v}{dx^2} - (\epsilon^2 - \alpha^2 + \beta^2)v = 0. \tag{B.11}$$

This is a fourth order Schrödinger like equation. We propose a solution of the form $e^{\gamma x}$ and by inserting this solution to differential equation, we can solve for γ.

$$\gamma^2 = \alpha \pm \sqrt{\epsilon^2 + \beta^2}. \tag{B.12}$$

We can treat v as a wave function, it has to satisfy boundary conditions of box. Thus, solution of v must be in the form of A$sin(kx)$ where $ik = \gamma$ and $k = n\pi$ due to boundary conditions. If we write γ^2 we can observe that

$$\gamma^2 = -n^2\pi^2 = \alpha \pm \sqrt{\epsilon^2 + \beta^2}. \tag{B.13}$$

Thus we can find an expression for eigenvalue ϵ as

$$\epsilon^2 = (n^2\pi^2 + \alpha)^2 - \beta^2. \tag{B.14}$$

because of the assumption of the constant ψ, we can approximate our new-defined parameters α and β as

$$\alpha = \beta \approx \widetilde{\mu}. \tag{B.15}$$

Hence, eigenvalue is approximated to

$$\epsilon \approx \pm\sqrt{2n^2\pi^2\widetilde{\mu} + n^4\pi^4}. \tag{B.16}$$

Calculated eigenvalues are

ϵ_1	± 444
ϵ_2	± 888
ϵ_3	± 1332
ϵ_4	± 1777

Table B.2: Approximated Bogoliubov energies of 1-d box

We can compare these results with Table 4.1. Error is in the order of $\sim 1.4\%$. Hence our numerical method succeeds in the solution of *Bogoliubov* equations.