HIGH FREQUENCY QUADRATURE AND ITS APPLICATION IN SCATTERING PROBLEMS

by

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ABSTRACT

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Highly oscillating integrals have drawn interest since the early stages of real and complex analysis. Their role in scattering theory, optics, fluid dynamics and many other applications increased this interest, and in the early 2000s study of numerical computation of such integrals leaped forward. Several methods are proposed for numerical methods, which we briefly compare in the thesis.

In this thesis we survey the Filon-Clenshaw-Curtis quadrature for highly oscillating integrals with algebraic and logarithmic singularities and oscillators with stationary points. In that matter, several algorithms are developed and error analysis is presented. The algorithms utilize the advantageous properties of Filon-Clenshaw-Curtis quadrature compared to previously proposed numerical methods. Numerical experiments which investigate extreme cases show the accuracy and stability of the proposed quadrature rule.

ÖZET

YÜKSEK FREKANSLI İNTEGRALLER VE SAÇILIM PROBLEMLERİNDE UYGULAMALARI

Yüksek frekanslı integraller reel ve karmaşık analizin ilk aşamalarından itibaren ilgi çekmiştir. Saçılma teorisi, optik, akışkanlar dinamiği ve diğer birçok alanlardaki uygulamaları bu ilgiyi arttırmış ve 2000'li yılların başından itibaren bu integrallerin sayısal hesaplanmasına yönelik çalışmalar büyük bir sıçrama yapmıştır. Tezde, sayısal yöntem olarak önerilen yöntemleri kısaca karşılaştıracağız.

Bu tezde cebirsel ve logaritmik tekillikleri ile durağan noktaları olan integrallerin hesaplanması için sunulan Filon-Clenshaw-Curtis yöntemini inceleyeceğiz. Bu bağlamda, çeşitli algoritmalar geliştirilmiş ve hata analizleri sunulmuştur. Algoritmalar, Filon-Clenshaw-Curtis yönteminin diğer sayısal yöntemlere kıyasla sahip olduğu avantajlı özelliklerinden yararlanmaktadır. Önerilen sayısal hesaplamanın doğruluğunu, stabilitesini inceleyen ve çeşitli kritik vakayı inceleyen deneyler sunulmuştur.

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LIST OF SYMBOLS

C	Continuous function space
$C[a,b]$	Continuous function space over the interval $[a,b]$
C^{∞}	Infinitely many differentiable
$C^{\infty}[a,b]$	Infinitely many differentiable on the interval $[a,b]$
$C^{m}_{\beta}[a,b]$	Set of all functions with $\|.\|_{m,\beta} < \infty$
$ci(x)$	Trigonometric integral defined in Lemma 4.8
$f_c(\theta)$	Function $f(\cos(\theta))$
${}_1F_1(a;b;x)$	Confluent hypergeometric function
H^m	Sobolev space of 2π periodic functions
$I_k(f)$	The oscillatory integral with linear oscillator in the interval $[-1,1]$ with oscillatory parameter k
$I^{\alpha}_k(f)$	The oscillatory integral with linear oscillator and logarithmic singularity in the interval $[-1,1]$ with oscillatory parameter k
$I_{k,N}(f)$	Approximation oscillatory integral with linear oscillator in the interval $[-1,1]$ with oscillatory parameter k evaluated on N quadrature points
$I^{[a,b]}_k(f)$	The oscillatory integral with linear oscillator in the interval $[a,b]$ with oscillatory parameter k
$I^{[a,b]}_{k,N}(f)$	Approximation of oscillatory integral with linear oscillator in the interval $[a,b]$ with oscillatory parameter k evaluated on N quadrature points
$I^{[a,b]}_k(f,g)$	The oscillatory integral with oscillator g in the interval $[a,b]$ with oscillatory parameter k
$I^{[a,b]}_{k,N}(f,g)$	Approximation of oscillatory integral with oscillator g in the interval $[a,b]$ with oscillatory parameter k evaluated on N quadrature points
i	Imaginary unit
$L^1(\Omega)$	Space of integrable functions over Ω
$L^2(\Omega)$	Space of square integrable functions over Ω

$L^p(\Omega)$	Space of p-integrable functions over Ω
$\mathbb{P}_N$	Algebraic polynomial of degree N
$\mathbb{N}$	Natural numbers
$\mathbb{N}_0$	Natural numbers including 0
$P_N f$	N^{th} degree polynomial interpolation of function f
$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Positive real numbers
Re $f(z)$	Real part of the complex function f
$S_J f$	Truncated Fourier series of f
$si(x)$	Trigonometric integral defined in Lemma 4.8
$T_n(t)$	Trigonometric Chebyshev polynomials of the first kind
$U_n(t)$	Chebyshev polynomials of the second kind
$Q_N f$	N^{th} degree interpolation of the function f at the points $\cos(j\pi/N)$
$\alpha_{n,N} f$	n^{th} coefficient of $Q_N f$
$\gamma(k)$	Integrals defined at (3.83)
$\Gamma(z)$	Gamma function
$\omega_n(k)$	Weights which are oscillatory linear integrals of n^{th} Chebyshev polynomial of the first kind with oscillatory parameter k
ξ_n^α	Weights with logarithmic singularity which are oscillatory linear integrals of n^{th} Chebyshev polynomial of the first kind with oscillatory parameter k, defined in Eq. (5.3)
$\sum''$	Finite summation whose the first and last term are halved
$\lVert.\rVert_{L^1[a,b]}$	L^1 norm on $[a,b]$
$\lVert.\rVert_{L^2[a,b]}$	L^2 norm on $[a,b]$
$\lVert.\rVert_{L^\infty[a,b]}$	L^∞ norm on $[a,b]$
$\lvert.\rvert_{H^m_\omega}$	Weighted seminorm of the functions in H^m on $[a,b]$
$\lVert.\rVert_{H^m}$	Norm of H^m
$\lVert.\rVert_{m,\beta}$	Special norm defined in section 4.1

$\lVert . \rVert_{(a,b],m,\beta}$	Generalised special norm defined in Definition 4.3
$\lesssim$	Notation for less than of equal
$\gtrapprox$	Notation for greater or approximatley equal
$\lceil . \rceil$	Ceiling function
$\ll$	Notation for much lesser
$\gg$	Notation for much greater

LIST OF ACRONYMS/ABBREVIATIONS

FCC	Filon-Clenshaw-Curtis

1. INTRODUCTION

In this thesis, we derive a two-phase algorithm depending on the Filon-Clenshaw-Curtis quadrature rule for the computation and error estimates of highly oscillatory integrals ($k \gg 1$) in the form

$$I_k^{[a,b]}(f,g) = \int_a^b f(x)\exp(ikg(x))dx \tag{1.1}$$

where f may have singularities and g may have finitely many stationary points ($g'(x) = 0$). We also consider oscillatory integrals in the form

$$I_k^{[a,b],\alpha}(f,g) = \int_a^b f(x)\log((x-\alpha)^2)\exp(ikx)dx \quad \text{where } \alpha \in [-1,1]. \tag{1.2}$$

These types of integrals are encountered in a wide range of studies; acoustic scattering, optics, image analysis, fluid dynamics, and so on [1] [3] [11]. Thus, the oscillatory integrals (1.1) and (1.2) has been a focus for many researchers in the last two decades.

Filon-type methods along with Levin's method and the steepest descent method are considered as the most efficient methods for calculating highly oscillatory integrals [12]. Filon-type methods interpolates the integrand f at distinct nodes $\{c_i\}_1^{N+1}$ with N^{th} degree polynomial $P_N f$ and calculates the integral as an approximation of (1.1)

$$I_{k,N}^{[a,b]}(f,g) = \int_a^b P_N f(x)\exp(ikg(x))dx \tag{1.3}$$

when g is the linear oscillator ($g(x) = x$) and $[a,b] = [-1,1]$, $I_{k,N}^{[a,b]}(f,g)$ is written as $I_{k,N}(f)$. The method's name comes from Louis Napoleon George Filon, who introduced the method in his seminal work in 1928 [8], fort the calculation the integral

$$\int_a^b f(x)\sin(kx)dx. \tag{1.4}$$

Contemporary works on Filon-type methods established asymptotic error analysis and convergence rate of the method. The paper [6] showed that the precision increases as the frequency of oscillation (namely k) increases and the decay rate of error is $\mathcal{O}(k^{-2})$ as $k \to \infty$. Later, Dominguez, Graham and Smyshlyaev in their paper 'Stability and error estimates for Filon-Clenshaw-Curtis Rules for highly oscillatory integrals' [3] improved the error analysis, which we included in this thesis.

Filon-type methods enjoy simplicity and flexibility [13]. However, the Filon method depends on the computability of moments in the form

$$I_k^{[-1,1]}(x^m, g) = \int_{-1}^{1} x^m \exp(ikg(x))dx, \tag{1.5}$$

which give rise to in closed form (incomplete) Gamma function and this, in turn, leads to unstable computation as the order of the interpolation polynomial's degree increases [7]. Later studies on the computation of moments, such as Olver that of [10] derives the moment-free Filon-type method to deal with the difficulty of computability of moments. For further discussions on interpolation polynomials, nodes, and moment computations we refer to [14].

The approach we discuss in this thesis for the numerical approximation of oscillatory integrals (1.1) and (1.2) is based on the Clenshaw-Curtis quadrature. The classical Clenshaw-Curtis method emerged in the paper [9], to deal with the integral (1.1) when $k = 0$. In Clenshaw-Curtis rule, the interpolant $P_N f$ (1.3) is computed using the Clenshaw points

$$x_j = \cos\left(\frac{j\pi}{n}\right), \quad j \in \{0, \dots, n\}. \tag{1.6}$$

For the linear oscillator ($g(x) = x$), the nonsingular integrand f and the interval $[-1, 1]$, Filon-Clenshaw-Curtis rule, using $N+1$ evaluations, approximates the integral (1.1) by replacing f with $P_N f$ [3] to have

$$I_{k,N}^{[-1,1]} = \int_{-1}^{1} P_N f(x) \exp(ikx) dx. \tag{1.7}$$

With FFT (Fast Fourier Transform), $P_N f$ can be expressed as a linear combination of Chebyshev polynomials of the first kind, which will be developed later in the thesis. Chebyshev polynomials admit recursive relations, which is essential for the calculation of the weights. Also, this recursive relation provides a better calculation time [3]. Therefore, our computation mainly depends on the calculating the weights

$$\omega_n(k) = \int_{-1}^{1} T_n(x) \exp(ikx) dx, \quad n \in \{0, \ldots, N\}. \tag{1.8}$$

In [3], Dominguéz presents estimates and algorithms for nonsingular f with linear oscillator. In [4], they extended the estimates to cover the case of f with algebraic singularities of order r and obtained estimates in the form

$$\left| I_k^{[-1,1]}(f) - I_{k,N}^{[-1,1]}(f) \right| \leq C k^{-r} N^{\rho(r)-m} \|f_c\|_{H^m} \tag{1.9}$$

for all $N \geq 1$ and $r \in [0, 2]$ (see section 3.2 Theorem 3.3).

The paper [4] also discusses the case where g has a stationary point and extends the FCC method to cover this case (see Chapter 4).

This thesis is organized as follows. In Chapter 2 we show and discuss different methods for numerical computation of the integral (1.1). In Chapter 3, the algorithm for the FCC rule is constructed, and the error estimates are derived. In Chapter 4 the composite algorithm is presented for oscillatory integrals with singularities and stationary points. Chapter 5 focuses on the integral (1.2) to obtain new weights for this type of integrals. In Chapter 6 numerical experiments are given for various settings which involve oscillatory integrals arising in scattering theory.

2. METHODS FOR COMPUTING HIGHLY OSCILLATING INTEGRALS

As mentioned in the introduction several different numerical algorithms are developed in the literature for the efficient evaluation of the oscillatory integrals (1.1). In this chapter, we provide a brief overview along with a comparison.

2.1. Asymptotic Expansion

The asymptotic expansion method is the basis of most of the methods developed for integrals (1.1) as it provides a general guideline. An important property of asymptotic expansion is that, as the frequency k increases, estimation accuracy increases. This property is crucial since our main interest is computing highly oscillating integrals.

For definiteness, let assume that $[a, b] = [-1, 1]$, $f \in C^\infty[-1, 1]$ and g has no stationary points. For convenience, we use the notation $I_k(f)$ for integrals on $[-1, 1]$. In this case, an integration by parts yields

$$\begin{aligned} I_k(f) = \int_{-1}^{1} f(x)\exp(ikg(x))dx &= \frac{1}{ik}\int_{-1}^{1} \frac{f(x)}{g'(x)}\frac{d}{dx}\left[\exp(ikg(x))\right]dx \\ &= \frac{1}{ik}\left[\frac{f(1)}{g'(1)}\exp(ikg(1)) - \frac{f(-1)}{g'(-1)}\exp(ikg(-1))\right] \\ &- \frac{1}{ik}\int_{-1}^{1}\frac{d}{dx}\left[\frac{f(x)}{g'(x)}\right]\exp(ikx)dx. \end{aligned} \tag{2.1}$$

This leads into the approximation

$$I_k(f) \sim \frac{1}{ik}\left[\frac{f(1)}{g'(1)}\exp(ikg(1)) - \frac{f(-1)}{g'(-1)}\exp(ikg(-1))\right]. \tag{2.2}$$

Indeed, as shown in [15] the remaining term in (2.1) decays like $\mathcal{O}(k^{-2})$ since the integral itself decays as $\mathcal{O}(k^{-1})$. Thus, the higher frequency leads to a more accurate approximation. This approach can be extended, by recursively defining,

$$f_0(x) = f(x), \quad f_t(x) = \frac{d}{dx}\frac{f_{t-1}(x)}{g'(x)}, \quad t = 1, 2, \dots \tag{2.3}$$

Repeating the process in Eq.(2.1), for $k \gg 1$, we obtain the asymptotic expansion

$$I_k(f) \sim -\sum_{t=0}^{\infty} \frac{1}{(-ik)^{t+1}} \left[\frac{f_t(1)}{g'(1)} \exp(ikg(1)) - \frac{f_t(-1)}{g'(-1)} \exp(ikg(-1)) \right]. \tag{2.4}$$

Approximating $I_k(f)$ with the partial sum

$$\begin{aligned} I_k(f) = &-\sum_{t=0}^{s} \frac{1}{(-ik)^{t+1}} \left[\frac{f_t(1)}{g'(1)} \exp(ikg(1)) - \frac{f_t(-1)}{g'(-1)} \exp(ikg(-1)) \right] \\ &+ \frac{1}{(-ik)^s} I_k(f_s) \end{aligned} \tag{2.5}$$

we thus see that the error decays at the rate $\mathcal{O}(k^{-s-2})$ as $k \to \infty$. Naturally, this approach is not suitable when $k \ll 1$ [16] [1]. In addition, even for small values satisfying $k > 1$, increasing s may not lead to more accurate approximations [17]. Thus, we can conclude that for a wide range of computations, we need a method that also works for $k \leq 1$. Also we need to know the derivatives of f at the endpoints $\{-1, 1\}$ in the asymptotic expansion method, which we may not be available.

When the oscillator g has a stationary point $c \in [-1, 1]$, the expansion (2.5) must be appropriately modified. Considering, for instance, the case of a first order stationary point ($g'(c) = 0$ and $g''(c) \neq 0$), we define the moments associated with the oscillator g as

$$\mu_m(k) = \int_{-1}^{1} x^m \exp(ikg(x))dx, \quad m \geq 0. \tag{2.6}$$

Following the same steps as in (2.1) and (2.4), we obtain,

$$I_k(f) = \mu_0(k) f(c) + \frac{1}{ik} \int_{-1}^{1} \frac{d}{dx} \left[\frac{f(x) - f(c)}{g'(x)} \right] \exp(ikx) dx. \tag{2.7}$$

The integral on the right-hand side has an integrand that has a removable singularity at $x = c$, hence we can use the same approach in the derivation of (2.4) by defining

$$f_0(x) = f(x), \quad f_t(x) = \frac{d}{dx} \left[\frac{f_{t-1}(x) - f_{t-1}(c)}{g'(x)} \right]. \tag{2.8}$$

This leads into (c.f. Theorem 2.3 [16])

$$\begin{aligned} I_k(f) &\sim \mu_0(k) \sum_{t=0}^{\infty} \frac{f_t(c)}{(-ik)^t} \\ &- \sum_{m=0}^{\infty} \frac{1}{(-ik)^{m+1}} \left[\frac{f_m(1) - f_m(c)}{g'(1)} \exp(ikg(1)) - \frac{f_m(-1) - f_m(c)}{g'(-1)} \exp(ikg(-1)) \right]. \end{aligned} \tag{2.9}$$

As above, one can then use partial sums of (2.9) to approximate $I_k(f)$. While, as shown in [15] this improves the error bound to $\mathcal{O}(k^{-s-3/2})$. Computation of $\mu_0(k)$ is usually problematic. See [16] for extension of this approach to cases where the order of the stationary point is larger.

2.2. Steepest Descent Method

The steepest descent method has a different approach than other methods as it deforms the integration path of the integral (1.1) to the complex plane.

Indeed, considering the case of the linear oscillator $g(x) = x$ and the interval $[-1, 1]$, one observes that

$$\exp(ikz) = \exp(ika) \exp(-kb) \quad \text{for } z = a + ib. \tag{2.10}$$

Thus, the oscillatory part comes from the real component of z, and the imaginary component is nonoscillatory and decaying with increasing k. Assuming that the f is analytic on the upper half plane, one can then invoke Cauchy's Theorem to have

$$
\begin{aligned}
I_k(f) = \int_{-1}^{1} f(x)\exp(ikx)dx &= \int_0^N f(-1+it)\exp(ik(-1+it))idt \\
&+ \int_{-1}^{1} f(t+iN)\exp(ik(t+iN))dt - \int_0^N f(1+it)\exp(ik(1+it))idt \\
&= i\exp(-ik)\int_0^N f(-1+it)\exp(-tk)dt \\
&+ \exp(-kN)\int_{-1}^{1} f(t+iN)\exp(ikt)dt \\
&- i\exp(ik)\int_0^N f(1+it)\exp(-tk)dt.
\end{aligned} \tag{2.11}
$$

The term involving $\exp(-kN)$ in this equation decays as N increases. Taking limit as $N \to \infty$ we obtain

$$
\begin{aligned}
I_k(f) = i\exp(-ik)\int_0^\infty f(-1+it)\exp(-tk)dt \\
- i\exp(ik)\int_0^\infty f(1+it)\exp(-tk)dt.
\end{aligned} \tag{2.12}
$$

While the Laplace integrals in (2.12) can be expanded into their asymptotic expansions [17], the approach in the steepest descent method [16] is based on the following approximation:

$$
\begin{aligned}
I_k(f) = i\exp(-ik)\int_0^N f(-1+it)\exp(-tk)dt \\
- i\exp(ik)\int_0^N f(1+it)\exp(-tk)dt + \mathcal{O}(\exp(-kN)).
\end{aligned} \tag{2.13}
$$

Considering next the case of a general oscillatory function g without a stationary point, observe that

$$
\exp(ikg(z)) = \exp(-k\,\mathrm{Im}[g(z)])\ exp(ik\,\mathrm{Re}[g(z)]). \tag{2.14}
$$

Thus, oscillatory component is $\exp(ik\,\mathrm{Re}[g(z)])$, and the component $\exp(-k\,\mathrm{Im}[g(z)])$ decays exponentially as $k \to \infty$ if $\mathrm{Im}[g(z)] > 0$ [17]. Choosing a path $\mathit{\Gamma}_a(t)$, originating from point a and parametrization as $\gamma_a(t)$, such that $\mathrm{Re}[g(z)]$ is fixed along the path, will thus remove the oscillating component [18]. That is to say, $\gamma_a(t)$ satisfies, for some interval $[0, N]$,

$$g(\gamma_a(t)) = g(a) + it \tag{2.15}$$

for all $t \in [0, N]$. Then, along the path γ_a, equation (2.14) takes on the form

$$\exp(ikg(z)) = \exp(ikg(\gamma_a(t))) = \exp(-tk)\ \exp(ikg(a)). \tag{2.16}$$

With this choice the integral (1.1) becomes

$$\int_{\mathit{\Gamma}_a} f(z)\exp(ikg(z))dz = \exp(ikg(a))\int_0^N f(\gamma_a(t))\gamma_a'(t)\exp(-kt)dt. \tag{2.17}$$

The integral on the right-hand side of (2.17) is nonoscillatory and decays exponentially with increasing k. Indeed, as shown in Proposition 5.4 in [16], when f and g are analytic in an open neighbourhood $[-1, 1]$, there exists $N > 0$ and paths $\mathit{\Gamma}_{\pm 1}(t)$ satisfying

$$I_k(f) = \left(\int_{\mathit{\Gamma}_{-1}} - \int_{\mathit{\Gamma}_1}\right) f(z)\exp(ikg(z))dz + \mathcal{O}(\exp(-kN)). \tag{2.18}$$

As this equation shows, the steepest descent method does not require computing moments, which is one of the advantages of the method. However, the implementation of the steepest descent method requires the determination of the path $\mathit{\Gamma}_a$ which, in turn, corresponds to the computation local inverse of g. Usually this leads into a nonlinear ordinary differential equation [16]. For instance, if

$$g'(\gamma_a(t))\gamma_a'(t) = i, \tag{2.19}$$

the corresponding nonlinear ordinary differential equation is

$$\gamma_a'(t) = \frac{i}{g'(\gamma_a(t))}. \tag{2.20}$$

A more precise form of these ideas is summarized in the following proposition (see Proposition 5.4 in [16]).

Proposition 2.1. *Consider an oscillatory integral $I_k(f,g)$, with f and g analytic in a neighborhood of $[-1,1]$, and with g real-valued on $[-1,1]$. Let $\{x_m\}_{m=1}^M$ be the ordered set of the endpoints ± 1 and all stationary points of g inside $[-1,1]$. Then there exist a sequence of finite contours $\mathit{\Gamma}_{x_m}$ of constant phase around x_m,*

$$\mathit{\Gamma}_{x_m} \cap \mathbb{R} = \{x_m\} \qquad \textit{and} \qquad \operatorname{Re} g(z) = g\left(x_m\right), \quad z \in \mathit{\Gamma}_{x_m}, \tag{2.21}$$

and a constant $P > 0$ such that

$$I_k(f,g) = \int_{-1}^{1} f(x) e^{ikg(x)} dx = \sum_{m=1}^{M} \int_{\mathit{\Gamma}_{x_m}} f(z) e^{ikg(z)} dz + \mathcal{O}\left(\mathrm{e}^{-kP}\right). \tag{2.22}$$

2.3. Filon Method

Plain (vanilla) Filon method interpolates the integrand f with a polynomial p of degree N, and approximates the integral the (1.1) with $I_k(p)$. As we have seen in asymptotic expansion method (c.f. (2.4)), the integral itself depends on f and its derivatives at the endpoints and, for a given $s \geq 0$, we interpolate f at these points as follows

$$p^{(j)}(a) = f^{(j)}(a) \text{ and } p^{(j)}(b) = f^{(j)}(b), \quad j \in \{0, \ldots, s\}. \tag{2.23}$$

Therefore p is the Hermite interpolant. We define the Filon operator of order s as

$$\mathcal{F}_s(f) = \int_a^b p(x)\exp(ikg(x)) = \sum_{t=0}^{2s+1} c_t I_k^{[a,b]}(x^t, g) \tag{2.24}$$

Again, by approximation (2.4), and the fact that first $s-1$ derivatives of f and p coincides at the endpoints, we obtain decay rate $\mathcal{O}(k^{-s-1})$.

The Filon method satisfies the properties of asymptotic expansion and also works for all k. Thus, the Filon method does not blow up for small values of k . As mentioned before, while the Filon method depends on calculating the moments, Olver [10] has also developed a moment-free Filon method. As can be seen in equation (2.23), the Filon method depends on the computability of derivatives of f at the endpoints, which can be unavailable in some cases. In [16] [13], it is advised that for a derivative-free approach, instead of calculating $f'(a)$, one can use the value $f(a+h)$ for sufficiently small h, and they established that this method works very well.

Extended Filon methods [16], internal points $\{c_j\}_{j=1}^{\nu}$ are also considered by adding the restrictions

$$f(c_j) = p(c_j) \tag{2.25}$$

to Eq.(2.23), for all $j \in \{1, \ldots \nu\}$. While the extended Filon method does not provide a better decay rate for the method, asymptotic order of a method is not the only property that provides good accuracy. Indeed as shown in [16] and [20], accuracy may be improved by choosing a noval choice of the internal points. More precisely, in [16] Jacobi nodes and Clenshaw-Curtis nodes are considered. While the Filon-Jacobi method provides a better decay rate, the method we discuss in this thesis is based later since the implementation of FCC is cheaper as it does not require the computation of the derivatives of f.

2.4. Levin's Method

David Levin, in his paper [19], constructed the Levin collocation method which does not depend on computing moments. The idea is determining a function F satisfying

$$\frac{d}{dx}\left[F(x)\exp(ikg(x))\right] = f(x)\exp(ikg(x)) \tag{2.26}$$

since, in this case, our integral simplifies into

$$\begin{aligned} I_k^{[a,b]}(f,g) &= \int_a^b f(x)\exp(ikg(x))dx = \int_a^b \frac{d}{dx}\left[F(x)\exp(ikg(x))\right]dx \\ &= F(b)\exp(ikg(b)) - F(a)\exp(ikg(a)). \end{aligned} \tag{2.27}$$

Motivated with this observation we define the differential operator $\mathcal{L}(F)$ as

$$\mathcal{L}(F) = F' + ikg'F, \tag{2.28}$$

and consider the differential equation

$$\mathcal{L}(F) = F' + ikg'F = f. \tag{2.29}$$

As (2.26) implies, there is no need to couple (2.29) with an initial condition. The idea is to approximate F in the form of $v = \sum_{t=1}^{\nu} c_t\psi_t$ where $\{\psi_1, \ldots, \psi_\nu\}$ is a basis for an appropriate space of functions, and the coefficients c_t are obtained by solving the linear system

$$\mathcal{L}(v)(x_t) = f(x_t) \tag{2.30}$$

for all $t \in \{1, \ldots, \nu\}$ with $\{x_1, \ldots, x_\nu\}$ denoting the collocation points. Thus, our approximation is

$$\mathcal{Q}_k^{\mathcal{L}}(f) = \int_a^b \mathcal{L}(v)(x) \exp(ikg(x))dx = \int_a^b \frac{d}{dx}\left[v \exp(ikg(x))\right] dx$$
$$= v(b) \exp(ikg(b)) - v(a) \exp(ikg(a)). \tag{2.31}$$

When the endpoints are included in collocation nodes, Levin's method decays as $\mathcal{O}(k^{-2})$. On the other hand the plain Filon method has the same decay rate with Levin's method when natural polynomial basis $\{1, x, x^2, \ldots, x^{N-1}\}$ is used and $g(x) = x$ [17]. This motivates the following definition

Definition 2.2. *Provided $g' \neq 0$ in $[a, b]$,*

$$\Psi_N^\star = \left\{g'(x), g'(x)g(x), g'(x)g^2(x), \ldots, g'(x)g^{N-1}(x)\right\} \tag{2.32}$$

is called the N^{th} natural basis of the oscillator g.

Indeed, as shown in Theorem 3.9 in [16], when this natural basis is used, Filon and Levin's methods coincide.

A Hermite type generaliziation to Levin's methods can be obtained by considering confluent collocation points, that is distinct points $\{x_1, \ldots, x_M\}$ with multiplicities $\{m_1, \ldots, m_M\}$ so that $\sum_{n=1}^{M} m_l = N$ and determine the coefficients c_t as the solutions of the Hermite linear system

$$\mathcal{L}(v)^{(p)}(x_t) = f^{(p)}(x_t) \tag{2.33}$$

for all $p \in \{0, \ldots, m_t - 1\}$. As shown in [17], [16] when g is nonstationary and $\{\psi_t\}_{t=1}^N$ are chosen as the Chebyshev polynomials, one has

$$\mathcal{Q}_k^{\mathcal{L}}(f) - I_k(f) = \mathcal{O}(k^{-s-1}). \tag{2.34}$$

Therefore the extended Levin's method has a similar asymptotic behavior when compared to the plain Filon method.

3. FILON–CLENSHAW–CURTIS RULES FOR HIGHLY OSCILLATORY INTEGRALS

3.1. Filon-Clenshaw-Curtis Rule

We consider the polynomial interpolation operator $Q_N : C[-1,1] \to \mathbb{P}_N$

$$Q_N f\left(t_{j,N}\right) = f\left(t_{j,N}\right) \tag{3.1}$$

where $j = 0, \ldots, N$ and

$$t_{j,N} = \cos\left(\frac{j\pi}{N}\right) \tag{3.2}$$

are the Clenshaw points. Using the trigonometric identity

$$\frac{2}{N}\sum_{n=0}^{N}{}'' \cos\left(\frac{j'n\pi}{N}\right)\cos\left(\frac{jn\pi}{N}\right) = \begin{cases} 1 \text{ if } j = j' \in \{1,2,\ldots,N-1\}, \\ 2 \text{ if } j = j' \in \{0,N\}, \\ 0 \text{ otherwise }, \end{cases} \tag{3.3}$$

we see that

$$Q_N f(s) = \sum_{n=0}^{N}{}'' \alpha_{n,N}(f) T_n(s) \tag{3.4}$$

where $T_n(s) = \cos(n \arccos(s))$ are the Chebyshev polynomials of the first kind (see Appendix A), and

$$\alpha_{n,N}(f) = \frac{2}{N}\sum_{j=0}^{N}{}'' \cos\left(\frac{jn\pi}{N}\right) f\left(t_{j,N}\right), \quad n = 0, \ldots, N. \tag{3.5}$$

Using Eq. (3.4) and Eq. (1.3), $I_{k,N}(f)$ can be written as

$$I_{k,N}(f) = \sum_{n=0}^{N}{}'' \alpha_{n,N}(f)\omega_n(k) \tag{3.6}$$

where the weights

$$\omega_n(k) := \int_{-1}^{1} T_n(s)\exp(iks)ds, \quad n \geq 0. \tag{3.7}$$

have to be computed. The weights $\omega_n(k)$ carry an essential property, namely

$$\begin{aligned}\omega_n(-k) &= \int_{-1}^{1} T_n(s)\exp(i(-k)s)ds = \int_{-1}^{1} T_n(s)\exp(-iks)ds \\ &= \int_{-1}^{1} T_n(s)\overline{\exp(iks)}ds = \overline{\int_{-1}^{1} T_n(s)\exp(iks)ds} \\ &= \overline{\omega_n(k)}.\end{aligned} \tag{3.8}$$

which allows us to expand our setup for $k \geq 0$ to all real numbers. Thus, we can construct an approximation for $k \geq 0$, and for negative values, we can take the conjugate of $\omega_n(k)$.

Next, we introduce the cosine transform of f which is the even 2π-periodic function

$$f_c(\theta) = f(\cos\theta) \tag{3.9}$$

where $\theta \in \mathbb{R}$. Getting back to summation (3.4), we have

$$\begin{aligned}(Q_N f(\theta))_c &= \sum_{n=0}^{N}{}'' \alpha_{n,N}(f)T_n(\cos(\theta)) \\ &= \sum_{n=0}^{N}{}'' \alpha_{n,N}(f)\cos(n\arccos(\cos(\theta)))\end{aligned} \tag{3.10}$$

which shows that $(Q_N f)_c \in \text{span}\{1, \cos\theta, \cos 2\theta, \ldots, \cos N\theta\}$ is the even trigonometric polynomial of degree N that interpolates f_c at the $N+1$ equally spaced points $j\pi/N$, $j = 0, \ldots, N$. In [3] an error analysis is carried out to obtain for $0 \leq \mu \leq v$ and $v \geq v_0 > 1/2$

$$\|f_c - (Q_N f)_c\|_{H^\mu} \leq C_{v_0,\mu} N^{\mu - v} \|f_c\|_{H^v} \tag{3.11}$$

for all $N \geq 2$ and for some constant $C_{v_0,\mu}$. In section 3.3 and 3.4, we follow [3] and provide an improved version of this estimate.

3.1.1. Evaluating Integrals With FCC

Let C be the $(N+1) \times (N+1)$ matrix with entries

$$C_{n,j} = (2/N)\cos(jn\pi/N) \tag{3.12}$$

where $n, j = 0, \ldots, N$, and define the column vectors

$$\boldsymbol{\alpha}_N(f) = [\alpha_{0,N}(f), \alpha_{1,N}(f), \ldots, \alpha_{N,N}(f)]^{\mathrm{T}} \tag{3.13}$$

$$\mathbf{f}_N = [f(t_{0,N})/2, f(t_{1,N}), \ldots, f(t_{N-1,N}), f(t_{N,N})/2]^{\mathrm{T}}. \tag{3.14}$$

By dicrete cosine transform of type I (see Section 4.7.25 in [28]) we can write $\boldsymbol{\alpha}_N(f) = C_N \mathbf{f}_N$ which can be computed by FFT in $\mathcal{O}(N \log N)$ time. Using the fact that C_N is symmetric we have

$$I_{k,N}(f) = \boldsymbol{\omega}_N^{\mathrm{T}} \boldsymbol{\alpha}_N(f) = \boldsymbol{\omega}_N^{\mathrm{T}} C_N \mathbf{f}_N = \left(C_N^{\mathrm{T}} \boldsymbol{\omega}_N\right)^{\mathrm{T}} \mathbf{f}_N = (C_N \boldsymbol{\omega}_N)^{\mathrm{T}} \mathbf{f}_N \tag{3.15}$$

where $\boldsymbol{\omega}_N = [\omega_0(k)/2, \omega_1(k), \ldots, \omega_{N-1}(k), \omega_N(k)/2]^{\mathrm{T}}$. Our focus is mainly on the weight $\boldsymbol{\omega}_N$, since the computation of $\mathbf{C}_N$ and $\mathbf{f}_N$ is straightforward.

3.2. Error Estimation of Integrals Over [-1,1]

In this section, we present error estimations for Eq.(1.3), for a function $f \in C[-1,1]$. We begin with the next result.

Theorem 3.1. *For $r = 0, 1, 2$ and for all $v \geq v_0 > \max\{1/2, \rho(r)\}$ there exists $C_{v_0} > 0$ such that for all $k > 0$*

$$|I_k(f) - I_{k,N}(f)| \leq C_{v_0} \left(\frac{1}{k^r}\right) \left(\frac{1}{N}\right)^{\nu-\rho(r)} \|f_c\|_{H^v} \tag{3.16}$$

where

$$\rho(r) = \begin{cases} 0, & r = 0, \\ 1, & r = 1, \\ 7/2, & r = 2. \end{cases} \tag{3.17}$$

Moreover, the inequality (3.16) *holds also for $v = r = 1$.*

Proof. We begin with introducing the 2π-periodic error function

$$e_N = f_c - (Q_N f)_c\,. \tag{3.18}$$

Since Chebyshev points include the endpoints ± 1 we have,

$$e_N(0) = e_N(\pi) = 0. \tag{3.19}$$

Using the cosine transform along with integration by parts and (3.19), we obtain

$$\begin{aligned} I_k(f) - I_{k,N}(f) &= \int_{-1}^{1} (f - Q_N f)(s) \exp(iks) ds \\ &= \int_0^{\pi} e_N(\theta) \exp(ik\cos\theta) \sin\theta d\theta \\ &= -\frac{\exp(ik\cos\theta) e_N(\theta)}{ik}\Big|_0^{\pi} + \frac{1}{ik}\int_0^{\pi} e_N'(\theta)\exp(ik\cos\theta) d\theta \\ &= \frac{1}{ik}\int_0^{\pi} e_N'(\theta)\exp(ik\cos\theta) d\theta. \end{aligned} \tag{3.20}$$

When $r = 0$, this entails

$$\begin{aligned} |I_k(f) - I_{k,N}(f)| &= \left| \int_0^{\pi} e_N(\theta)\exp(ik\cos\theta)\sin\theta d\theta \right| \\ &\leq \int_0^{\pi} |e_N(\theta)| d\theta = ||e_N||_{L^1(0,\pi)} \end{aligned} \tag{3.21}$$

and (3.11), in turn, implies

$$\begin{aligned} ||e_N||_{L^1(0,\pi)} &\leq \sqrt{\pi}||e_N||_{L^2(0,\pi)} = \sqrt{\pi}||e_N||_{H^0(0,\pi)} \\ &\leq \sqrt{\pi} C_{v_0}(N)^{0-\nu} \|f_{\mathrm{c}}\|_{H^v} \\ &= \widetilde{C}_{v_0}(N)^{-\nu}\left(\frac{1}{k^r}\right)\left(\frac{1}{N}\right)^{\nu-\rho(r)} \|f_{\mathrm{c}}\|_{H^v}. \end{aligned} \tag{3.22}$$

For the case $r = 1$, using the definitions (B.3) and (B.4) of Sobolev norms, we see that

$$||e_N||_{H^1} = \left(||e_N||^2_{L^2(0,\pi)} + ||e_N'||^2_{L^2(0,\pi)}\right)^{1/2} \geq \left(||e_N'||^2_{L^2(0,\pi)}\right)^{1/2} = ||e_N'||_{L^2(0,\pi)}. \tag{3.23}$$

Therefore, (3.20) implies

$$\begin{aligned}|I_k(f) - I_{k,N}(f)| &\leq \frac{1}{|k|}\left|\int_0^\pi e'_N(\theta)\exp(ik\cos\theta)d\theta\right| \leq \frac{1}{|k|}\int_0^\pi |e'_N(\theta)||\exp(ik\cos\theta)|d\theta \\ &= \frac{1}{|k|}\int_0^\pi |e'_N(\theta)|d\theta = \frac{1}{|k|}||e'_N||_{L^1(0,\pi)} \leq \frac{1}{|k|}\sqrt{\pi}||e'_N||_{L^2(0,\pi)} \\ &\leq \sqrt{\pi}\frac{1}{|k|}||e'_N||_{H^1(0,\pi)} \leq \sqrt{\pi}C_{v_0}(N)^{0-\nu}\left(\frac{1}{|k|^1}\right)\left(\frac{1}{N}\right)^{\nu-1}\|f_c\|_{H^v} \\ &\leq \widetilde{C}_{v_0}(N)^{0-\nu}\left(\frac{1}{|k|^r}\right)\left(\frac{1}{N}\right)^{\nu-\rho(r)}\|f_c\|_{H^v}\,. \end{aligned} \tag{3.24}$$

For $r = 2$, we observe that $e'_N(\theta) = -\left(f - Q_N f\right)'(\cos\theta)\sin\theta$, so $e'_N(\theta)$ vanishes also at 0 and π. Hence, we can insert function

$$\varphi_N(\theta) = \frac{e'_N(\theta)}{\sin\theta} = -\left(f - Q_N f\right)'(\cos\theta) \tag{3.25}$$

into (3.20) and then perform another integration by parts to obtain

$$\begin{aligned}I_k(f) - I_{k,N}(f) &= \frac{1}{k^2}\left[\varphi_N(\theta)\exp(ik\cos\theta)|_0^\pi - \int_0^\pi \varphi'_N(\theta)\exp(ik\cos\theta)d\theta\right] \\ &= \frac{1}{k^2}\left[E_1 - E_2\right]. \end{aligned} \tag{3.26}$$

Using L'Hôpital's rule and (3.19) we see that $\varphi_N(0) = e''_N(0)$ and $\varphi_N(\pi) = -e''_N(\pi)$, and using the Sobolev embedding theorem and inequality (3.11) again, we obtain

$$|E_1| \leq |\varphi_N(\pi) + \varphi_N(0)| \leq |e''_N(0)| + |e''_N(\pi)| \leq C\left\|e_N\right\|_{H^3} \leq C_{\nu_0}\frac{1}{N^{\nu-3}}\left\|f_c\right\|_{H^\nu} \tag{3.27}$$

for all $v \geq v_0 \geq 3$. Now we turn our focus to E_2. To this end, we begin by writing e_N as a cosine series

$$e_N(\theta) = \sum_{m=1}^{\infty} \widehat{e}_N(m) \cos m\theta \tag{3.28}$$

where

$$\widehat{e}_N(m) = \begin{cases} \dfrac{1}{\pi} \displaystyle\int_0^{\pi} e_N(\theta) d\theta, & m = 0, \\ \dfrac{2}{\pi} \displaystyle\int_0^{\pi} e_N(\theta) \cos m\theta d\theta, & m \geq 1. \end{cases} \tag{3.29}$$

Hence,

$$\varphi_N(\theta) = -\sum_{m=1}^{\infty} m\widehat{e}_N(m) \frac{\sin m\theta}{\sin\theta}. \tag{3.30}$$

Then considering the bounded $\mathcal{C}^\infty$ function $\sigma(\theta) = (\sin\theta)/\theta$, we observe that $\sigma(\theta) \geq 2/\pi$ for $\theta \in [-\pi/2, \pi/2]$ and so for $\theta \in [0, \pi/2]$ and $m \geq 1$, we establish the estimate

$$\left| \left(\frac{\sin m\theta}{\sin\theta} \right)' \right| = \left| m \left(\frac{\sigma(m\theta)}{\sigma(\theta)} \right)' \right| = \left| m^2 \frac{\sigma'(m\theta)}{\sigma(\theta)} - m \frac{\sigma(m\theta)\sigma'(\theta)}{\sigma^2(\theta)} \right| \leq Cm^2 \tag{3.31}$$

for some constant C. The trigonometric identity

$$\frac{\sin m\theta}{\sin\theta} = (-1)^{m-1} \frac{\sin m(\theta - \pi)}{\sin(\theta - \pi)} \tag{3.32}$$

allows us to extend (3.31) to $\theta \in [0, \pi]$. Therefore,

$$|E_2| \leq \pi \left\| \varphi_N' \right\|_{L^\infty(0,\pi)} \leq C \sum_{m=1}^{\infty} m^3 \left| \widehat{e}_N(m) \right|. \tag{3.33}$$

To estimate the right-hand side of (3.33) we use elementary estimates

$$\sum_{m=1}^{N} m^6 = \frac{N^7}{7} + \frac{N^6}{2} + \frac{N^6}{2} - \frac{N^3}{6} + \frac{N}{42} < \frac{(N+1)^7}{7}, \tag{3.34}$$

$$\sum_{m=N+1}^{\infty} \frac{1}{m^{1+\alpha}} < \int_N^{\infty} \frac{1}{m^{1+\alpha}} = \frac{1}{\alpha N^{\alpha}}, \quad \alpha > 0. \tag{3.35}$$

In detail, we divide the sum in (3.33) into two parts as $m \leq N$ and $m \geq N+1$, and use the Cauchy-Schwarz inequality, (3.34), (3.11) and (B.3) to deduce, for all $v \geq v_0 > 7/2$,

$$\begin{aligned} |E_2| &\leq C\Big\{ \left[\textstyle\sum_{m=1}^{N} m^6\right]^{1/2} \left[\textstyle\sum_{m=1}^{N} |\widehat{e}_N(m)|^2\right]^{1/2} + \left[\textstyle\sum_{m=N+1}^{\infty} \frac{1}{m^{2v-6}}\right]^{1/2} \left[\textstyle\sum_{m=N+1}^{\infty} m^{2v} |\widehat{e}_N(m)|^2\right]^{1/2} \Big\} \\ &\leq C\left\{ N^{-1(-7/2)} \|e_N\|_{H^0} + N^{-1(v-7/2)} \|e_N\|_{H^v} \right\} \\ &\leq C_{v_0} N^{-1(v-7/2)} \|f_{\mathrm{c}}\|_{H^v} \end{aligned} \tag{3.36}$$

with C denoting a generic constant. Combining the estimates in (3.27) and (3.36) gives the desired result. □

The error estimate in the preceding theorem is not desirable for small values of k, a case that arises in implementations. Thus, we introduce the following corollary, which gives us a better error estimation when k is small.

Corollary 3.2. *Under the conditions of Theorem 3.1, for* $r = 0, 1, 2$ *and for all* $v \geq v_0 > \max\{1/2, \rho(r)\}$, *there exists* $C_{v_0} > 0$

$$|I_k(f) - I_{k,N}(f)| \leq C_{v_0} \min\left\{1, k^{-r}\right\} N^{-(v-\rho(r))} \|f_{\mathrm{c}}\|_{H^v}. \tag{3.37}$$

Proof. For the case $k \geq 1$ the result is directly obtained from Theorem 3.1 since $k^{-r} \leq 1$ which yields the desired result. When $k < 1$ it follows from inequalities (3.11) and (3.21) that, for $0 \leq \mu \leq v$ and $v \geq v_0 > 1/2$,

$$\begin{aligned} |I_k(f) - I_{k,N}(f)| &\leq ||e_N||_{L^1(0,\pi)} \leq \sqrt{\pi}\, \|e_N\|_{L^2(0,\pi)} \leq \sqrt{\pi}\, \|e_N\|_{H^0} \\ &\leq C_{v_0,\mu} N^{\mu-v} \|f_{\mathrm{c}}\|_{H^v} \\ &\leq_{v_0} \min\left\{1, k^{-r}\right\} N^{-(v-\rho(r))} \|f_{\mathrm{c}}\|_{H^v} \end{aligned} \tag{3.38}$$

which yields the result. □

In Theorem 3.1 provides the necessary estimates when $r = 0, 1, 2$. The next result presents the error estimates for $r \in [0, 2]$.

Theorem 3.3. *There exists a constant $C > 0$ such that for all $r \in [0, 2]$ and all integers $m > \max\{1/2, \rho(r)\}$,$\rho(r) = 5r/2 - 3/2, r \in [1, 2]$, the estimate*

$$|I_k(f) - I_{k,N}(f)| \leq C \left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} \|f_c\|_{H^m} \tag{3.39}$$

where $N \geq 1$ holds when $f_c \in H^m$.

Proof. In Theorem 3.1, the estimate above was obtained for $k \geq 1/2$ and $r = 0, 1, 2$. Now, for $\theta \in [0, 1]$ we write

$$|I_k(f) - I_{k,N}(f)| = |I_k(f) - I_{k,N}(f)|^\theta\, |I_k(f) - I_{k,N}(f)|^{1-\theta}, \tag{3.40}$$

and estimate the first factor on the right-hand side using the bound obtained for $r = 0$ and the second term the bound for $r = 1$ to deduce

$$\begin{aligned} &|I_k(f) - I_{k,N}(f)|^\theta\, |I_k(f) - I_{k,N}(f)|^{1-\theta} \leq \\ &\quad \left(C^\theta \left(\frac{1}{N}\right)^{m\theta} \|f_c\|^\theta_{H^m}\right) \left(C^{1-\theta} \left(\frac{1}{k}\right)^{(1-\theta)} \left(\frac{1}{N}\right)^{(m-1)(1-\theta)} \|f_c\|^{1-\theta}_{H^m}\right). \end{aligned} \tag{3.41}$$

In other words,

$$|I_k(f) - I_{k,N}(f)| \leq C \left(\frac{1}{k}\right)^{1-\theta} \left(\frac{1}{N}\right)^{m-1+\theta} \|f_c\|_{H^m} . \tag{3.42}$$

Setting $\theta = 1 - r$, this entails

$$|I_k(f) - I_{k,N}(f)| \leq C \left(\frac{1}{k}\right)^{r} \left(\frac{1}{N}\right)^{m-\rho(r)} \|f_c\|_{H^m} . \tag{3.43}$$

Thus, obtained the result for $r \in [0, 1]$. Now consider the case $r \in [1, 2]$. We will follow the same approach, bound the first factor of the right-hand side with the bound we obtained for $r = 1$ and the second factor for $r = 2$ to get

$$\begin{aligned} &|I_k(f) - I_{k,N}(f)|^{\theta} \, |I_k(f) - I_{k,N}(f)|^{1-\theta} \\ &\leq \left(C^{\theta} \left(\frac{1}{k}\right)^{\theta} \left(\frac{1}{N}\right)^{(m-1)\theta} \|f_c\|_{H^m}^{\theta}\right) \left(C^{1-\theta} \left(\frac{1}{k}\right)^{2(1-\theta)} \left(\frac{1}{N}\right)^{(m-7/2)(1-\theta)} \|f_c\|_{H^m}^{1-\theta}\right) . \end{aligned} \tag{3.44}$$

Therefore,

$$|I_k(f) - I_{k,N}(f)| \leq C \left(\frac{1}{k}\right)^{2-\theta} \left(\frac{1}{N}\right)^{m-7/2+5\theta/2} \|f_c\|_{H^m} . \tag{3.45}$$

Setting $\theta = 2 - r$, we obtain the desired result for $k \geq 1/2$ while $r \in [1, 2]$. The case $k \leq 1/2$ follows from Theorem 3.1 and Corollary 3.2 since k bounded by $1/2$. □

With these theorems, we obtained a bound depending on f_c. To obtain a bound that depends on the derivatives of f, not f_c, we need to lemmas.

Lemma 3.4. *Let f be such that $(f')_c \in L^1[-\pi,\pi]$. Then*

$$\widehat{f_c}(\mu) = \frac{1}{2\mu}\left[\widehat{(f')_c}(\mu-1) - \widehat{(f')_c}(\mu+1)\right] \qquad \textit{for} \quad \mu \neq 0. \tag{3.46}$$

Proof. Since f_c is even we can use the Sobolev norm (B.3) and integrate by parts to get

$$\begin{aligned}
\widehat{f_c}(\mu) &= \frac{1}{\pi}\int_0^\pi f_c(\theta)\cos(\mu\theta)d\theta = -\frac{1}{\pi\mu}\int_0^\pi (f_c)'(\theta)\sin(\mu\theta)d\theta \\
&= \frac{1}{\mu\pi}\int_0^\pi f'(\cos\theta)\sin\theta\sin(\mu\theta)d\theta \\
&= \frac{1}{2\mu\pi}\int_0^\pi (f')_c(\theta)[\cos((\mu-1)\theta) - \cos((\mu+1)\theta)]d\theta \\
&= \frac{1}{2\mu\pi}\int_0^\pi f'\cos((\mu-1)\theta)d\theta - \frac{1}{2\mu\pi}\int_0^\pi f'\cos((\mu+1)\theta)d\theta \\
&= \frac{1}{2\mu}\left[\widehat{(f')_c}(\mu-1) - \widehat{(f')_c}(\mu+1)\right]
\end{aligned} \tag{3.47}$$

which delivers the desired result. □

Lemma 3.5. *For $0 \leq m \leq N+1$ there exist constants $\sigma_{m,N} > 0$ such that*

$$\|(I - \mathcal{S}_N)\, f_c\|_{H^m} \leq \sigma_{m,N} \left\|\left(f^{(m)}\right)_c\right\|_{H^0} \tag{3.48}$$

Proof. Since $\mathcal{S}_N$ is the orthogonal projection of H^0 onto $\operatorname{span}\{\exp(ij\theta) : 0 \leq |j| \leq N\}$, the result is straightforward when $m = 0$. So we assume $m \geq 1$. Since f_c is even, from (B.3) and (B.5) we have, for all $J \geq 0$,

$$(I - \mathcal{S}_J)\, f_c = 2\sum_{\mu=1}^{\infty} \widehat{f_c}(\mu)\cos(\mu\theta) - 2\sum_{\mu=1}^{J} \widehat{f_c}(\mu)\cos(\mu\theta) = 2\sum_{\mu=J+1}^{\infty} \widehat{f_c}(\mu)\cos(\mu\theta) \tag{3.49}$$

so that

$$\|(I - \mathcal{S}_J)\, f_c\|_{H^m}^2 = 2\sum_{\mu \geq J+1} \mu^{2m}\left|\widehat{f_c}(\mu)\right|^2. \tag{3.50}$$

Using Lemma 3.4, this entails

$$
\begin{aligned}
\left\|(I-\mathcal{S}_J)\,f_c\right\|_{H^m}^2 &\leq \frac{1}{2}\sum_{\mu\geq J+1}\mu^{2m-2}\left|\widehat{(f')}_c(\mu-1)-\widehat{(f')}_c(\mu+1)\right|^2 \\
&\leq \sum_{\mu\geq J+1}\mu^{2m-2}\left|\widehat{(f')}_c(\mu-1)\right|^2+\sum_{\mu\geq J+1}\mu^{2m-2}\left|\widehat{(f')}_c(\mu+1)\right|^2 \\
&= \sum_{\mu\geq J}(\mu+1)^{2m-2}\left|\widehat{(f')}_c(\mu)\right|^2+\sum_{\mu\geq J+2}(\mu-1)^{2m-2}\left|\widehat{(f')}_c(\mu)\right|^2 \\
&\leq 2\sum_{\mu\geq J}(\mu+1)^{2m-2}\left|\widehat{(f')}_c(\mu)\right|^2. \qquad (3.51)
\end{aligned}
$$

For $\mu\geq J$ we have $\dfrac{J+1}{J}\mu\geq\mu+1$ so that

$$
\begin{aligned}
\left\|(I-\mathcal{S}_J)\,f_c\right\|_{H^m}^2 &\leq 2\left(\frac{J+1}{J}\right)^{2m-2}\sum_{\mu\geq J}\mu^{2(m-1)}\left|\widehat{(f')}_c(\mu)\right|^2 \\
&\leq 2\left(\frac{J+1}{J}\right)^{2m-2}\sum_{\mu\geq 1}\mu^{2(m-1)}\left|\widehat{(f')}_c(\mu)\right|^2 \\
&= \left(\frac{J+1}{J}\right)^{2m-2}\left\|(I-\mathcal{S}_{J-1})\,(f')_c\right\|_{H^{m-1}}^2. \qquad (3.52)
\end{aligned}
$$

Since $m\leq N+1$, we can iteratively use this inequality $m-1$ times to have

$$
\begin{aligned}
&\left\|(I-\mathcal{S}_N)\,f_c\right\|_{H^m} \\
&\leq \left(\frac{N+1}{N}\right)^{m-1}\left(\frac{N}{N-1}\right)^{m-2}\cdots\left(\frac{N-m+3}{N-m+2}\right)\left\|(I-\mathcal{S}_{N-m+1})\left(f^{(m-1)}\right)_c\right\|_{H^1} \qquad (3.53)
\end{aligned}
$$

which we write as

$$
\left\|(I-\mathcal{S}_N)\,f_c\right\|_{H^m}\leq\sigma_{m,N}\left\|(I-\mathcal{S}_{N-m+1})\left(f^{(m-1)}\right)_c\right\|_{H^1} \qquad (3.54)
$$

where

$$
\sigma_{m,N}=\frac{(N+1)^{m-1}}{N(N-1)\ldots(N-m+2)}=\prod_{j=1}^{m-1}\left(\frac{N+1}{N+1-j}\right). \qquad (3.55)
$$

For $m < N+1$, we apply (3.52) to the right-hand side of (3.54) to get

$$\left\|(I-\mathcal{S}_N)\, f_c\right\|_{H^m} \leq \sigma_{m,N} \left\|\left(f^{(m)}\right)_c\right\|_{H^0}. \tag{3.56}$$

For the case $m = N+1$,

$$\begin{aligned}
&\left\|(I-\mathcal{S}_0)\left(f^{(m-1)}\right)_c\right\|_{H^1}^2 \\
&\leq \sum_{\mu\geq 0}\left|\widehat{(f^{(m)})}_c(\mu)\right|^2 + \sum_{\mu\geq 2}\left|\widehat{(f^{(m)})}_c(\mu)\right|^2 \\
&\leq |\widehat{(f^{(m)})}_c(0)|^2 + 2\sum_{\mu\geq 1}\left|\widehat{(f^{(m)})}_c(\mu)\right|^2 = \left\|\left(f^{(m)}\right)_c\right\|_{H^0}^2,
\end{aligned} \tag{3.57}$$

where the last equality comes from (B.2). Hence

$$\left\|(I-\mathcal{S}_N)\, f_c\right\|_{H^m} \leq \sigma_{m,N}\left\|(I-\mathcal{S}_0)\left(f^{(m-1)}\right)_c\right\|_{H^1} \leq \sigma_{m,N}\left\|\left(f^{(m)}\right)_c\right\|_{H^0} \tag{3.58}$$

and this completes the proof. □

Remark 3.6. *For fixed $m \geq 1$ $\sigma_{m,N}$ bounded and, $\sigma_{m,N}$ bounded and $\sigma_{m,N} \to 1$ as $N \to \infty$. Furthermore, if m grows with N (for example, set $m = N+1$)*

$$\sigma_{N+1,N} = \frac{(N+1)^N}{N!} = \frac{N^N}{N!}\left(1+\frac{1}{N}\right)^N \sim \frac{e^{N+1}}{\sqrt{2\pi N}}. \tag{3.59}$$

The previous lemmas and theorems were constructed for improving the bound on $|I_k(f) - I_{k,N}(f)|$ we obtained Theorem 3.1 and Theorem 3.3. To state the next theorem, we define, for integer $m \geq 1$ the weighted seminorm on the interval $[a,b]$ as

$$|f|_{H_w^m[a,b]} = \left\{\int_a^b \frac{\left|f^{(m)}(x)\right|^2}{\sqrt{(b-x)(x-a)}}dx\right\}^{1/2}. \tag{3.60}$$

When $[a,b]=[-1,1]$ we just write $|\cdot|_{H_w^m}$ and we note that

$$|f|_{H_w^m} = \left\{ \int_0^\pi \left| \left(f^{(m)}\right)_c(\theta) \right|^2 d\theta \right\}^{1/2} = \sqrt{\pi} \left\| \left(f^{(m)}\right)_c \right\|_{H^0}. \tag{3.61}$$

Theorem 3.7. *Let $r \in [0,2]$ and $0 \le m \le N+1$. Then there exist constants $\sigma'_{m,N}$ such that*

$$|I_k(f) - I_{k,N}(f)| \le \sigma'_{m,N} \left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} |f|_{H_w^m} \tag{3.62}$$

when $|f|_{H_w^m} < \infty$. Moreover $\sigma'_{m,N} = C\sigma_{m,N}$ with C independent of m, N.

Proof. Assuming that $|f|_{H_w^m} < \infty$, we have $f \in L^2[-1,1]$, and therefore we can define an algebraic polynomial p of degree N as

$$p(x) = \widehat{f_c}(0) + 2\sum_{n=1}^{N} \widehat{f_c}(n) T_n(x). \tag{3.63}$$

Using definition (B.6) we obtain

$$p_c(\theta) = \widehat{f_c}(0) + 2\sum_{n=1}^{N} \widehat{f_c}(n)\cos(n\theta) = (S_N f_c)(\theta) \tag{3.64}$$

for all $\theta \in [-\pi,\pi]$. Since $I_{k,N}$ is exact for polynomials of degree up to N, Using Theorem 3.3, we get

$$\begin{aligned} |I_k(f) - I_{k,N}(f)| &= |I_k(f-p) - I_{k,N}(f-p)| \\ &\le C \left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} \|(f-p)_c\|_{H^m} \\ &= C \left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} \|(I-\mathcal{S}_N) f_c\|_{H^m}. \end{aligned} \tag{3.65}$$

By Lemma 3.5 we have

$$|I_k(f) - I_{k,N}(f)| \leq C\sigma_{m,N} \left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} \left\| \left(f^{(m)}\right)_c \right\|_{H^0}$$
$$= C\sigma_{m,N} \left(\frac{1}{k}\right)^r \left(\frac{1}{N}\right)^{m-\rho(r)} \frac{1}{\sqrt{\pi}} |f|_{H_w^m} \tag{3.66}$$

and the proof is complete. □

3.3. Error Estimation of Integrals Over [a,b]

In previous sections we worked on the interval $[-1, 1]$. Now we investigate arbitrary intervals $[a, b]$. To this end, we utilize the linear change of variables

$$x = \alpha + \delta t, \quad t \in [-1, 1] \tag{3.67}$$

where $\alpha = \frac{b+a}{2}$ and $\delta = \frac{b-a}{2}$, to deduce

$$I_k^{[a,b]}(f) = \delta \exp(ik\alpha) I_{\delta k}(\tilde{f}) \tag{3.68}$$

where $\tilde{f}(t) = f(\alpha + \delta t), t \in [-1, 1]$. So, we can apply the FCC quadrature to the right-hand side to approximate $I_k^{[a,b]}(f)$ as

$$I_{k,N}^{[a,b]}(f) = \delta \exp(ik\alpha) I_{\delta k,N}(\tilde{f}) \approx I_k^{[a,b]}(f). \tag{3.69}$$

We now consider an extension of Theorem 3.7.

Theorem 3.8. *Let $r \in [0, 2]$ and $0 \leq m \leq N + 1$ and suppose that $|f|_{H_w^m[a,b]} < \infty$. Then*

$$\left| I_k^{[a,b]}(f) - I_{k,N}^{[a,b]}(f) \right| \leq \sigma'_{m,N} \left(\frac{1}{k}\right)^r \left(\frac{b-a}{2}\right)^{m+1-r} \left(\frac{1}{N}\right)^{m-\rho(r)} |f|_{H_w^m[a,b]}. \tag{3.70}$$

Proof. Upon noting $\delta = \frac{b-a}{2}$, we get

$$\left| I_k^{[a,b]}(f) - I_{k,N}^{[a,b]}(f) \right| = \delta \left| I_{\delta k}(\tilde{f}) - I_{\delta k,N}(\tilde{f}) \right|. \tag{3.71}$$

Therefore Theorem 3.7 implies

$$\begin{aligned} \delta \left| I_{\delta k}(\tilde{f}) - I_{\delta k,N}(\tilde{f}) \right| &\leq \sigma'_{m,N} \delta \left(\frac{1}{\delta k} \right)^r \left(\frac{1}{N} \right)^{m-\rho(r)} |\tilde{f}|_{H_w^m} \\ &= \sigma'_{m,N} \left(\frac{1}{k} \right)^r \delta^{1-r} \left(\frac{1}{N} \right)^{m-\rho(r)} |\tilde{f}|_{H_w^m}. \end{aligned} \tag{3.72}$$

Since $\tilde{f}^{(m)}(t) = \delta^m f^{(m)}(\alpha + \delta t)$, we observe that

$$|\tilde{f}|_{H_w^m}^2 = \delta^{2m} \int_{-1}^{1} \frac{\left| f^{(m)}(\alpha + \delta t) \right|^2}{\sqrt{1-t^2}} dt = \delta^{2m} |f|_{H_w^m[a,b]}^2. \tag{3.73}$$

Thus

$$\left| I_k^{[a,b]}(f) - I_{k,N}^{[a,b]}(f) \right| \leq \sigma'_{m,N} \left(\frac{1}{k} \right)^r \left(\frac{b-a}{2} \right)^{m+1-r} \left(\frac{1}{N} \right)^{m-\rho(r)} |f|_{H_w^m[a,b]}. \tag{3.74}$$

□

This theorem is the most useful when we assume N is fixed. Observe that convergence is satisfied by letting $b - a \to 0$, hence $\delta \to 0$.

Corollary 3.9. *Let $r \in [0, 2]$. For each $N \geq 1$, there exists a constant $c_N = C\sigma'_{N+1,N}(1/N)^{N+1-\rho(r)}$ such that*

$$\left| I_k^{[a,b]}(f) - I_{k,N}^{[a,b]}(f) \right| \leq c_N \left(\frac{1}{k} \right)^r \delta^{N+2-r} \max_{x \in [a,b]} \left| f^{(N+1)}(x) \right| \tag{3.75}$$

when $f \in C^{N+1}[a, b]$.

Proof. Theorem 3.8 entails for $m = N+1$

$$\left|I_k^{[a,b]}(f) - I_{k,N}^{[a,b]}(f)\right| \le \sigma'_{N+1,N}\left(\frac{1}{k}\right)^r \left(\frac{b-a}{2}\right)^{N+2-r} \left(\frac{1}{N}\right)^{N+1-\rho(r)} |f|_{H_w^{N+1}[a,b]}. \quad (3.76)$$

By (3.73), we also have,

$$\begin{aligned} |f|_{H_w^{N+1}[a,b]} &= \left(\int_{-1}^{1} \frac{\left|f^{(N+1)}(\alpha+\delta t)\right|^2}{\sqrt{1-t^2}} dt\right)^{1/2} \\ &\le \max_{x\in[a,b]} \left|f^{(N+1)}(x)\right| \left(\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} dt\right)^{1/2} \\ &= \sqrt{\pi} \max_{x\in[a,b]} \left|f^{(N+1)}(x)\right|. \end{aligned} \quad (3.77)$$

Therefore the choice $c_N = \left(\frac{1}{k}\right)^r \delta^{N+2-r}$ yields

$$\left|I_k^{[a,b]}(f) - I_{k,N}^{[a,b]}(f)\right| \le c_N \left(\frac{1}{k}\right)^r \delta^{N+2-r} \max_{x\in[a,b]} \left|f^{(N+1)}(x)\right| \quad (3.78)$$

which is the desired result. □

3.4. Accurate Computation of the Weights

The FCC quadrature rule highly depends on the accurate computation of the weights. In this section, we introduce a two-phase algorithm.

In this connection, we first note that Eq. (A.2) implies

$$2\omega_n(k) = \rho_{n+1}(k) - \rho_{n-1}(k), \quad n \ge 2 \quad (3.79)$$

where, for $n \ge 1$,

$$\rho_n(k) = \int_{-1}^{1} U_{n-1}(s)\exp(iks)ds \quad (3.80)$$

$$= \frac{1}{n}\int_{-1}^{1} T_n'(s)\exp(iks)ds. \tag{3.81}$$

Thus by integration by parts

$$\begin{aligned}\rho_n(k) &= \frac{1}{n}\int_{-1}^{1} T_n'(s)\exp(iks)ds \\ &= T_n(s)\exp(iks)\Big|_{s=-1}^{s=1} - \frac{ik}{n}\int_{-1}^{1} T_n(s)\end{aligned} \tag{3.82}$$

where $n \geq 1$. Setting

$$\gamma_n(k) = \frac{1}{ik}T_n(s)\exp(iks)\Bigg|_{s=-1}^{s=1}, \tag{3.83}$$

we obtain the relation

$$\omega_0(k) = \gamma_0(k), \quad \omega_n(k) = \gamma_n(k) - \frac{n}{ik}\rho_n(k), \tag{3.84}$$

where $n \geq 1$. Combining Eq. (3.79) and (3.80) we obtain the recurrence relation

$$2\gamma_n(k) - \frac{2n}{ik}\rho_n(k) = \rho_{n+1}(k) - \rho_{n-1}(k), \quad n \geq 2. \tag{3.85}$$

Moreover, since $U_0(s) = 1$ and $U_1(s) = 2s$, we have

$$\rho_1(k) = \gamma_0(k) \tag{3.86}$$

$$\rho_2(k) = 2\gamma_1(k) - \frac{2}{ik}\gamma_0(k). \tag{3.87}$$

For computational purposes, it is important to observe that

$$\gamma_n(k) = \frac{1}{ik}\left[\exp(ik) - (-1)^n \exp(-ik)\right]$$
$$= \begin{cases} \dfrac{2\sin k}{k} & \text{for even } \; n, \\ \dfrac{2\cos k}{ik} & \text{for odd } \; n \end{cases} \tag{3.88}$$

which follows from Eq. 22.4.4 in [21].

3.4.1. Algorithm I: For $n \leq \min\{N, k\}$

The first phase of the algorithm is constructed as follows.

Algorithm I: for $n \leq \min\{N, k\}$

Result: Weights ω_n defined as Eq. (3.7)

Initial: Compute ρ_1 and ρ_2 defined in Eq.(3.80) using γ_n as in Eq. (3.88)

$\rho_1(k) = \gamma_0(k),$

$\rho_2(k) = 2\gamma_1(k) - \frac{2}{ik}\gamma_0(k)$

for $n = 2$ **to** $\min\{N, k\} - 1$ **do**

$$\rho_{n+1}(k) = 2\gamma_n(k) - \frac{2n}{ik}\rho_n(k) + \rho_{n-1}(k)$$

end

Set: $\omega_0(k) = \rho_1(k)$

for $n = 1$ **to** $\min\{N, k\}$ **do**

$\omega_n(k) = \gamma_n(k) - \frac{n}{ik}\rho_n(k)$

end

Algorithm 1: First phase of the algorithm

For the sake of stability, the restriction $n \leq k$ must be added to the first phase. If $N > k$, then the second phase must be implemented. To this end, we set and define

$n_0 = \lceil k \rceil$, choose $M \geq n_0$, tridiagonal matrix

$$A_M(k) = \begin{bmatrix} \frac{2n_0}{ik} & 1 & & & \\ -1 & \frac{2(n_0+1)}{ik} & 1 & & \\ & -1 & \frac{2(n_0+2)}{ik} & 1 & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \frac{2(2M-1)}{ik} & \end{bmatrix} \tag{3.89}$$

and the right-hand side vector

$$\mathbf{b}_M(k) = \begin{bmatrix} 2\gamma_{n_0}(k) + \rho_{n_0-1}(k) \\ 2\gamma_{n_0+1}(k) \\ 2\gamma_{n_0+2}(k) \\ \vdots \\ 2\gamma_{2M-1}(k) - \rho_{2M}(k) \end{bmatrix}. \tag{3.90}$$

Observe that

$$\boldsymbol{\rho_M}(k) = [\rho_{n_0}(k)\rho_{n_0+1}(k)\rho_{n_0+2}(k), \ldots, \rho_{2M-1}(k)]^{\mathrm{T}} \tag{3.91}$$

is a solution of the equation

$$A_M(k)\mathbf{x} = \mathbf{b}_M(k). \tag{3.92}$$

With that linear system, the variables ρ appearing in the equation (3.92) computation of the weights can then be found by solving the tridiagonal linear system. Note in this connection that the entries of $\mathbf{b}_M(k)$ are given by (3.88) except for ρ_{n_0-1} and ρ_{2M}. The former is computed in the first phase of the algorithm, and for the latter we use approximation with sufficiently large M. This approximation is given at the of Section 3.5.

3.4.2. Algorithm II: For $k < n \leq N$

The first phase of the algorithm is constructed as follows.

Algorithm II: for $k < n \leq N$

Set: $n_0 = \lceil k \rceil$ and take $M \geq \max\{n_0/2, N/2\}$

Result: Weights ω_n defined as Eq. (3.7)

Initial: Compute matrix $A_M(k)$ defined in (3.89) and column vector $\mathbf{b}_M(k)$ defined in (3.90)

Solve: Obtain $\boldsymbol{\rho}_M(k)$ by solving $A_M(k)\boldsymbol{\rho}_M(k) = \mathbf{b}_M(k)$ where

$$\boldsymbol{\rho}_M(k) = [\rho_{n_0}(k)\rho_{n_0+1}(k)\rho_{n_0+2}(k), \ldots, \rho_{2M-1}(k)]^{\mathrm{T}}$$

for $n = n_0$ **to** N **do**

  $\omega_n(k) = \gamma_n(k) - \frac{n}{ik}\rho_n(k)$ `/*` γ_n `defined as Eq.` (3.88) `*/`

end

Algorithm 2: Second phase of the algorithm

After establishing our two-phase algorithm, a natural question is why the second phase of the algorithm is not used for computing all the weights $\rho_n(k)$. The reason is that the first phase of the algorithm is faster since we only use a forward recurrence relation, whereas the second phase involves the solution of a tridiagonal system. Additionally, the proof of the stability of the second phase of the algorithm, depends on the diagonal dominance of the matrix. This property holds only if the second phase of the algorithm is restricted to the computation of the $A_m(k)$ the coefficients $\rho_n(k)$ for $n \geq k$.

Remark 3.10. *The first phase of the algorithm deals with a forward recurrence relation, hence has complexity $\mathcal{O}(N)$. The second phase solves a tridiagonal system, which is solved by Thomas' algorithm which has complexity $\mathcal{O}(N)$ when M is proportional to N. Also, we obtained $I_{k,N}(f)$ by applying the discrete cosine transform which has computational complexity $\mathcal{O}(N \log N)$. Hence our quadrature rule has complexity $\mathcal{O}(N \log N)$.*

3.5. Asymptotic Expansion of ρ_{2M}

In this section, we study the asymptotic expansion of ρ_{2M} which is essential for the second phase of the algorithm. Let M be given, and introduce for $M \geq k$

$$\rho_{2M}(k) = \int_{-1}^{1} U_{2M-1}(s)\exp(iks)ds = \int_{0}^{\pi} \exp(ik\cos\theta)\sin(2M\theta)d\theta \tag{3.93}$$

where the identity follows from (A.2) and (A.3). Therefore

$$\begin{aligned}\rho_{2M}(k) &= -\frac{i}{2}\left[\int_{0}^{\pi} \exp(ik\cos\theta)(\exp(2iM\theta) - \exp(-2iM\theta))d\theta\right] \\ &= -\frac{i}{2}\left[I_k^{+}(M) - I_k^{-}(M)\right]\end{aligned} \tag{3.94}$$

where we have set

$$I_k^{\pm}(M) := \int_{0}^{\pi} \exp\left(iS_{\pm}(\theta)\right)d\theta, \quad S_{\pm}(\theta) := (\pm 2M\theta + k\cos\theta). \tag{3.95}$$

The families of smooth functions

$$p_0^{\pm}(\theta) = \frac{1}{iS'_{\pm}(\theta)} = \frac{1}{i(\pm 2M - k\sin\theta)} \tag{3.96}$$

$$p_r^{\pm}(\theta) = \frac{1}{iS'_{\pm}(\theta)}\frac{dp_{r-1}^{\pm}(\theta)}{d\theta}. \quad (r \in \mathbb{N}) \tag{3.97}$$

will be used in the next Lemmas and Theorems.

Lemma 3.11. *For all $r \geq 0$,*

$$p_r^{\pm}(\theta) = \frac{q_r^{\pm}(\theta)}{i^{r+1}\left(S'_{\pm}(\theta)\right)^{2r+1}} \tag{3.98}$$

where $q_r^{\pm}(\theta)$ is a trigonometric polynomial in θ defined recursively as

$$q_0^{\pm} \equiv 1, \quad q_{r+1}^{\pm} = \left(q_r^{\pm}\right)' S_{\pm}' - (2r+1) q_r^{\pm} S_{\pm}'', \quad r = 0, 1, 2, \ldots \tag{3.99}$$

In addition

$$p_r^{\pm}(0) = (-1)^r p_r^{\pm}(\pi) \tag{3.100}$$

$$p_r^{+}(0) = -p_r^{-}(0), \quad p_r^{+}(\pi) = -p_r^{-}(\pi). \tag{3.101}$$

Proof. For $r = 0$,

$$p_0^{\pm}(0) = \frac{1}{iS_{\pm}'(\theta)} = \frac{1}{i(\pm 2M - k\sin(0))} = \frac{1}{i(\pm 2M - k\sin(\pi))} = p_0^{\pm}(\pi) \tag{3.102}$$

and also

$$p_0^{\pm}(0) = \frac{1}{iS_{\pm}'(\theta)} = p_0^{\pm}(0) = \frac{q_0^{\pm}}{iS_{\pm}'(\theta)}. \tag{3.103}$$

Assume by induction for some r, Eq. (3.98) holds. Then

$$\begin{aligned} p_{r+1}^{\pm}(\theta) &= \frac{1}{iS_{\pm}'(\theta)} \frac{d}{d\theta} \left[\frac{q_r^{\pm}(\theta)}{i^{r+1} \left(S_{\pm}'(\theta)\right)^{2r+1}} \right] \\ &= \frac{1}{i^{r+2} S_{\pm}'(\theta)} \left[\frac{\left(q_r^{\pm}\right)'(\theta)}{\left(S_{\pm}'(\theta)\right)^{2r+1}} - \frac{(2r+1) q_r^{\pm}(\theta) S_{\pm}''(\theta)}{\left(S_{\pm}'(\theta)\right)^{2r+2}} \right] \\ &= \frac{1}{i^{r+2} \left(S_{\pm}'(\theta)\right)^{2r+3}} \left[\left(q_r^{\pm}\right)'(\theta) S_{\pm}'(\theta) - (2r+1) q_r^{\pm}(\theta) S_{\pm}''(\theta) \right]. \end{aligned} \tag{3.104}$$

So, Eq. (3.98) is proven. Considering equations (3.100) and (3.101), we observe that

$$S_{\pm}^{(r)}(\theta) = (-1)^{r+1} S_{\pm}^{(r)}(\pi - \theta) = (-1)^r S_{\mp}^{(r)}(-\theta), \quad r = 1, 2, \ldots \tag{3.105}$$

We also need the identities

$$q_r^{\pm}(\theta) = (-1)^r q_r^{\pm}(\pi - \theta), \tag{3.106}$$

$$q_r^{+}(\theta) = q_r^{-}(-\theta), \tag{3.107}$$

which clearly holds for $r = 0$. To see these we argue as follows. By induction we assume, (3.106) and (3.107) hold for some r, and compute

$$\begin{aligned} q_{r+1}^{\pm}(\theta) &= \left(q_r^{\pm}\right)'(\theta) S_{\pm}'(\theta) - (2r+1) q_r^{\pm}(\theta) S_{\pm}''(\theta) \\ &= (-1)^{r+1} \left[\left(q_r^{\pm}\right)'(\pi - \theta)\right] (-1)^2 S_{\pm}'(\pi - \theta) \\ &- (2r+1)(-1)^r \left[q_r^{\pm}(\pi - \theta)\right] (-1)^3 S_{\pm}''(\pi - \theta) \\ &= (-1)^{r+3} \left(\left(q_r^{\pm}\right)'(\pi - \theta) S_{\pm}'(\pi - \theta) - (2r+1) q_r^{\pm}(\pi - \theta) S_{\pm}''(\pi - \theta)\right) \\ &= (-1)^{r+1} q_{r+1}^{\pm}(\pi - \theta). \end{aligned} \tag{3.108}$$

and

$$\begin{aligned} q_{r+1}^{+}(\theta) &= \left(q_r^{+}\right)'(\theta) S_{+}'(\theta) - (2r+1) q_r^{+}(\theta) S_{+}''(\theta) \\ &= \left[-\left(q_r^{-}\right)'(-\theta)\right] \left(-S_{-}'(-\theta)\right) - (2r+1) \left[q_r^{-}(-\theta)\right] (-1)^2 S_{-}''(-\theta) \\ &= \left(q_r^{-}\right)'(-\theta) S_{-}'(-\theta) - (2r+1) q_r^{-}(-\theta) S_{-}''(-\theta) \\ &= q_{r+1}^{-}(-\theta). \end{aligned} \tag{3.109}$$

$\square$

Corollary 3.12. *For all $M \geq k$ and $r \geq 1$ there exists $C_r > 0$ independent of M and k such that*

$$\left|p_r^{\pm}(\theta)\right| + \left|\left(p_r^{\pm}\right)'(\theta)\right| \leq C_r k M^{-r-2} \text{ for all } \theta \in [0, \pi]. \tag{3.110}$$

Proof. First we note

$$q_1^{\pm}(\theta) = -1q_0^{\pm}(\theta)S''_{\pm} = k\cos\theta. \tag{3.111}$$

So, by recurrence relation in (3.99), we can see that every $q_r^{\pm}(\theta)$ has k as a common factor. Also, observe that

$$q_{r+1}^{\pm} = \left(q_r^{\pm}\right)'(\pm M - k\sin\theta) - (2r+1)q_r^{\pm}(-k\cos\theta). \tag{3.112}$$

Now, fix θ, then we have $q_r^{\pm}$ as polynomial in M and k which have total degree of r. So we can deduce for $r \geq 1$

$$\left|q_r^{\pm}(\theta)\right| \leq C'_r k \sum_{p\geq 1, q\geq 0, p+q\leq r} k^{p-1}M^q = C'_r kM^{r-1} \sum_{p\geq 1, q\geq 0, p+q\leq r} \left(\frac{k}{M}\right)^{p-1} M^{q+p-r}. \tag{3.113}$$

Now, since $M > k$ we have, for all $\theta \in [0, \pi]$,

$$\left|q_r^{\pm}(\theta)\right| \leq C'_r kM^{r-1}. \tag{3.114}$$

Also

$$\left|S'_{\pm}(\theta)\right| = |\pm 2M - k\sin\theta| \geq 2M - k \geq M. \tag{3.115}$$

Combine these, we obtain

$$\left|p_r^{\pm}(\theta)\right| = \left|\frac{q_r^{\pm}(\theta)}{\left(S'_{\pm}(\theta)\right)^{2r+1}}\right| \leq \frac{C'_r kM^{r-1}}{M^{2r+1}} = C'_r kM^{-r-2} \tag{3.116}$$

where $\theta \in [0, \pi]$. For estimation of $\left|\left(p_r^{\pm}\right)'(\theta)\right|$, we use the definition of $p_r^{\pm}$ in equations (3.96) and (3.97) to have

$$\left|\left(p_r^{\pm}\right)'(\theta)\right| = \left|iS'_{\pm}(\theta)p_{r+1}^{\pm}(\theta)\right| \leq C'_{r+1}(2M+k)kM^{-r-3} \leq 3C'_{r+1}kM^{-r-2} \tag{3.117}$$

for $\theta \in [0, \pi]$. Combining these estimates, we conclude

$$\left|p_r^{\pm}(\theta)\right| + \left|\left(p_r^{\pm}\right)'(\theta)\right| \leq C_r k M^{-r-2}, \quad \text{for all } \theta \in [0, \pi]. \tag{3.118}$$

□

Theorem 3.13. *For all $M \geq k$ we have*

$$\rho_{2M}(k) = -2\left[\sum_{r=0}^{J} p_{2r}^{+}(0)\sin k + i\sum_{r=0}^{J} p_{2r+1}^{+}(0)\cos k\right] + R_J(M, k) \tag{3.119}$$

with

$$|R_J(M, k)| \leq C_J k M^{-2J-4} \tag{3.120}$$

and C_J is independent of M and k.

Proof. Remark that

$$I_k^{\pm}(M) = \int_0^{\pi} \exp\left(iS_{\pm}(\theta)\right) d\theta. \tag{3.121}$$

Now, by integration by parts twice,

$$\begin{aligned}
I_k^{\pm}(M) &= \int_0^{\pi} \frac{1}{iS'_{\pm}(\theta)}\left[iS'_{\pm}(\theta)\exp\left(iS_{\pm}(\theta)\right)\right] d\theta = \int_0^{\pi} p_0^{\pm}(\theta)\left[iS'_{\pm}(\theta)\exp\left(iS_{\pm}(\theta)\right)\right] d\theta \\
&= p_0^{\pm}(\theta)\exp\left(iS_{\pm}(\theta)\right)\Big|_0^{\pi} - \int_0^{\pi} \frac{dp_0^{\pm}(\theta)}{d\theta}\exp\left(iS_{\pm}(\theta)\right) d\theta \\
&= p_0^{\pm}(\theta)\exp\left(iS_{\pm}(\theta)\right)\Big|_0^{\pi} - \int_0^{\pi} p_1^{\pm}(\theta)\left[iS'_{\pm}(\theta)\exp\left(iS_{\pm}(\theta)\right)\right] d\theta \\
&= \sum_{r=0}^{1}(-1)^r p_r^{\pm}(\theta)\exp\left(iS_{\pm}(\theta)\right)\Bigg|_0^{\pi} + \int_0^{\pi} \frac{dp_1^{\pm}(\theta)}{d\theta}\exp\left(iS_{\pm}(\theta)\right) d\theta.
\end{aligned} \tag{3.122}$$

We can repeat the integration by parts argument and with $S_{\pm}(0) = k, S_{\pm}(\pi) = \pm 2M\pi - k$, we have

$$\begin{aligned}
I_k^{\pm}(M) =& \sum_{r=0}^{2J+2} (-1)^r p_r^{\pm}(\theta) \exp\left(iS_{\pm}(\theta)\right)\Bigg|_0^{\pi} - \int_0^{\pi} \frac{dp_{2J+2}^{\pm}(\theta)}{d\theta} \exp\left(iS_{\pm}(\theta)\right) d\theta \\
=& - \sum_{r=0}^{2J+2} (-1)^r p_r^{\pm}(0) \left[\exp(ik) + (-1)^{r+1} \exp(-ik)\right] \\
& - \int_0^{\pi} \frac{dp_{2J+2}^{\pm}(\theta)}{d\theta} \exp\left(iS_{\pm}(\theta)\right) d\theta
\end{aligned} \tag{3.123}$$

where we have now used equations (3.98) and (3.100). So by equations (3.94), (3.98) and (3.100)

$$\begin{aligned}
\rho_{2M}(k) =& -\frac{i}{2}\left[I_k^{+}(M) - I_k^{-}(M)\right] \\
=& \frac{i}{2}\sum_{r=0}^{2J+2} (-1)^r \left(p_r^{+}(0) - p_r^{-}(0)\right) \left[\exp(ik) + (-1)^{r+1} \exp(-ik)\right] \\
& + \frac{i}{2}\int_0^{\pi} \left[\frac{dp_{2J+2}^{+}(\theta)}{d\theta} \exp\left(iS_{+}(\theta)\right) - \frac{dp_{2J+2}^{-}(\theta)}{d\theta} \exp\left(iS_{-}(\theta)\right)\right] d\theta \\
=& -2\left[\sum_{r=0}^{J} p_{2r}^{+}(0) \sin k + i\sum_{r=0}^{J} p_{2r+1}^{+}(0) \cos k\right] + R_J(M,k),
\end{aligned} \tag{3.124}$$

where we have used equation (3.100) in the last step. If we now define

$$\begin{aligned}
R_J(M,k) =& -2p_{2J+2}^{+}(0) \sin k \\
& + \frac{i}{2}\int_0^{\pi} \left[\frac{dp_{2J+2}^{+}(\theta)}{d\theta} \exp\left(iS_{+}(\theta)\right) - \frac{dp_{2J+2}^{-}(\theta)}{d\theta} \exp\left(iS_{-}(\theta)\right)\right] d\theta
\end{aligned} \tag{3.125}$$

then we have

$$|R_J(M,k)| \leq C_1 \left|p_{2J+2}^{+}(0)\right| + C_2 max_{\theta\in[0,\pi]} \left|(p_{2J+2}^{+})'(\theta)\right|. \tag{3.126}$$

Therefore Corollary 3.12 entails

$$|R_J(M,k)| \leq C_J k M^{-2J-4} \tag{3.127}$$

where constant C_J is independent of M and k. □

Now we have the final theorem of the section.

Theorem 3.14. *Let M be an integer with $M \geq k$ and define*

$$p_0(\theta) = \frac{1}{(2M - k\sin\theta)}, \quad p_r(\theta) = p_0(\theta)\frac{d}{d\theta}p_{r-1}(\theta), \quad r = 1, 2, \ldots \tag{3.128}$$

Then

$$\rho_{2M}(k) = 2i\left[\sum_{r=0}^{J}(-1)^r p_{2r}(0)\sin k + \sum_{r=0}^{J}(-1)^r p_{2r+1}(0)\cos k\right] + R_J(M,k) \tag{3.129}$$

where

$$|R_J(M,k)| \leq C_J k M^{-2J-4} \tag{3.130}$$

and C_J is independent of M and k.

Proof. Observe that

$$p_0^+(\theta) = \frac{1}{S'_+(\theta)} = \frac{p_0}{i} \tag{3.131}$$

and

$$\begin{aligned} p_r^+(\theta) &= \frac{1}{iS'_+(\theta)}\frac{dp_{r-1}^+(\theta)}{d\theta} \\ &= p_0^+(\theta)\frac{dp_{r-1}^+(\theta)}{d\theta} = \frac{1}{i^{r+1}}p_0(\theta)\frac{dp_{r-1}(\theta)}{d\theta} \\ &= \frac{p_r(\theta)}{i^{r+1}}. \end{aligned} \tag{3.132}$$

So we have establish

$$p_r = i^{r+1} p_r^+ \quad \text{ and } p_{2r} = i(-1)^r p_{2r}^+. \tag{3.133}$$

So the result follows from the last theorem □

For computational purposes, we give the first seven asymptotics

$$\begin{aligned}
p_0(0) &= \frac{1}{2M}, \\
p_1(0) &= \frac{k}{(2M)^3}, \\
p_2(0) &= \frac{3k^2}{(2M)^5}, \\
p_3(0) &= \frac{(15k^2 - 4M^2)\, k}{(2M)^7}, \\
p_4(0) &= \frac{(105k^2 - 60M^2)\, k^2}{(2M)^9}, \\
p_5(0) &= \frac{(945k^4 - 840k^2M^2 + 16M^4)\, k}{(2M)^{11}}, \\
p_6(0) &= \frac{(-12600k^2M^2 + 1008M^4 + 10395k^4)\, k^2}{(2M)^{13}}.
\end{aligned} \tag{3.134}$$

3.6. Stability of the Algorithm

In this section, we study the stability of the algorithm, starting with the first phase.

Theorem 3.15. *Let $(\varepsilon_m)_m \subset \mathbb{C}$ with $|\varepsilon_m| \leq \varepsilon$ and define*

$$\begin{aligned}
\widetilde{\rho}_1(k) &= \rho_1(k) + \varepsilon_1 \\
\widetilde{\rho}_2(k) &= \rho_2(k) + \varepsilon_2 \\
\widetilde{\rho}_{n+1}(k) &= 2\gamma_n(k) - \frac{2n}{ik}\widetilde{\rho}_n(k) + \widetilde{\rho}_{n-1}(k) + \varepsilon_{n+1}, \quad n = 2, 3, \ldots
\end{aligned} \tag{3.135}$$

Then for all $2 < n < k$

$$|\widetilde{\rho}_n(k) - \rho_n(k)| \leq \left[1 + \frac{4}{3}\frac{nk^{1/2}}{\left(k^2 - n^2\right)^{1/4}}\right]\varepsilon. \tag{3.136}$$

Proof. For $\delta_n = \widetilde{\rho}_n(k) - \rho_n(k)$, we have

$$\begin{aligned} \delta_1 &= \varepsilon_1 \\ \delta_2 &= \varepsilon_2 \\ \delta_n &= -\frac{2(n-1)}{ik}\delta_{n-1} + \delta_{n-2} + \varepsilon_n, \quad n = 3, 4, \ldots \end{aligned} \tag{3.137}$$

which can be re-written as

$$\begin{bmatrix} \delta_n \\ \delta_{n-1} \end{bmatrix} = \begin{bmatrix} -\frac{2(n-1)}{ik} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_{n-1} \\ \delta_{n-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_n \\ 0 \end{bmatrix}, \quad n \geq 3 \tag{3.138}$$

Introducing the notation

$$\boldsymbol{\delta}_n = \begin{bmatrix} \delta_n \\ \delta_{n-1} \end{bmatrix}, \quad \boldsymbol{\varepsilon}_2 = \begin{bmatrix} \varepsilon_2 \\ \varepsilon_1 \end{bmatrix}, \quad \boldsymbol{\varepsilon}_n = \begin{bmatrix} \varepsilon_n \\ 0 \end{bmatrix} \quad \text{for } n = 3, 4, \ldots \tag{3.139}$$

and

$$D_n = \begin{bmatrix} -\frac{2(n-1)}{ik} & 1 \\ 1 & 0 \end{bmatrix} \tag{3.140}$$

we see that $\boldsymbol{\delta}_n$ satisfies

$$\boldsymbol{\delta}_n = D_n \boldsymbol{\delta}_{n-1} + \boldsymbol{\varepsilon}_n \quad \text{for } n \geq 3 \tag{3.141}$$

Repeating the argument, we get

$$\boldsymbol{\delta}_n = D_n \left(D_{n-1} \boldsymbol{\delta}_{n-2} + \boldsymbol{\varepsilon}_{n-1} \right) + \boldsymbol{\varepsilon}_n \quad \text{for } n \geq 3 \tag{3.142}$$

Thus, inductively

$$\boldsymbol{\delta}_n = \sum_{j=2}^{n-1} \left[\prod_{i=j}^{n-1} D_{i+1} \right] \varepsilon_j + \varepsilon_n \quad \text{for } n \geq 2 \tag{3.143}$$

where the sum on the right-hand side vanishes for $n = 2$ and

$$\prod_{i=j}^{n-1} D_{i+1} = D_n D_{n-1} \ldots D_j . \tag{3.144}$$

Introducing the sequence $\left\{ \boldsymbol{\delta}_n^{(j)} \right\}_{n=j}^{\infty}$, $j \geq 2$ defined as

$$\boldsymbol{\delta}_n^{(j)} = \left[\prod_{i=j}^{n-1} D_{i+1} \right] \boldsymbol{\varepsilon}_j , \quad n \geq j+1 \tag{3.145}$$

and

$$\boldsymbol{\delta}_n^{(n)} = \boldsymbol{\varepsilon}_n , \tag{3.146}$$

we re-write the relation (3.143) as

$$\boldsymbol{\delta}_n = \sum_{j=2}^{n-1} \boldsymbol{\delta}_n^{(j)} + \boldsymbol{\varepsilon}_n . \tag{3.147}$$

Writing

$$\boldsymbol{\delta}_n^{(j)} = \begin{bmatrix} \sigma_n^{(j)} \\ \sigma_{n-1}^{(j)} \end{bmatrix} , \tag{3.148}$$

the recurrence relation $\boldsymbol{\delta}_n^{(j)} = D_n \boldsymbol{\delta}_{n-1}^{(j)}$ implies for $n \geq j+1$ and $j \geq 2$

$$\sigma_n^{(j)} = -\frac{2(n-1)}{ik}\sigma_{n-1}^{(j)} + \sigma_{n-2}^{(j)} \tag{3.149}$$

Initial conditions are,

$$\sigma_1^{(2)} = \varepsilon_1, \qquad \sigma_2^{(2)} = \varepsilon_2. \tag{3.150}$$

and with the relation,

$$\boldsymbol{\delta}_j^{(j)} = \begin{bmatrix} \varepsilon_j \\ 0 \end{bmatrix} \tag{3.151}$$

thus, starting conditions

$$\sigma_{j-1}^{(j)} = 0, \quad \sigma_j^{(j)} = \varepsilon_j \quad \text{ for all } j \geq 3. \tag{3.152}$$

Equation (3.149) has a form similar to

$$a_n - \frac{2(n-1)}{x}a_{n-1} + a_{n-2} = 0 \tag{3.153}$$

By equations 9.1.27 and 9.1.29 in [21], the sequence of Bessel functions $\{J_n(x), Y_n(x)\}_{n \geq 1}$ is a fundamental system of solutions for (3.153). Therefore

$$\widetilde{J}_n(k) = i^n J_n(k), \quad \widetilde{Y}_n(k) := i^n Y_n(k) \tag{3.154}$$

are independent solutions for Eq. (3.149). Hence, for $j \geq 3$, the solution of Eq. (3.149) may be written as

$$\delta_n^{(j)} = \alpha^{(j)} \widetilde{J}_n(k) + \beta^{(j)} \widetilde{Y}_n(k) \quad \text{ for } n \geq j+1. \tag{3.155}$$

By the initial conditions of $\sigma_n^{(j)}$, we also have

$$\begin{bmatrix} \widetilde{J}_{j-1}(k) & \widetilde{Y}_{j-1}(k) \\ \widetilde{J}_j(k) & \widetilde{Y}_j(k) \end{bmatrix} \begin{bmatrix} \alpha^{(j)} \\ \beta^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon_j \end{bmatrix} \tag{3.156}$$

so that

$$\begin{bmatrix} \alpha^{(j)} \\ \beta^{(j)} \end{bmatrix} = \begin{bmatrix} \widetilde{J}_{j-1}(k) & \widetilde{Y}_{j-1}(k) \\ \widetilde{J}_j(k) & \widetilde{Y}_j(k) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \varepsilon_j \end{bmatrix}. \tag{3.157}$$

By Eq. (9.1.16) in [21], for $j \geq 3$ and $n \geq j+1$

$$\begin{aligned} \sigma_n^{(j)} &= \left[\widetilde{J}_n(k)\widetilde{Y}_n(k)\right] \begin{bmatrix} \widetilde{J}_{j-1}(k) & \widetilde{Y}_{j-1}(k) \\ \widetilde{J}_j(k) & \widetilde{Y}_j(k) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \varepsilon_j \end{bmatrix} \\ &= \frac{(-1)^{j+1}\pi k i}{2} \left[\widetilde{J}_n(k)\widetilde{Y}_n(k)\right] \begin{bmatrix} \widetilde{Y}_j(k) & -\widetilde{Y}_{j-1}(k) \\ -\widetilde{J}_j(k) & \widetilde{J}_{j-1}(k) \end{bmatrix} \begin{bmatrix} 0 \\ \varepsilon_j \end{bmatrix}. \end{aligned} \tag{3.158}$$

Therefore, defining $M_n(k) = \sqrt{J_n(k)^2 + Y_n(k)^2}$, using the Cauchy-Schwarz inequality and $|\varepsilon_j| \leq \varepsilon$, we arrive at

$$\left|\sigma_n^{(j)}\right| = \left| \frac{(-1)^{j+1}\pi k i}{2} \left[\widetilde{J}_n(k)\widetilde{Y}_n(k)\right] \begin{bmatrix} -\widetilde{Y}_{j-1}(k) \\ \widetilde{J}_{j-1}(k) \end{bmatrix} \varepsilon_j \right| \leq \frac{\pi k}{2} M_n(k) M_{j-1}(k) \varepsilon \tag{3.159}$$

for $j \geq 3$. Proceeding similarly, for $j = 2$, we find

$$\left|\sigma_n^{(2)}\right| \leq \frac{\pi k}{2} M_n(k) \left(M_1(k) + M_2(k)\right) \varepsilon \tag{3.160}$$

Since (c.f. section 13.74 in [22])

$$M_n^2(x) \leq \frac{2}{\pi} \frac{1}{\sqrt{x^2 - n^2}} \qquad \text{for } x > n > 1/2, \tag{3.161}$$

and $2 < n < k$, it follows for $3 \leq j \leq n$ that

$$
\begin{aligned}
|\sigma_n^{(j)}| &\leq \frac{\pi k \varepsilon}{2} \left(\frac{\sqrt{2}}{\sqrt{\pi}(k^2 - n^2)^{1/4}} \frac{\sqrt{2}}{\sqrt{\pi}(k^2 - (j-1)^2)^{1/4}} \right) \\
|\sigma_n^{(j)}| &\leq \frac{k\varepsilon}{(k^2 - n^2)^{1/4}} \left[\frac{1}{(k^2 - (j-1)^2)^{1/4}} \right] \\
|\sigma_n^{(2)}| &\leq \frac{k\varepsilon}{(k^2 - n^2)^{1/4}} \left[\frac{1}{(k^2 - 1)^{1/4}} + \frac{1}{(k^2 - 4)^{1/4}} \right].
\end{aligned}
\tag{3.162}
$$

After establishing these estimates, we can use (3.147) to deduce for $n \geq 3$

$$
\begin{aligned}
|\delta_n| &\leq \varepsilon + \frac{k\varepsilon}{(k^2 - n^2)^{1/4}} \left[\frac{1}{(k^2 - 1)^{1/4}} + \frac{1}{(k^2 - 4)^{1/4}} + \sum_{j=3}^{n-2} \frac{1}{(k^2 - j^2)^{1/4}} \right] \\
&\leq \varepsilon + \frac{k\varepsilon}{(k^2 - n^2)^{1/4}} \left[\frac{1}{(k^2 - 1)^{1/4}} + \frac{1}{(k^2 - (n-1)^2)^{1/4}} + \sum_{j=3}^{n-2} \frac{1}{(k^2 - j^2)^{1/4}} \right] \\
&= \varepsilon + \frac{k\varepsilon}{(k^2 - n^2)^{1/4}} \left[\sum_{j=1}^{n-1} \frac{1}{(k^2 - j^2)^{1/4}} \right].
\end{aligned}
\tag{3.163}
$$

Since $(k^2 - x^2)^{-1/4}$ is an increasing function for $x \in [0, k)$, we have

$$
\begin{aligned}
\sum_{j=1}^{n-1} \frac{1}{(k^2 - j^2)^{1/4}} &\leq \int_0^n \frac{dx}{(k^2 - x^2)^{1/4}} < \frac{1}{k^{1/4}} \int_0^n \frac{dx}{(k - x)^{1/4}} \\
&= \frac{1}{k^{1/4}} \left[-\frac{4}{3}(k - x)^{3/4} \right]_0^n = \frac{4}{3k^{1/4}} \left[k^{3/4} - (k - n)^{3/4} \right] \\
&= \frac{4k^{1/2}}{3} \left[1 - \left(\frac{k - n}{k} \right)^{3/4} \right] < \frac{4k^{1/2}}{3} \left[1 - \frac{k - n}{k} \right] \\
&= \frac{4}{3} k^{-1/2} n
\end{aligned}
\tag{3.164}
$$

so that

$$
\begin{aligned}
|\widetilde{\rho}_n(k) - \rho_n(k)| &= |\delta_n| \\
&\leq \varepsilon + \frac{k\varepsilon}{(k^2-n^2)^{1/4}} \left[\sum_{j=1}^{n-1} \frac{1}{(k^2-j^2)^{1/4}} \right] \\
&\leq \varepsilon + \frac{k\varepsilon}{(k^2-n^2)^{1/4}} \left[\frac{4}{3} k^{-1/2} n \right] \\
&\leq \left[1 + \frac{4}{3} \frac{nk^{1/2}}{(k^2-n^2)^{1/4}} \right] \varepsilon. \qquad (3.165)
\end{aligned}
$$

This completes the proof. □

For $k \gg n$, the preceding theorem entails $|\widetilde{\rho}_n(k) - \rho_n(k)| \lesssim n\varepsilon \leq k\varepsilon$. Next we consider the case where n is close to k.

Corollary 3.16. *Under the same assumptions as in Theorem 3.15, for all* $2 < n \leq k$, *we have*

$$
|\widetilde{\rho}_n(k) - \rho_n(k)| \leq \left[4 + 2^{7/4} k^{5/4} \right] \varepsilon. \qquad (3.166)
$$

Proof. By the previous theorem, for $n \leq k-1$,

$$
\begin{aligned}
|\delta_n| &\leq \left[1 + \frac{4}{3} \frac{k^{1/2}(k-1)}{(k^2-(k-1)^2)^{1/4}} \right] \varepsilon = \left[1 + \frac{2^{7/4}}{3} \frac{k^{1/2}(k-1)}{(k-1/2)^{1/4}} \right] \varepsilon \\
&\leq \left[1 + \frac{2^{7/4} k^{5/4}}{3} \right] \varepsilon \leq \left[4 + 2^{7/4} k^{5/4} \right] \varepsilon. \qquad (3.167)
\end{aligned}
$$

For $k-1 < n \leq k$, we estimate

$$\begin{aligned} |\delta_n| &\leq \varepsilon + \frac{2n}{k}|\delta_{n-1}| + |\delta_{n-2}| \\ &\leq \varepsilon + 2|\delta_{n-1}| + |\delta_{n-2}| \\ &\leq \varepsilon + 3\left[1 + \frac{2^{7/4}k^{5/4}}{3}\right]\varepsilon \leq \left[4 + 2^{7/4}k^{5/4}\right]\varepsilon. \end{aligned} \tag{3.168}$$

Therefore

$$|\delta_n| \leq \left[4 + 2^{7/4}k^{5/4}\right]\varepsilon \tag{3.169}$$

and the result follows □

Corollary 3.16 gives us an error bound for n close to k, but it is unstable for the case $n > k$. The second phase of the algorithm deals with that instability.

Proposition 3.17. *Let $M \geq n_0 = \lceil k \rceil$ and $A_M(k)$ and $\mathbf{b}_M(k)$ be defined as in* (3.89) *and* (3.90). *If*

$$A_M(k)\boldsymbol{\rho}_M(k) = \mathbf{b}_M(k), \quad A_M(k)\widetilde{\boldsymbol{\rho}}_M(k) = \widetilde{\mathbf{b}}_M(k) \tag{3.170}$$

with

$$\left\|\widetilde{\mathbf{b}}_M(k) - \mathbf{b}_M(k)\right\|_\infty \leq \varepsilon \tag{3.171}$$

then

$$\|\rho_M(k) - \tilde{\boldsymbol{\rho}}_M(k)\|_\infty \leq \left(\frac{n_0+2}{2}\right)\varepsilon \tag{3.172}$$

Proof. The proof mainly depends on bounding $A_M(k)$.To this end, we use (3.170) to write

$$(\boldsymbol{\rho}_M(k) - \widetilde{\boldsymbol{\rho}}_M(k)) = A_M^{-1}(k) \left(\widetilde{\mathbf{b}}_M(k) - \mathbf{b}_M(k)\right) \tag{3.173}$$

so that

$$\begin{aligned} \|\widetilde{\boldsymbol{\rho}}_M(k) - \boldsymbol{\rho}_M(k)\|_\infty &\leq \left\|A_M^{-1}(k)\right\|_\infty \left\|\widetilde{\mathbf{b}}_M(k) - \mathbf{b}_M(k)\right\|_\infty \\ &\leq \left\|A_M^{-1}(k)\right\|_\infty \varepsilon. \end{aligned} \tag{3.174}$$

Next we observe that if

$$D_M(k) = \begin{bmatrix} \frac{2n_0}{ik} & & & & \\ & \frac{2(n_0+1)}{ik} & & & \\ & & \frac{2(n_0+2)}{ik} & & \\ & & & \ddots & \\ & & & & \frac{2(2M-1)}{ik} \end{bmatrix}, \tag{3.175}$$

then

$$A_M(k) = (I_M + K_M(k))\, D_M(k) \tag{3.176}$$

where I_M is the identity matrix of order $2M - n_0$ and

$$K_M(k) = \begin{bmatrix} 0 & \frac{ik}{2(n_0+1)} & & & & \\ -\frac{ik}{2n_0} & 0 & \frac{ik}{2(n_0+2)} & & & \\ & -\frac{ik}{2(n_0+1)} & 0 & \frac{ik}{2(n_0+3)} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & -\frac{ik}{4(M-1)} & 0 \end{bmatrix}. \tag{3.177}$$

Note that

$$\|K_M(k)\|_\infty = \frac{(n_0+1)\,k}{n_0\,(n_0+2)} \le \frac{n_0+1}{n_0+2} < 1. \tag{3.178}$$

Therefore,

$$\begin{aligned}\left\|A_M^{-1}(k)\right\|_\infty &\le \left\|D_M^{-1}(k)\right\|_\infty \left\|(I_M+K_M(k))^{-1}\right\|_\infty \le \frac{k}{2n_0}\frac{1}{1-\|K_M(k)\|_\infty}\\ &\le \frac{n_0+2}{2}\end{aligned} \tag{3.179}$$

Hence we established,

$$\|\rho_M(k)-\tilde{\boldsymbol{\rho}}_M(k)\|_\infty \le \left(\frac{n_0+2}{2}\right)\varepsilon \tag{3.180}$$

and the proof is complete. □

From Proposition 3.17 we see that error bound for ρ_m contains the constant n_0. Therefore, we can choose a large value of M to have a better approximation for **b**, and this will increase our accuracy [7].

4. COMPOSITE CLENSHAW-CURTIS RULES

In this chapter, we will consider integrals with algebraic singularities on an interval $[a, b]$. Applying the FCC quadrature rule directly on $[a, b]$ would lead to an unreliable computation. Hence, we develop a graded mesh approach, and assume that the singularity is at the endpoint of a mesh. Without loss of generality we assume $[a, b] = [0, 1]$ and the singularity occurs at 0.

First, we define three function spaces that have singularity order β.

Definition 4.1. *Let $m \geq 1$, $\beta \in (0, 1)$, and define $C^m_\beta[0, 1]$ to be the space of all functions $v \in C[0, 1]$ such that $\|v\|_{m,\beta} < \infty$ where*

$$\|v\|_{m,\beta} = \max\left\{ \sup_{x\in[0,1]} |v(x)|, \sup_{x\in(0,1]} \left|x^{(j-\beta)} v^{(j)}(x)\right|, \quad j = 1, \ldots, m \right\}. \tag{4.1}$$

Definition 4.2. *Let $m \geq 1$, $\beta \in (-1, 0)$ and define $C^m_\beta[0, 1]$ to be the space of all functions $v \in C(0, 1]$ such that $\|v\|_{m,\beta} < \infty$ where*

$$\|v\|_{m,\beta} = \max\left\{ \sup_{x\in(0,1]} \left|x^{(j-\beta)} v^{(j)}(x)\right|, \quad j = 0, \ldots, m \right\}. \tag{4.2}$$

We now extend these definitions over an arbitrary interval $[a, b]$.

Definition 4.3. *Let $[a, b]$ be an arbitrary interval, and let $\beta \in (-1, 0) \cup (0, 1)$. We define*

$$\|f\|_{(a,b],m,\beta} = \begin{cases} \max\begin{cases} (b-a)^{-\beta} \sup_{v\in[a,b]} |f(v)|, \\ \max_{j=1,\ldots,m} \sup_{v\in(a,b]} \left|(v-a)^{j-\beta} f^{(j)}(v)\right| \end{cases} & \beta \in (0, 1), \\ \max_{j=0,\ldots,m} \sup_{v\in(a,b]} \left|(v-a)^{j-\beta} f^{(j)}(v)\right|, & \beta \in (-1, 0). \end{cases} \tag{4.3}$$

Now we define the function space appropriate for logarithmic singularities

Definition 4.4. *For $m \geq 1$, we define $C_0^m[0,1]$ to be the space of all functions $v \in C(0,1]$ such that $\|v\|_{m,0} < \infty$ where*

$$\|v\|_{m,0} = \max\left\{ \sup_{x\in[0,1]} \left|(|\log x| + 1)^{-1} v(x)\right|, \sup_{x\in[0,1]} \left|x^j v^{(j)}(x)\right|, \quad j = 1,\dots,m \right\}. \quad (4.4)$$

Observe that $C_\beta^m[0,1] \subset L^1[0,1]$, for $\beta \in (-1,1)$.

4.1. The Composite Algorithm

Throughout this section, we assume $f \in C_\beta^m[0,1]$ where $\beta \in (-1,1)$ and that singularity occurs at 0, which can be ensured through with an affine change of variables. For the intervals that contain singularity we develop a graded mesh approach, thus we will have small subintervals that are near the singularity which increases the accuracy. The definition of graded mesh is as follows.

Definition 4.5. *For $M \in \mathbb{N}_0$ and $q \in \mathbb{R}^+$ we define the graded mesh*

$$\Pi_{M,q} = \left\{ x_j = \left(\frac{j}{M}\right)^q : j = 0,1,\dots,M \right\}. \quad (4.5)$$

With this definition, we write our integral as

$$I_k^{[0,1]}(f) = I_k^{[x_0,x_1]}(f) + \sum_{j=2}^{M} I_k^{[x_{j-1},x_j]}(f). \quad (4.6)$$

The first term in this expression needs a special treatment since depending on whether $\beta > 0$ or $\beta \leq 0$, we will have different approximations. For integrals in the summation, we can directly use the FCC quadrature rule, since these intervals, by assumption, do not contain a singularity.

For the first subinterval, we introduce the approximation

$$\widetilde{I}_k^{[x_0,x_1]}(f) = \begin{cases} I_{k,1}^{[x_0,x_1]}(f) & \text{if } \beta \in (0,1) \\ 0 & \text{if } \beta \in (-1,0] \end{cases} \tag{4.7}$$

where

$$I_{k,1}^{[x_0,x_1]}(f) = \begin{cases} \int_{x_0}^{x_1} \left(Q_1^{[x_0,x_1]} f\right)(x) \exp(ikx)dx & \text{if } x_1 k \geq 1 \\ \int_{x_0}^{x_1} \left(Q_1^{[x_0,x_1]}(f \exp(ik\cdot))\right)(x)dx & \text{if } x_1 k < 1. \end{cases} \tag{4.8}$$

Here $Q_1^{[x_0,x_1]} f$ is the linear function interpolating $(x_0, f(x_0))$ and $(x_1, f(x_1))$. Putting this approximation in place of first subinterval computation, we get

$$I_{k,N,M,q}^{[0,1]}(f) = \tilde{I}_k^{[x_0,x_1]}(f) + \sum_{j=2}^{M} I_{k,N}^{[x_{j-1},x_j]}(f). \tag{4.9}$$

We define the total error $E_{k,N,M,q}(f)$ as

$$E_{k,N,M,q}(f) = \left| I_k^{[0,1]}(f) - I_{k,N,M,q}^{[0,1]}(f) \right| \tag{4.10}$$

where the local errors are given by

$$\begin{aligned} \tilde{e}_1 &= I_k^{[x_0,x_1]}(f) - \tilde{I}_k^{[x_0,x_1]}(f) \\ e_j &= I_k^{[x_{j-1},x_j]}(f) - I_{k,N}^{[x_{j-1},x_j]}(f) \quad \text{for} \quad j = 2, \ldots, M. \end{aligned} \tag{4.11}$$

Observe that

$$E_{k,N,M,q}(f) \leq |\tilde{e}_1| + \sum_{j=2}^{M} |e_j| \tag{4.12}$$

4.2. Estimates on the Size of the Integrals

In this section we derive estimates on the integrals $I_k^{[0,\varepsilon]}(f)$. This will be useful on estimating the total error $E_{k,N,M,q}(f)$.

Lemma 4.6. *For any $\beta \in (-1,0)$ and any $f \in C_\beta^1[0,1]$ there exists $C_\beta > 0$ such that, for $\varepsilon \in (0,1]$*

$$\left| I_k^{[0,\varepsilon]}(f) \right| \leq C_\beta \varepsilon^{1+\beta-s} \left(\frac{1}{k}\right)^s \|f\|_{1,\beta} \tag{4.13}$$

where $s \in [0, 1+\beta]$. Furthermore, for any $f \in C_0^1[0,1]$ there exists $C_0 > 0$ such that, for $\varepsilon \in (0,1]$

$$\left| I_k^{[0,\varepsilon]}(f) \right| \leq C_0(\varepsilon + \varepsilon|\log \varepsilon|)^{1-s} \left(\frac{1+\log k}{k}\right)^s \|f\|_{1,0} \tag{4.14}$$

where $s \in [0,1]$.

Proof. First we consider the case when $\beta \in (-1,0)$. The case $\beta = 0$ is similar. First note that, for all $\varepsilon \in (0,1]$,

$$\left| I_k^{[0,\varepsilon]}(f) \right| \leq \left[\int_0^\varepsilon x^\beta dx\right] \|f\|_{0,\beta} = \frac{1}{1+\beta}\varepsilon^{1+\beta}\|f\|_{0,\beta} \leq \frac{1}{1-|\beta|}\varepsilon^{1+\beta}\|f\|_{1,\beta}. \tag{4.15}$$

For $\varepsilon k \leq 1$, we have

$$\left| I_k^{[0,\varepsilon]}(f) \right| \leq \left(\frac{1}{1-|\beta|}\varepsilon^{1+\beta-s}\|f\|_{1,\beta}\right) \left(\frac{1}{k}\right)^s = C_\beta \varepsilon^{1+\beta-s} \left(\frac{1}{k}\right)^s \|f\|_{1,\beta} \tag{4.16}$$

where $C_\beta = (1-|\beta|)^{-1}$. Considering the case $\varepsilon k > 1$, we observe that $|f(x)| = x^\beta |f(x)x^{(-\beta)}| \leq x^\beta \|f\|_{1,\beta}$, and show that

$$\left| I_k^{[0,\varepsilon]}(f) \right| \leq \left[\frac{1}{1-|\beta|}\right] \left[\frac{2+|\beta|}{|\beta|}\right] \left(\frac{1}{k}\right)^{1+\beta} \|f\|_{1,\beta}. \tag{4.17}$$

To this end, we write

$$I_k^{[0,\varepsilon]}(f) = I_k^{[0,1/k]}(f) + I_k^{[1/k,\varepsilon]}(f). \tag{4.18}$$

Using (4.15), we estimate

$$\left| I_k^{[0,1/k]}(f) \right| \leq \frac{1}{1-|\beta|} \left(\frac{1}{k}\right)^{1+\beta} \|f\|_{0,\beta}. \tag{4.19}$$

On the other hand, integration by parts yields

$$I_k^{[1/k,\varepsilon]}(f) = \frac{1}{ik}[f(x)\exp(ikx)]_{x=1/k}\Big|^{x=\varepsilon} - \frac{1}{ik}\int_{1/k}^{\varepsilon} f'(x)\exp(ikx)dx. \tag{4.20}$$

Thus we have

$$\left| I_k^{[1/k,\varepsilon]}(f) \right| \leq \frac{1}{k}\left[|f(\varepsilon)| + |f(1/k)| + \int_{1/k}^{\varepsilon} |f'(x)|\, dx \right]. \tag{4.21}$$

Since , $|f(x)| \leq x^{\beta}\|f\|_{1,\beta}$ for $x < 0$, we see that

$$\int_{1/k}^{\varepsilon} |f'(x)|\, dx \leq \int_{1/k}^{\varepsilon} x^{\beta-1} dx \|f\|_{1,\beta} \leq \frac{1}{|\beta|}\left[\varepsilon^{\beta} + \left(\frac{1}{k}\right)^{\beta}\right] \|f\|_{1,\beta}. \tag{4.22}$$

Thus

$$\left| I_k^{[1/k,\varepsilon]}(f) \right| \leq \frac{1}{k}\left[1 + \frac{1}{|\beta|}\right]\left[\varepsilon^{\beta} + \left(\frac{1}{k}\right)^{\beta}\right] \|f\|_{1,\beta} \tag{4.23}$$

and, since $\varepsilon^{\beta} < (1/k)^{\beta}$, we obtain

$$\left| I_k^{[1/k,\varepsilon]}(f) \right| \leq 2\left[\frac{1+|\beta|}{|\beta|}\right]\left(\frac{1}{k}\right)^{1+\beta} \|f\|_{1,\beta}. \tag{4.24}$$

Substituting (4.24) and (4.19) into (4.18) we obtain

$$\begin{aligned} I_k^{[0,\varepsilon]}(f) &\leq \left[\frac{|\beta| + 2\,(1-|\beta|^2)}{(1-|\beta|)|\beta|}\right]\left(\frac{1}{k}\right)^{1+\beta}\|f\|_{1,\beta} \\ &\leq \left[\frac{|\beta| + 2\,(1-|\beta|^2)}{(1-|\beta|)|\beta|}\right]\varepsilon^{1+\beta-s}\left(\frac{1}{k}\right)^{s}\|f\|_{1,\beta} \end{aligned} \tag{4.25}$$

Considering the case $\beta = 0$, we observe that $1+|\log(x)| = 1-\log(x)$ for $x \in (0,1]$ and, $|f(x)| = (1+|\log(x)|)\,[(1+|\log(x)|)^{-1}f(x)] \leq (1-\log(x))\|f\|_{1,\beta}$. Therefore

$$\begin{aligned} \left|I_k^{[0,\varepsilon]}(f)\right| &\leq \int_0^{\varepsilon} 1-\log(x)dx\|f\|_{1,\beta} = (2\varepsilon - \varepsilon\log(\varepsilon))\|f\|_{1,\beta} \\ &\leq (2\varepsilon + \varepsilon\log(\varepsilon))\|f\|_{1,\beta}. \end{aligned} \tag{4.26}$$

Now if $\varepsilon k \leq 1$ we can obtain the desired result since

$$\begin{aligned} \left|I_k^{[0,\varepsilon]}(f)\right| &\leq (2\varepsilon + \varepsilon\log(\varepsilon))^{1-s}(2\varepsilon + \varepsilon\log(\varepsilon))^{s}\|f\|_{1,\beta} \\ &\leq C_0(\varepsilon + \varepsilon|\log\varepsilon|)^{1-s}\left(\frac{1+\log k}{k}\right)^{s}\|f\|_{1,0}. \end{aligned} \tag{4.27}$$

If $\varepsilon k > 1$ we follow the technique as in the case $\beta < 0$ to have

$$\left|I_k^{[0,1/k]}(f)\right| \leq \int_0^{1/k} 1-\log(x)dx\|f\|_{1,\beta} \leq \left(\frac{2+\log k}{k}\right)\|f\|_{1,0}. \tag{4.28}$$

and

$$\left|I_k^{[1/k,\varepsilon]}(f)\right| \leq \frac{1}{k}\left[|f(\varepsilon)| + |f(1/k)| + \int_{1/k}^{\varepsilon}|f'(x)|\,dx\right]. \tag{4.29}$$

Since

$$\frac{1}{k}|f(\varepsilon)| + |f(1/k)| \leq \frac{1}{k}\left(1 + |\log(\varepsilon)| + 1 + |\log(1/k)|\right)\|f\|_{1,\beta}$$
$$= \frac{1}{k}\left(1 - \log(\varepsilon) + 1 + |\log(k)|\right)\|f\|_{1,\beta}. \tag{4.30}$$

and

$$\frac{1}{k}\int_{1/k}^{\varepsilon} |f'(x)| \leq \frac{1}{k}\int_{1/k}^{\varepsilon} x^{-1}dx\|f\|_{1,\beta} = \frac{1}{k}\log(x)|_{1/k}^{\varepsilon}$$
$$\leq \frac{1}{k}(\log(\varepsilon) + |\log(1/k)|)\|f\|_{1,\beta}. \tag{4.31}$$

Inequalities (4.28) and (4.29) entail

$$\left|I_k^{[1/k,\varepsilon]}(f)\right| \leq \left(\frac{2 + 2|\log(1/k)|}{k}\right)\|f\|_{1,\beta}. \tag{4.32}$$

$$\left|I_k^{[0,\varepsilon]}(f)\right| \leq \left(\frac{4 + 3|\log(1/k)|}{k}\right)\|f\|_{1,\beta} \leq 4\left(\frac{1 + |\log(1/k)|}{k}\right)\|f\|_{1,\beta}. \tag{4.33}$$

Since $|\log(\varepsilon)| \geq |\log(1/k)| = \log(k)$ for $\varepsilon \in [0,1]$, we get

$$\left|I_k^{[0,\varepsilon]}(f)\right| \leq C\left(\frac{1 + \log(k)}{k}\right)^s (\varepsilon + \varepsilon\log(\varepsilon))^{1-s}\|f\|_{1,\beta} \tag{4.34}$$

and the proof is complete. $\square$

Lemma 4.7. *For any $\beta \in (0,1)$ and for any $f \in C^2_\beta[0,1]$ there exists $C_\beta > 0$ such that for $\varepsilon \in (0,1]$ and $s \in [0,1]$*

$$\left| I_k^{[0,\varepsilon]}(f) \right| \leq C_\beta \varepsilon^{1-s} \left(\frac{1}{k}\right)^s \|f\|_{1,\beta}. \tag{4.35}$$

Moreover, if $s \in [1, 1+\beta]$,

$$\begin{aligned} \left| I_k^{[0,\varepsilon]}(f) \right| &\leq \frac{1}{k}[|f(0)| + |f(\varepsilon)|] + C_\beta \varepsilon^{1+\beta-s} \left(\frac{1}{k}\right)^s \|f'\|_{1,\beta-1} \\ &\leq \frac{2}{k}\|f\|_{0,\beta} + C_\beta \varepsilon^{1+\beta-s} \left(\frac{1}{k}\right)^s \|f\|_{2,\beta}. \end{aligned} \tag{4.36}$$

Proof. We truly have

$$\left| I_k^{[0,\varepsilon]}(f) \right| \leq \varepsilon \|f\|_{0,\beta}. \tag{4.37}$$

On the other hand, since $|f(x)| \leq \|f\|_{0,\beta}$ for $x \in [0,1]$, an integration by parts yields

$$\begin{aligned} \left| I_k^{[0,\varepsilon]}(f) \right| &\leq \frac{1}{k}[|f(0)| + |f(\varepsilon)|] + \frac{1}{k} \left| \int_0^\varepsilon f'(x) \exp(ikx) dx \right| \\ &\leq \frac{2}{k}\|f\|_{0,\beta} + \frac{1}{k} \left| \int_0^\varepsilon f'(x) \exp(ikx) dx \right|. \end{aligned} \tag{4.38}$$

Since $|f'(x)| \leq x^{\beta-1}\|f\|_{1,\beta}$ for $x < 0$, we also have

$$\left| \int_0^\varepsilon f'(x) \exp(ikx) dx \right| \leq \left[\int_0^\varepsilon x^{\beta-1} dx \right] \|f\|_{1,\beta} = \frac{1}{\beta} \varepsilon^\beta \|f\|_{1,\beta}. \tag{4.39}$$

Therefore

$$\left| I_k^{[0,\varepsilon]}(f) \right| \leq \left(2 + \frac{1}{\beta}\right) \frac{1}{k} \|f\|_{1,\beta}. \tag{4.40}$$

It follows that

$$\begin{aligned}
\left| I_k^{[0,\varepsilon]}(f) \right| &= \left| I_k^{[0,\varepsilon]}(f) \right|^s \left| I_k^{[0,\varepsilon]}(f) \right|^{1-s} \\
&\leq \left(\left(2 + \frac{1}{\beta} \right) \frac{1}{k} \|f\|_{1,\beta} \right)^s (\varepsilon \|f\|_{1,\beta})^{1-s} \\
&\leq C_\beta \varepsilon^{1-s} \left(\frac{1}{k} \right)^s \|f\|_{1,\beta}. \qquad (4.41)
\end{aligned}$$

As for the second inequality in the lemma, observe that $\beta - 1 \in (-1, 0)$, and since $f \in C^2_\beta[0,1]$ we have $f' \in C^1_{\beta-1}[0,1]$, Lemma 4.6 entails

$$\left| \int_0^\varepsilon f'(x) \exp(ikx) dx \right| \leq C_{\beta-1} \varepsilon^{\beta-t} \left(\frac{1}{k} \right)^t \|f'\|_{1,\beta-1} \leq C_{\beta-1} \varepsilon^{\beta-t} \left(\frac{1}{k} \right)^t \|f\|_{2,\beta} \qquad (4.42)$$

for all $t \in [0, \beta]$. Setting $s = 1 + t$, the result follows. $\square$

Concerning the sharpness of the estimates in Lemma 4.6 and Lemma 4.7 as $k \to \infty$, we define the family of functions

$$f_\beta(x) = \begin{cases} x^\beta, & \beta \in (-1,0) \cup (0,1) \\ \log x, & \beta = 0. \end{cases} \qquad (4.43)$$

Lemma 4.8. *For all $\beta \in (-1, 1)$, there exists a constant $A_\beta > 0$ such that, for all k sufficiently large,*

$$\begin{aligned}
k^{\min\{1+\beta, 1\}} \left| I_k^{[0,1]}(f_\beta) \right| &\geq A_\beta \quad \textit{when } \beta \in (-1,0) \cup (0,1) \\
\frac{k}{\log k} \left| I_k^{[0,1]}(f_0) \right| &\geq A_0 \quad \textit{when } \beta = 0. \qquad (4.44)
\end{aligned}$$

Proof. First we note that (c.f. Eq. (3.761.1) and Eq. (3.761.6) in [24]),

$$\int_0^1 x^{\mu-1}\sin(ax)dx = \frac{-i}{2\mu}\left[{}_1F_1(\mu;\mu+1;ia) - {}_1F_1(\mu;\mu+1;-ia)\right]$$
$$\text{for } [a>0, \quad \operatorname{Re}\mu > -1, \quad \mu \neq 0],$$
$$\int_0^1 x^{\mu-1}\cos(ax)dx = \frac{1}{2\mu}\left[{}_1F_1(\mu;\mu+1;ia) + {}_1F_1(\mu;\mu+1;-ia)\right]$$
$$\text{for } [a>0, \quad \operatorname{Re}\mu > 0] \tag{4.45}$$

where ${}_1F_1(a,b,z)$ denotes the confluent hypergeometric function as defined in (C.2). For $\beta \neq 0$,

$$\begin{aligned} I_k^{[0,1]}(f_\beta) &= \int_0^1 x^\beta \exp(ikx)dx = i\int_0^1 x^\beta \sin(kx)dx + \int_0^1 x^\beta \cos(kx)dx \\ &= \frac{1}{2(\beta+1)}\left[{}_1F_1(\beta+1,\beta+2,ik) - {}_1F_1(\beta+1,\beta+2,-ik)\right] \\ &+ \frac{1}{2(\beta+1)}\left[{}_1F_1(\beta+1,\beta+2,ik) + {}_1F_1(\beta+1,\beta+2,-ik)\right] \end{aligned} \tag{4.46}$$

so that

$$I_k^{[0,1]}(f_\beta) = \frac{1}{\beta+1}{}_1F_1(\beta+1,\beta+2,ik). \tag{4.47}$$

The confluent hypergeometric function ${}_1F_1(a,b,z)$ has the following asymptotic property mentioned in the Eq. (13.5.1) [21], at large values of $|z|$, with $-\pi/2 < \arg z < 3\pi/2$, for fixed a and b:

$${}_1F_1(a,b,z) = \Gamma(b)\left(\frac{e^{i\pi a}z^{-a}}{\Gamma(b-a)} + \frac{e^z z^{a-b}}{\Gamma(a)}\right)\left(1 + O\left(|z|^{-1}\right)\right) \tag{4.48}$$

Hence, for our integral, we have the asymptotic behavior

$$I_k^{[0,1]}(f_\beta) = \left(\Gamma(1+\beta)e^{i\pi(1+\beta)}(ik)^{-1-\beta} + e^{ik}(ik)^{-1}\right)\left(1 + O\left(k^{-1}\right)\right). \tag{4.49}$$

So, for k sufficiently large and $\beta \in (-1, 0)$,

$$\left| I_k^{[0,1]}(f_\beta) \right| \geq \frac{1}{2}\left(\Gamma(1+\beta)k^{-1-\beta} - k^{-1}\right) \geq \frac{1}{4}\Gamma(1+\beta)k^{-1-\beta}, \tag{4.50}$$

and for $\beta \in (0, 1)$,

$$\left| I_k^{[0,1]}(f_\beta) \right| \geq \frac{1}{2}\left(k^{-1} - \Gamma(1+\beta)k^{-1-\beta}\right) \geq \frac{1}{4}k^{-1}. \tag{4.51}$$

Thus, we obtained the first inequality in the lemma.

For the second inequality, let $\beta = 0$. By Eq. (4.381.1) and Eq. (4.381.2) in [24] ,

$$I_k^{[0,1]}(f_0) = \int_0^1 \log(x)\exp(ikx)dx = i\int_0^1 \log(x)\sin(kx)dx + \int_0^1 \log(x)\cos(kx)dx \tag{4.52}$$

where si and ci are the sine and cosine integrals

$$\mathrm{si}(x) = -\frac{\pi}{2} + \int_0^x \frac{\sin t}{t}dt, \quad \mathrm{ci}(x) = \gamma + \log x + \int_0^x \frac{\cos t - 1}{t}dt \tag{4.53}$$

and $\gamma \approx 0.5772$ is the Euler-Macheroti constant. So we have

$$I_k^{[0,1]}(f_0) = -\frac{1}{k}\left(\frac{\pi}{2} + i\gamma + i\log k\right) + \frac{i}{k}(\mathrm{ci}(k) + i\mathrm{textsi}(k)). \tag{4.54}$$

Since (see equations (5.2 .8) (5.2 .9)] in [21])

$$\begin{aligned} si(z) &= -f(z)\cos(z) - g(z)\sin(z), \\ ci(z) &= f(z)\sin(z) - g(z)\cos(z). \end{aligned} \tag{4.55}$$

with (c.f. equations (5.2 .34) and (5.2 .35) in [21]),

$$f(z) \approx \frac{1}{z}\left(1 - \frac{2!}{z^2} + \frac{4!}{z^4} - \frac{6!}{z^6}\dots\right) \text{ where } |arg(z)|, < \pi$$

$$g(z) \approx \frac{1}{z}\left(1 - \frac{3!}{z^2} + \frac{5!}{z^4} - \frac{7!}{z^6} \dots\right) \text{ where } |arg(z)| < \pi \tag{4.56}$$

as $|z| \to \infty$ we have

$$\mathrm{si}(k) = \mathcal{O}(1/k), \quad \mathrm{ci}(k) = \mathcal{O}(1/k) \quad \text{as } k \to \infty. \tag{4.57}$$

Hence, we obtain

$$I_k^{[0,1]}(f_0) = -\frac{1}{k}\left(\frac{\pi}{2} + i\gamma + i\log k\right) + O\left(\frac{1}{k^2}\right). \tag{4.58}$$

Therefore for all k sufficiently large,

$$\left|I_k^{[0,1]}(f_0)\right| \geq \frac{\log k}{2k}, \tag{4.59}$$

and the proof is complete. □

4.3. The Total Error for the Composite FCC Method

In this section, we estimate the total error, defined in (4.10), of the Filon-Clenshaw-Curtis rule. Using Lemma 4.6 and Lemma 4.7, we can estimate $|\tilde{e}_1|$. In the following lemma, we estimate the summation part of the error in (4.10). Finally we derive an estimation for $|\tilde{e}_1|$.

Lemma 4.9. *Let $f \in C_\beta^{N+1}[0,1], \beta \in (-1,1)$, let $r \geq 0$, and choose*

$$q > (N+1-r)/(\beta+1-r), \quad \textit{for } r < 1+\beta. \tag{4.60}$$

Then there exists a constant C which depends on $N, \beta,$ and q such that

$$\sum_{j=2}^{M} |e_j| \leq C \left(\frac{1}{k}\right)^{r} \left(\frac{1}{M}\right)^{N+1-r} \|f\|_{N+1,\beta}. \tag{4.61}$$

Proof. We let C denote a generic constant and it may depend on N, β, and q. Using the bound developed in Corollary 3.9 , and denoting $\delta_j = (x_j - x_{j-1})/2$, we have

$$\begin{aligned} \sum_{j=2}^{M} |e_j| &\leq C \left(\frac{1}{k}\right)^{r} \sum_{j=2}^{M} \delta_j^{N+2-r} \max_{x\in[x_{j-1},x_j]} \left|f^{(N+1)}(x)\right| \\ &\leq C \left(\frac{1}{k}\right)^{r} \left\{\sum_{j=2}^{M} \delta_j^{N+2-r} x_{j-1}^{\beta-N-1}\right\} \|f\|_{N+1,\beta}. \end{aligned} \tag{4.62}$$

The mean-value theorem shows that

$$\delta_j \leq C \frac{1}{M} \left(\frac{j-1}{M}\right)^{q-1} \qquad \text{for} \quad j = 2, \ldots, M \tag{4.63}$$

so we have

$$\delta_j^{N+2-r} x_{j-1}^{\beta-N-1} \leq C \left(\frac{1}{M}\right)^{N+2-r} \left(\frac{j-1}{M}\right)^{\alpha} \tag{4.64}$$

where $\alpha = q(\beta + 1 - r) - (N + 2 - r)$. Therefore

$$\sum_{j=2}^{M} |e_j| \leq C \left(\frac{1}{k}\right)^{r} \left(\frac{1}{M}\right)^{N+1-r} \left\{\sum_{j=2}^{M} \frac{1}{M} \left(\frac{j-1}{M}\right)^{\alpha}\right\} \|f\|_{N+1,\beta}. \tag{4.65}$$

The sum here corresponds to a Riemann sum for the integral $\int_0^1 x^{\alpha} dx$, which is finite since $\alpha > -1$. $\square$

Lemma 4.10. *Under the same hypothesis as the Lemma 4.9, there exists a constant C which depends on β and q such that for $N \geq 1$*

$$|\widetilde{e}_1| \leq \; C\left(\frac{1}{k}\right)^r\left(\frac{1}{M}\right)^{N+1-r} \begin{cases} \|f\|_{2,\beta} & \textit{when } \beta\in(-1,0)\cup(0,1) \\ (1+\log k)^r(\log M)^{1-r}\|f\|_{2,0} & \textit{when } \beta=0. \end{cases} \tag{4.66}$$

Proof. In this proof we take $x_0 = 0$ since the first point is 0. Considering $\beta \in (0,1)$, we have

$$\begin{aligned} \left|\left(Q_1^{[0,x_1]}f\right)'\right| &= \left|\frac{f\left(x_1\right)-f(0)}{x_1}\right| = \left|\frac{1}{x_1}\int_0^{x_1} f'(x)dx\right| \leq \frac{1}{x_1}\int_0^{x_1} |f'(x)|\, dx \\ &\leq \frac{1}{x_1}\left[\int_0^{x_1} x^{\beta-1}dx\right]\|f\|_{1,\beta} = \frac{1}{\beta}x_1^{\beta-1}\|f\|_{1,\beta}. \end{aligned} \tag{4.67}$$

Moreover, for any $t \in [0,x_1]$ and any $f \in C^1_\beta[0,1]$, we have

$$\begin{aligned} \left|f(t)-Q_1^{[0,x_1]}f(t)\right| &= \left|\int_0^t \left(f-Q_1^{[0,x_1]}f\right)'(x)dx\right| \\ &\leq \int_0^t |f'(x)|dx + \int_0^t \left|\left(Q_1^{[0,x_1]}f\right)'\right| dx \\ &\leq \int_0^{x_1} |f'(x)|\, dx + x_1\left|\left(Q_1^{[0,x_1]}f\right)'\right| \\ &\leq \frac{2}{\beta}x_1^{\beta}\|f\|_{1,\beta} \end{aligned} \tag{4.68}$$

so that

$$|\widetilde{e}_1| = \left|\int_0^{x_1}\left(f(x)-Q_1^{[0,x_1]}f(x)\right)\exp(ikx)dx\right| \leq \frac{2}{\beta}x_1^{1+\beta}\|f\|_{1,\beta}. \tag{4.69}$$

By integration by parts, we also have

$$\widetilde{e}_1 = -\frac{1}{ik}\int_0^{x_1}\left(f(x)-\left(Q_1^{[0,x_1]}f\right)(x)\right)'\exp(ikx)dx. \tag{4.70}$$

Since $f' \in C^N_{\beta-1}$ and $\beta - 1 \in (-1, 0)$, we can observe that Lemma 4.6 implies

$$\left| \int_0^{x_1} f'(x) \exp(ikx) dx \right| \leq C_\beta \left(\frac{1}{k} \right)^\beta \|f'\|_{1,\beta-1} \leq C_\beta \left(\frac{1}{k} \right)^\beta \|f\|_{2,\beta}. \tag{4.71}$$

Furthermore, $\left(Q_1^{[0,x_1]} f \right)'$ is constant, so we have

$$\begin{aligned} \left| \int_0^{x_1} \left(Q_1^{[0,x_1]} f \right)' \exp(ikx) dx \right| &= \left| \left(Q_1^{[0,x_1]} f \right)' \right| \left| \int_0^{x_1} \exp(ikx) dx \right| \\ &\leq \frac{x_1^{\beta-1}}{\beta k} \|f\|_{1,\beta}. \end{aligned} \tag{4.72}$$

Combining (4.71) and (4.72) with (4.70), we arrive at

$$|\widetilde{e}_1| \leq C'_\beta \left[\left(\frac{1}{k} \right)^{1+\beta} + \frac{x_1^{\beta-1}}{k^2} \right] \|f\|_{2,\beta} \leq C'_\beta \left(\frac{1}{k} \right)^{1+\beta} \left[1 + (x_1 k)^{\beta-1} \right] \|f\|_{2,\beta}. \tag{4.73}$$

Hence if $x_1 k \geq 1$ we can interpolate (4.69) and (4.73), to obtain for any $r \in [0, 1+\beta]$

$$|\widetilde{e}_1| \leq C''_\beta \left(\frac{1}{k} \right)^r x_1^{1+\beta-r} \|f\|_{N+1,\beta} \leq C''_\beta \left(\frac{1}{k} \right)^r \left(\frac{1}{M} \right)^{N+1-r} \|f\|_{2,\beta}. \tag{4.74}$$

If $x_1 k < 1$, define $f_k(x) = f(x) \exp(ikx)$ and use(4.68) to have

$$\begin{aligned} |\widetilde{e}_1| &= \left| \int_0^{x_1} \left(f_k(x) - Q_1^{[0,x_1]} f_k(x) \right) dx \right| \\ &\leq \frac{2}{\beta} x_1^{1+\beta} \|f_k\|_{1,\beta} \\ &\leq \frac{2}{\beta} \left(\frac{1}{k} \right)^r x_1^{1+\beta-r} \|f\|_{1,\beta} \\ &\leq \frac{2}{\beta} \left(\frac{1}{k} \right)^r \left(\frac{1}{M} \right)^{N+1-r} \|f\|_{1,\beta}. \end{aligned} \tag{4.75}$$

So the result follows for $\beta \in (0, 1)$.

For $\beta \in (-1, 0]$, we approximated $|\widetilde{e}_1|$ as zero, thus the result directly follows from Lemma 4.6 with $\varepsilon = x_1$ and $s = r$. □

Theorem 4.11. *Under the same hypothesis as Lemma 4.9, there exists a constant C which depends on N, β, and q such that*

$$E_{k,N,M,q}(f) \leq C \left(\frac{1}{k}\right)^r \left(\frac{1}{M}\right)^{N+1-r} \times \begin{cases} \|f\|_{N+1,\beta} & \text{when } \beta \in (-1,0) \cup (0,1) \\ (1+\log k)^r (\log M)^{1-r} \|f\|_{N+1,0} & \text{when } \beta = 0 \end{cases} \tag{4.76}$$

Proof. We have the following inequality,

$$|E_{k,N,M,q}(f)| \leq |\tilde{e}_1| + \sum_{j=2}^{M} |e_j| \,. \tag{4.77}$$

Since is $\|f\|_{N,\beta} \leq \|f\|_{N+l,\beta}$ for $l > 0$, Lemma 4.10 yields for $\beta \in (-1,0) \cup (0,1)$

$$|\widetilde{e}_1| \leq C \left(\frac{1}{k}\right)^r \left(\frac{1}{M}\right)^{N+1-r} \|f\|_{2,\beta} \leq C \left(\frac{1}{k}\right)^r \left(\frac{1}{M}\right)^{N+1-r} \|f\|_{N+1,\beta}. \tag{4.78}$$

Combining this with Lemma 4.9 directly gives the result. The case $\beta = 0$ is treated similarly. □

One may extend the analysis we developed above to functions that have more complicated singularities than we considered. See [25] for more information and examples reader can check. To exemplify one such case, consider functions that behave similarly to $x^\beta \log(x)$ with $\beta \in (-1,0)$. The relevant function space in this case is defined as follows.

Definition 4.12. *Let $m \geq 1$, $\beta \in (-1,0)$ we define $\widetilde{C}^m_\beta[0,1]$ to be the space of all functions $v \in C(0,1]$ such that $\|v\|_{m,\beta} < \infty$ where*

$$\|v\|_{m,\beta} = \max \begin{cases} \sup_{x\in[0,1]} \left| \left[x^\beta(|\log x|+1)\right]^{-1} v(x) \right|, \\ \sup_{x\in[0,1]} \left| \left[x^{\beta-j}(|\log x|+j)\right]^{-1} v^{(j)}(x) \right|, \qquad j=1,\ldots,m \end{cases} \tag{4.79}$$

As shown in [25], a similar error analysis yields, for $s \in [0,1]$, $q > (N+1-r)/(\beta+1-r)$ and $r \in [0,1+\beta)$,

$$E_{k,N,M,q}(f) \leq C_0(f) \left(\frac{1+|\log k|}{k^{1+\beta}}\right)^r (\log M)^{1+\beta-r} \left(\frac{1}{M}\right)^{N+1-r}. \tag{4.80}$$

where $C_0(f)$ is a constant.

4.4. Nonlinear Oscillators

Next we consider the case nonlinear oscillators, that is, $g(x) \neq x$.

4.4.1. A Nonlinear Change of Variables

In this section, we consider the integral

$$I_k^{[a,b]}(f,g) = \int_a^b f(x)\exp(ikg(x))dx. \tag{4.81}$$

with a nonlinear oscillator g. We assume $g \in C^\infty$ and introduce the change of variables

$$\tau = g(x). \tag{4.82}$$

If g has no stationary point, meaning $g'(x) \neq 0$ for all x, we have $g^{-1} \in C^{\infty}[a,b]$, $(g^{-1})' = 1/\left(g' \circ g^{-1}\right)$. Now, with the change of variable, our new integrand becomes

$$F = \left(f \circ g^{-1}\right) \left|\left(g^{-1}\right)'\right| = \frac{\left(f \circ g^{-1}\right)}{\left|\left(g' \circ g^{-1}\right)\right|}. \tag{4.83}$$

So, in the end, we have

$$I_k^{[a,b]}(f,g) = I_k^{[g(a),g(b)]}(F). \tag{4.84}$$

Hence, our integral is similar to what we worked on in previous sections. If $f \in C^{\infty}[a,b]$, we can directly use FCC rules as we did before. If our integrand f has singularities, we can also use the methods (involving graded mesh) to compute the integral. One must see that, there is an extra computational cost added here due to the inverse g^{-1}, which needs special treatment in some cases. Another important aspect of this change of variable is that if $g(a) > g(b)$ while $a < b$, we should transfer our integral with the affine change of variables to $[g(b), g(a)]$.

4.4.2. Nonlinear Oscillator With Oscillatory Points

In this section, we work on the case where g has finitely many stationary points $\{\xi_i\}_1^N$, which will drastically expand our use of the method. Use of the composite rule in this case needs special attention since stationary points lead to singularities in F.

We introduce subintervals of $[a,b]$ such that every subinterval has at most 1 stationary point or 1 singularity point (of f); however, stationary and singularity points can coincide in some cases, but it does not pollute the computation. With this approach, we obtain a monotone oscillator on each one of these subintervals which allows us to utilize the approach in Section Section 4.4.1.

This reduces the problem to the treatment of an interval with a singularity or stationary point. In the rest of this section, we assume that the oscillator g has one stationary point $\xi \in [a, b]$ such that

$$g'(\xi) = g''(\xi) = \cdots = g^{(n)}(\xi) = 0, \quad g^{(n+1)}(\xi) \neq 0 \text{ and } g'(x) \neq 0 for x \in [a, b] \backslash \{\xi\} \tag{4.85}$$

for some $n \geq 1$. Thus, ξ is a stationary point of order n.

Now, we separate our integral into two integrals as

$$I_k^{[a,b]}(f, g) = \int_{g(a)}^{g(\xi)} F(\tau) \exp(ik\tau) d\tau + \int_{g(\xi)}^{g(b)} F(\tau) \exp(ik\tau) d\tau. \tag{4.86}$$

For the following results, we need the Faà di Bruno formula which concerns the derivatives of the composition. In detail, for sufficiently smooth functions ϕ, ψ

$$(\phi \circ \psi)^{(p)} = \sum \frac{p!}{m!} \left(\phi^{(|m|)} \circ \psi\right) \left(\prod_{j=1}^{p} \left(\psi^{(j)}\right)^{m_j}\right) \tag{4.87}$$

where $p \in \mathbb{N}_0$, m is multiindice with $\sum_{k=1}^{p} k m_k = p$ and $|m| = \sum_{k=1}^{p} m_k$ and $m! = \mathit{\Pi}_{k=1}^{p} m_k!$.

Lemma 4.13. *Assume $g \in C^{\infty}[a, b]$ and that it has a single stationary point ξ of order n. Then for all $p \in \mathbb{N}$, there exists a constant $C_p > 0$ such that*

$$\left| \left(g^{-1}\right)^{(p)} (\tau) \right| \leq C_p |\tau - g(\xi)|^{\alpha - p}. \tag{4.88}$$

Proof. We assume without loss of generality, $g^{(n+1)}(\xi) > 0$ and $\xi < b$. Hence, we study the case where $\tau \in (\xi, b]$ only. By Taylor's theorem, we have

$$g(x) = g(\xi) + (x - \xi) g'(\xi) + \cdots + \frac{(x - \xi)^n}{n!} g^{(n)}(\xi) + R_\xi(x) = g(\xi) + R_\xi(x) \tag{4.89}$$

where

$$R_\xi(x) = \frac{1}{n!}\int_\xi^x (x-t)^n g^{(n+1)}(t)dt. \tag{4.90}$$

Introduce the change of variables as

$$y = \frac{t-\xi}{x-\xi} \tag{4.91}$$

so that

$$t = y(x-\xi)+\xi \quad \text{and} \quad dt = (x-\xi)dy, \tag{4.92}$$

we obtain

$$\begin{aligned} R_\xi(x) &= \frac{1}{n!}\int_0^1 [x-y(x-\xi)-\xi]^n \, g^{(n+1)}(y(x-\xi)+\xi)(x-\xi)dy \\ &= \frac{1}{n!}\int_0^1 [(x-\xi)(1-y)]^n \, g^{(n+1)}(y(x-\xi)+\xi)(x-\xi)dy \\ &= (x-\xi)^{n+1}\frac{1}{n!}\int_0^1 (1-y)^n g^{(n+1)}(y(x-\xi)+\xi)dy. \end{aligned} \tag{4.93}$$

Therefore

$$R_\xi(x) = (x-\xi)^{n+1}T_\xi(x) \quad \text{where} \quad T_\xi(x) = \frac{1}{n!}\int_0^1 (1-y)^n g^{(n+1)}(\xi+y(x-\xi))dy. \tag{4.94}$$

It follows that, for all $x \in (\xi, b], R_\xi(x) > 0$ and $T_\xi(x) > 0$. Also, since $g \in C^\infty[a,b]$, we have $T_\xi \in C^\infty[\xi, b]$ and

$$T_\xi(\xi) = \frac{1}{n!}\int_0^1 (1-y)^n g^{(n+1)}(\xi)dy = \frac{1}{(n+1)!}g^{(n+1)}(\xi) > 0 \text{ since } g^{(n+1)}(\xi) > 0. \tag{4.95}$$

Now, set $\alpha = 1/(n+1)$ and define

$$h_\xi(x) = (g(x) - g(\xi))^\alpha = (R_\xi(x))^\alpha = (T_\xi(x))^\alpha (x - \xi). \tag{4.96}$$

Then $h_\xi \in C^\infty(\xi, b]$, since $g \in C^\infty(\xi, b]$ and $h'_\xi(x) = \alpha(g(x) - g(\xi))^{\alpha-1} g'(x) > 0$ for $x \in (\xi, b]$.

Furthermore, since $h'_\xi(\xi) = (T_\xi(\xi))^\alpha > 0$ and h'_ξ is positive valued on $[\xi, b]$, it follows that $h_\xi : [\xi, b] \to \mathbb{R}$ is invertible and $(h_\xi)^{-1} \in C^\infty [h_\xi(\xi), h_\xi(b)]$.

Using the notation $\phi = h_\xi^{-1}$ and $\psi(\tau) = (\tau - g(\xi))^\alpha$ and we apply Faà di Bruno formula to obtain

$$\begin{aligned}(\phi \circ \psi)^{(p)} &= h_\xi^{-1} \left((\tau - g(\xi))^\alpha\right)^{(p)} \\ &= \sum \frac{p!}{m!} \left((h_\xi^{-1})^{(|m|)} \circ (\tau - g(\xi))^\alpha\right) \left(\prod_{j=1}^{p} \left(((\tau - g(\xi))^\alpha)^{(j)}\right)^{m_j}\right).\end{aligned} \tag{4.97}$$

Since h_ξ^{-1} is smooth, $\left((h_\xi^{-1})^{(|m|)} \circ (\tau - g(\xi))^\alpha\right)$ is bounded, and we observe that

$$\prod_{j=1}^{p} \left(((\tau - g(\xi))^\alpha)^{(j)})^{m_j}\right) = C|\tau - g(\xi)|^{(\alpha-1)m_1 + (\alpha-2)m_2 + \cdots + (\alpha-p)m_p} \tag{4.98}$$

where C is a constant. Since $\alpha|m| - p \geq \alpha - p$, the desired result follows. $\square$

Theorem 4.14. *Assume that f and g are in $C^\infty[a, b]$ and that g has a single stationary point ξ of order n. Then for each $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, there exists $C_p > 0$ such that*

$$\left|F^{(p)}(\tau)\right| \leq C_p |\tau - g(\xi)|^{\alpha-p-1}, \quad \tau \in [g(a), g(\xi)) \cup (g(\xi), g(b)]. \tag{4.99}$$

Proof. Remark that our integrand F is the following,

$$F = \left(f \circ g^{-1}\right)\left(g^{-1}\right)'. \tag{4.100}$$

By the Leibnitz rule, $F^{(p)}$ is a linear combination of terms of the form

$$\left(f \circ g^{-1}\right)^{(l)}\left(g^{-1}\right)^{(p-l+1)}, \quad l = 0, \ldots, p \tag{4.101}$$

We already developed in Lemma 4.13 that

$$\left|\left(g^{-1}\right)^{(p-l+1)}(\tau)\right| \leq C_1|\tau - g(\xi)|^{\alpha-p+l-1}. \tag{4.102}$$

Since f is smooth, by Faà di Bruno formula we can estimate the first term as

$$C\left|\prod_{j=1}^{l}\left(\left(g^{-1}\right)^{(j)}\right)^{m_j}\right| \tag{4.103}$$

where C is constant. Again, for the product part, we can use the Lemma 4.14's proof and estimate it with $|\tau - g(\xi)|^{\alpha-l}$.

After establishing the bound, we return to the products, for $l \neq 0$ each of these can have the terms of the form

$$C|\tau - g(\xi)|^{\alpha-l}C_1|\tau - g(\xi)|^{\alpha-p+l-1} = C_p|\tau - g(\xi)|^{2\alpha-p-1}. \tag{4.104}$$

where C_p is constant. For the case $l = 0$ the bound is $C_p|\tau - g(\xi)|^{\alpha-p-1}$ and since $\alpha > 0$, we obtain the desired result. $\square$

As in Remark 4.12, the tools we have developed to prove Theorem 4.15 can be easily extended to the case of functions with an algebraic or logarithmic singularity at points where g is also stationary. Let us assume that g is an increasing function on

$[a,b]$ with a stationary point of order $n+1$ at point a. When $\beta \in (0,1)$ the result comes directly since $f \in C[a,b]$. For the case $\beta(-1,0]$, we present the following theorem.

Theorem 4.15. *Let g be an increasing function on $[a,b]$ with a stationary point of order $n+1$ at point a and let $\beta \in (-1,0]$. Suppose that $f \in C^{\infty}(a,b] \cap C^n_\beta[a,b]$ satisfies*

$$|f(x)| \leq C \begin{cases} 1 + |\log(x-a)| & \text{if } \beta = 0 \\ (x-a)^{\beta} & \text{if } \beta \in (-1,0). \end{cases} \quad \text{for all } x \in (a,b] \tag{4.105}$$

and for $n \geq 1$,

$$\sup_{x \in (a,b]} \left|(x-a)^{p-\beta} f^{(p)}(x)\right| \leq C_p \quad \text{for } 0 \leq p \leq n. \tag{4.106}$$

Then, for $\tau \in (g(a), g(b)]$, we have

$$\begin{aligned} &\left|F^{(p)}(\tau)\right| \\ &\leq C_p \begin{cases} |\tau - g(a)|^{\alpha+\beta-p-1} & \text{if } \beta \in (-1,0), \\ |\tau - g(a)|^{\alpha-p-1}(1 + |\log(|\tau - g(a)|)|) & \text{if } \beta = 0, \end{cases} \end{aligned} \tag{4.107}$$

for all $\tau \in (g(a), g(b)]$.

Proof. By (4.106), for $\beta \in (-1,0]$ and $p \leq n$, we have

$$\left|f^{(p)}(x)\right| \leq C_p \left|(x-a)^{\beta-p}\right| \quad \text{for all } x \in (a,b]. \tag{4.108}$$

Fix $p \leq n$. Using the same idea as in the previous theorem, we see that $F^{(p)}$ is a linear combination of

$$(f \circ g^{-1})^{(l)} (g^{-1})^{(p-l+1)} \quad l = 0, \ldots, p. \tag{4.109}$$

The term on the right can be bounded by Lemma 4.14 as $|\tau - g(a)|^{\alpha-p+l-1}$. When $l > 0$, for the first term, use the Faa di Bruno formula to write

$$(f \circ g^{-1})^{(l)} = \sum \frac{l!}{m!} \left(f^{(|m|)} \circ g^{-1}\right) \left(\prod_{j=1}^{l} \left((g^{-1})^{(j)}\right)^{m_j}\right). \tag{4.110}$$

Again, by Theorem 4.15, we can use the estimate for the term in the product by $|\tau - g(a)|^{\alpha-l}$. Thus we have

$$\left|f^{(|m|)} \circ g^{-1}(x)\right| \leq C_p \left|\tau - g(a)\right|^{\beta-|m|}. \tag{4.111}$$

Since $\sum jm_j = l$, and $|m| = \sum m_j \leq l$, assuming $|\tau - g(a)| < 1$, we have

$$\left|\tau - g(a)\right|^{\beta-|m|} \leq \left|\tau - g(a)\right|^{\beta-l}. \tag{4.112}$$

Hence, we have a bound for the first term in (4.110), hence our overall estimate for the first term in (4.109) is

$$\left|(f \circ g^{-1})^{(l)}\right| \leq \left|\tau - g(a)\right|^{\alpha+\beta-2l}. \tag{4.113}$$

Then, we have

$$\left|(f \circ g^{-1})^{(l)}(g^{-1})^{(p-l+1)}\right| \leq \left|\tau - g(a)\right|^{2\alpha-p+\beta-l-1}. \tag{4.114}$$

Now, we check $l = 0$. Consider $\beta = 0$, using (4.105)

$$\left|(f \circ g^{-1})(g^{-1})^{(p+1)}\right| \leq (1 + |\log(|\tau - g(a)|)|) \left|\tau - g(a)\right|^{\alpha-p-1}. \tag{4.115}$$

For $\beta \in (-1,0)$ we have

$$\begin{aligned}\left|(f \circ g^{-1})\ (g^{-1})^{(p+1)}\right| & \\ &\leq |\tau - g(a)|^{\beta} |\tau - g(a)|^{\alpha-p-1} \\ &= |\tau - g(a)|^{\alpha+\beta-p-1} \end{aligned} \tag{4.116}$$

and we're done. □

4.4.3. Implementation for Integrals With Stationary Point

In this section, we give details for the implementation. For $\xi \in [a,b]$ be stationary point, and write

$$I_k^{[a,b]}(f,g) = \int_{g(a)}^{g(\xi)} F(\tau)\exp(ik\tau)d\tau + \int_{g(\xi)}^{g(b)} F(\tau)\exp(ik\tau)d\tau. \tag{4.117}$$

As we mentioned, F has singularity points at the stationary points of g. With a linear change of variables, we can transform these integrals to integrals on the interval $[0,1]$ where 0 is the singularity point of F (hence the stationary point of g). After that, we can implement the composite FCC method we developed for integrals with a singularity. For example, considering the second integral, introduce a new change of variable

$$\tau = g(\xi) + c\widehat{x} \qquad \text{where } c = g(b) - g(\xi) \tag{4.118}$$

where $c = g(b) - g(\xi)$ and $\widehat{x} \in [0,1]$. This entails

$$I_k^{[a,b]}(f,g) = I_{kc}^{[0,1]}(\widehat{F}) \tag{4.119}$$

with

$$\widehat{F}(\widehat{x}) = c\exp(ikg(\xi))F(g(\xi) + c\widehat{x}) \tag{4.120}$$

so that, by Theorem 4.15, we have $|\widehat{F}(\widehat{x})^{(p)}(\widehat{x})^{p+1-\alpha}| \leq C_p$ where C_p is constant. We can see that $\widehat{F} \in C_\beta[0,1]$ with $\beta = \alpha - 1 = -n/(n+1) \in (-1, 0)$.

In the composite FCC method, we used graded mesh. For large q and M values our first subintervals become too small which may lead to rounding errors computing $g(\xi)+c\widehat{x}$ while $g(\xi) \gg c\widehat{x}$. This causes polluted and troublesome evaluation of $F(g(\xi)+c\widehat{x})$. In the implementation of the algorithm, we frequently encounter the problem of accurate evaluation of $x := g^{-1}(g(\xi) + \epsilon)$ when ε is small (and for our case, $\varepsilon = c\widehat{x}$).

Observe that $g(x) - g(\xi) = \varepsilon$ and from Lemma 4.14

$$R_\xi(x) = g(x) - g(\xi) \quad \text{and} \quad R_\xi(x) = (x - \xi)^{n+1}T_\xi(x). \tag{4.121}$$

with $T_\xi(\xi)$. Therefore

$$\varepsilon = (x - \xi)^{n+1}T_\xi(x), \tag{4.122}$$

So that, for $\alpha = 1/(n+1)$,

$$\varepsilon^\alpha = (x - \xi)(T_\xi(x))^\alpha. \tag{4.123}$$

Therefore x is the solution to the nonlinear parameter-dependent problem

$$G(x, \varepsilon) = (T_\xi(x))^\alpha (x - \xi) - \varepsilon^\alpha = 0. \tag{4.124}$$

Since $G(\xi, 0) = 0 \neq G_x(\xi, 0)$, by the implicit function theorem near $\varepsilon = 0$, there exists unique x that is smooth function of ε^α and there exists a constant C_1 so that

$|x-\xi| \leq C_1 \varepsilon^\alpha$ for small enough ε. Furthermore, x is also a solution to the fixed point problem

$$x = \xi + \left(\frac{\varepsilon}{T_\xi(x)}\right)^\alpha = H(x). \tag{4.125}$$

Since $T_\xi(\xi) > 0$ and T_ξ is smooth in a neighborhood of ξ, it is clear that H is Lipschitz in a ball centred on ξ and its Lipschitz constant is $C_2\epsilon^\alpha$ for some constant C_2. So for small enough ϵ, x is the unique fixed point of H and fixed point iteration converges. Thus, for small ε we can approximate x with $\tilde{x} := H(\xi)$ and our error is

$$|\tilde{x} - x| = |H(\xi) - H(x)| \leq C_2 \epsilon^\alpha |\xi - x| = \mathcal{O}\left(\epsilon^{2\alpha}\right). \tag{4.126}$$

For the case where $g(\xi) > g(a)$ the approximations above takes the form

$$x = \xi - \left(\frac{\varepsilon}{T_\xi(x)}\right)^\alpha = H(x). \tag{4.127}$$

5. METHOD FOR LOGARITHMIC SINGULARITIES

In this chapter, we are concerned with the approximation of integrals of the form

$$I_k^\alpha(f) = \int_{-1}^{1} f(x) \log\left((x-\alpha)^2\right) \exp(ikx)dx \tag{5.1}$$

where $\alpha \in [-1, 1]$ and $k \in \mathbb{R}$.

The approach of the FCC quadrature for the integral (5.1) is similar as we mentioned in Chapter 2. In the case of logarithmic singularities, we use the approximation

$$I_{k,N}^\alpha(f) = \int_{-1}^{1} P_N f(x) \log\left((x-\alpha)^2\right) \exp(ikx)dx \approx I_k^\alpha(f) \tag{5.2}$$

where P_N is the N^{th} degree polynomial that interpolates f at the Chebyshev points.

The main concern in the estimation of the integral (5.1) is, once again, to find the weights,

$$\xi_n^\alpha(k) = \int_{-1}^{1} T_n(x) \log\left((x-\alpha)^2\right) \exp(ikx)dx. \tag{5.3}$$

Observe that, for the nonoscillatory case, the weights are

$$\xi_n^\alpha = \xi_n^\alpha(0) = \int_{-1}^{1} T_n(x) \log\left((x-\alpha)^2\right) dx. \tag{5.4}$$

The blessing of the three-term recurrence relation for the computation of the weights in Section 3.1 is again applicable here, weights ξ_n^α (nonoscillatory case) also obey the three-term recurrence relation, which opens ways to fast and accurate computation. For the oscillatory case we will derive a two-step algorithm, the first algorithm ($n < k$ case) will use the three-term recurrence, and the second algorithm ($n \geq k$ case) will deal with a matrix equation.

5.1. Error Estimation

We write the difference as

$$I_{k,N}^{\alpha} - I_k^{\alpha}(f) = \int_{-1}^{1} E_N(x) \log\left((x-\alpha)^2\right) \exp(ikx) dx \tag{5.5}$$

where

$$E_N = \mathcal{Q}_N f - f. \tag{5.6}$$

The convergence estimates for the trigonometric interpolant in Sobolev spaces can be straightforwardly adapted to prove that, as established in Theorem 8.3.1 in [26],

$$\left\|(E_N)_c\right\|_{H^s} \leq C_{s,r_0} N^{s-r} \left\|f_c\right\|_{H^r} \tag{5.7}$$

where $r \geq s \geq 0$ with $r \geq r_0 > 1/2$ and C_{s,r_0} independent of f, N and r.

Set $w(x) = (1-x^2)^{1/2}$ and define for $\beta \in \{-1, 1\}$

$$\|f\|_{\beta,w} = \left\|f w^{\beta/2}\right\|_{L_2(-1,1)} = \left[\int_{-1}^{1} |f(x)|^2 \left(1-x^2\right)^{\beta/2} dx\right]^{1/2}. \tag{5.8}$$

Observe that, by Hölder's inequality

$$\begin{aligned} \left|\int_{-1}^{1} f(x) g(x) dx\right| &= \left|\int_{-1}^{1} f(x) \left(1-x^2\right)^{1/4} g(x) \left(1-x^2\right)^{-1/4} dx\right| \\ &\leq \|f\|_{-1,w} \|g\|_{1,w} \end{aligned} \tag{5.9}$$

and also we have the bound

$$\begin{aligned}\|g\|_{1,w} &= \left|\int_{-1}^{1} g(x)\left(1-x^2\right)^{1/2} dx\right|^{1/2} \\ &\le \|g\|_{L_\infty(-1,1)} \left|\int_{-1}^{1} \left(1-x^2\right)^{1/2} dx\right|^{1/2} \\ &= \sqrt{\pi}\|g\|_{L_\infty(-1,1)}. \end{aligned} \tag{5.10}$$

From the relations

$$\|f\|_{-1,w} = \|f_c\|_{L_2(0,\pi)} \quad \text{and} \quad \|f'\|_{1,w} = \left\|(f_c)'\right\|_{L_2(0,\pi)}, \tag{5.11}$$

estimates (5.7), and the Sobolev embedding theorem [24, Lemma 5.3.3], we can easily obtain the estimate

$$\|E_N\|_{-1,w} + N^{-1}\|E_N'\|_{1,w} + N^{-1/2-\varepsilon}\|E_N\|_{L_\infty(-1,1)} \le C_\varepsilon N^{-r}\|f_c\|_{H^r} \tag{5.12}$$

valid for any $r \ge 1$ with C_ε depending only on $\varepsilon > 0$.

In the next lemma, we consider the asymptotics of $\xi_0^\alpha(k)$.

Lemma 5.1. *For all $\alpha \in (-1,1)$ there exists $C_\alpha > 0$ so that for all $k \ge 2$*

$$|\xi_0^\alpha(k)| \le C_\alpha\left(1 + \left|\log\left(1-\alpha^2\right)\right|\right) k^{-1}. \tag{5.13}$$

Moreover, for $\alpha = \pm 1$,

$$\left|\xi_0^{\pm 1}(k)\right| \le C_1 k^{-1} \log k \tag{5.14}$$

with $C_1 > 0$ independent of $k \ge 2$.

Proof. For the proof, we borrow straightforward calculations from Subsection 5.2.3, equations (5.78) and (5.78) and consider $k \to \infty$. These entails

$$\lim_{k\to\infty} |\xi_0^{\alpha}(k)| = \lim_{k\to\infty} \left| \frac{2}{k} \left[\log\left(1-\alpha^2\right)\sin k + \sin(\alpha k)\left(\mathrm{Ci}((\alpha+1)k)\right) - \mathrm{Ci}((1-\alpha)k)\right.\right.$$
$$\left. - \cos(\alpha k)\left(\mathrm{Si}((\alpha+1)k) + \mathrm{Si}((1-\alpha)k)\right)\right]$$
$$+ \frac{2i}{k}\left[\log\left(\frac{1+\alpha}{1-\alpha}\right)\cos k + \cos(\alpha k)\left(\mathrm{Ci}((1-\alpha)k) - \mathrm{Ci}((1+\alpha)k)\right)\right.$$
$$\left.\left. - \sin(\alpha k)\left(\mathrm{Si}((1-\alpha)k) + \mathrm{Si}((1+\alpha)k)\right)\right] \right|. \tag{5.15}$$

Similarly, for $\alpha \neq \pm 1$, and for $\alpha = 0$,

$$\lim_{k\to\infty} |\xi_0^{\pm 1}(k)| = \lim_{k\to\infty} \left| \frac{2}{k}[-(\gamma - \mathrm{Ci}(2k) + \log(k/2))\sin k - \mathrm{Si}(2k)\cos k]\right.$$
$$\left. \pm \frac{2i}{k}[(\gamma - \mathrm{Ci}(2k) + \log(2k))\cos k - \mathrm{Si}(2k)\sin k] \right|. \tag{5.16}$$

□

Lemma 5.2. *Define the $e_N^{\alpha}(x)$ as:*

$$e_N^{\alpha}(x) = \frac{E_N(x) - E_N(\alpha)}{x - \alpha}. \tag{5.17}$$

Then for all $r \geq s_0 > 5/2$, there exists C_{s_0} independent of f, N and r so that

$$\|E_N'\|_{L_\infty(-1,1)} + \|e_N^{\alpha}\|_{L_\infty(-1,1)} \leq C_{s_0} N^{s_0 - r} \|f_c\|_{H^r}. \tag{5.18}$$

Moreover,

$$\|wE_N''\|_{L_\infty(-1,1)} + \left\|w\left(e_N^{\alpha}\right)'\right\|_{L_\infty(-1,1)} \leq C_{s_1} N^{s_1 - r} \|f_c\|_{H^r} \tag{5.19}$$

for $r \geq s_1 > 7/2$, with C_{s_1} independent also of f, N and r.

Proof. By Eq.(A.11),

$$p_n^\alpha(x) = \frac{T_n(x) - T_n(\alpha)}{x - \alpha} \in \mathbb{P}_{n-1} \tag{5.20}$$

is a polynomial in $\mathbb{P}_{n-1}$.

By the mean value theorem, we therefore have

$$\|p_n^\alpha\|_{L^\infty(-1,1)} \leq \|T_n'\|_{L^\infty(-1,1)} \leq n^2. \tag{5.21}$$

Using (A.11) once more, we also have

$$\left(p_n^\alpha\right)'(x) = 2\sum_{j=0}^{n-2}(j+1)^{-1}T_{j+1}'(\alpha)T_{n-1-j}'(x). \tag{5.22}$$

Combining these, we obtain

$$\begin{aligned}\left\|w\left(p_n^\alpha\right)'\right\|_{L^\infty(-1,1)} &\leq 2\sum_{j=0}^{n-2}\left|(j+1)^{-1}T_{j+1}'(\alpha)\right|\left\|wT_{n-1-j}'\right\|_{L^\infty(-1,1)} \\ &= 2\sum_{j=0}^{n-2}\left|(j+1)^{-1}\right|\left|(j+1)^2\right|(n-1-j) \\ &= 2\sum_{j=0}^{n-2}(j+1)(n-1-j) = \frac{1}{3}(n-1)n(n+1) \\ &< \frac{n^3}{3}. \end{aligned} \tag{5.23}$$

On the other hand, for all f smooth enough,

$$f = \widehat{f}_c(0) + 2\sum_{n=1}^{\infty}\widehat{f}_c(n)T_n \tag{5.24}$$

where

$$\widehat{f}_c(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}_c(\theta)\exp(in\theta)d\theta = \frac{1}{\pi}\int_{-1}^{1} f(x)T_n(x)dx \tag{5.25}$$

and, using Sobolev norm defined in (B.3),

$$\|f_c\|_{H^r}^2 = \left|\widehat{f}_c(0)\right|^2 + 2\sum_{n=1}^{\infty}\left|\widehat{f}_c(n)\right|^2 |n|^{2r}. \tag{5.26}$$

Using Fourier representation, we have

$$E_N(x) = \widehat{(E_N)_c}(0) + 2\sum_{n=1}^{\infty}\widehat{(E_N)_c}(n)T_n(x). \tag{5.27}$$

So that

$$E_N'(x) = 2\sum_{n=1}^{\infty}\widehat{(E_N)_c}(n)T_n'(x). \tag{5.28}$$

Also

$$\begin{aligned} e_N^{\alpha}(x)(x-\alpha) &= E_N(x) - E_N(\alpha) \\ &= \widehat{(E_N)_c}(0) + 2\sum_{n=1}^{\infty}\widehat{(E_N)_c}(n)T_n(x) - \widehat{(E_N)_c}(0) + 2\sum_{n=1}^{\infty}\widehat{(E_N)_c}(n)T_n(\alpha) \\ &= 2\sum_{n=1}^{\infty}\widehat{(E_N)_c}(T_n(x) - T_n(\alpha)). \end{aligned} \tag{5.29}$$

It follows that

$$e_N^{\alpha}(x) = 2\sum_{n=1}^{\infty}\widehat{(E_N)_c}\left(\frac{T_n(x)-T_n(\alpha)}{x-\alpha}\right) = \sum_{n=1}^{\infty}\widehat{(E_N)_c}(n)p_n^{\alpha}(x). \tag{5.30}$$

Considering now the first inequality, we estimate

$$
\begin{aligned}
\|e_N^\alpha\|_{L_\infty(-1,1)} + \|E_N'\|_{L_\infty(-1,1)} &= 2\left\|\sum_{n=1}^{\infty}\widehat{(E_N)_c}(n)p_n^\alpha\right\|_{L_\infty(-1,1)} + 2\left\|\sum_{n=1}^{\infty}\widehat{(E_N)_c}(n)T_n'\right\|_{L_\infty(-1,1)} \\
&\le 2\sum_{n=1}^{\infty}\left|\widehat{(E_N)_c}(n)\right|\left(\|p_n^\alpha\|_{L_\infty(-1,1)} + \|T_n'\|_{L_\infty(-1,1)}\right) \\
&\le 2\sum_{n=1}^{\infty}\left|\widehat{(E_N)_c}(n)\right|(2n^2) \\
&\le 2\sum_{n=1}^{\infty}\left|\widehat{(E_N)_c}(n)\right|(2n^{5/2+\epsilon})(n^{-1/2+\epsilon}) \\
&\le 2\left[\sum_{n=1}^{\infty}n^{-1-2\epsilon}\right]^{1/2}\left[2\sum_{n=1}^{\infty}\left|\widehat{(E_N)_c}(n)\right|^2 n^{5+2\epsilon}\right]^{1/2} \\
&= 2\left[\sum_{n=1}^{\infty}n^{-1-2\epsilon}\right]^{1/2}\|(E_N)_c\|_{H^{5/2+\epsilon}} = C_\varepsilon\,\|(E_N)_c\|_{H^{5/2+\epsilon}}
\end{aligned}
\tag{5.31}
$$

where the last identity follows from Eq. (B.2). Also, by inequality (5.7), we have

$$\|(E_N)_c\|_{H^{5/2}} \le C_{s_0}N^{5/2-r}\,\|f_c\|_{H^r} \quad \text{where } r \ge 5/2. \tag{5.32}$$

Therefore, the first inequality in the lemma is established. Differentiating E_n and e_N^α, we have

$$wE_N''(x) = 2\sum_{n=1}^{\infty}\widehat{(E_N)_c}(n)wT_n''(x) \tag{5.33}$$

and

$$w(e_N^\alpha)'(x) = \sum_{n=1}^{\infty}\widehat{(E_N)_c}(n)w(p_n^\alpha)'(x). \tag{5.34}$$

We now use these observations to estimate

$$
\begin{aligned}
\|w(e_N^\alpha)\|_{L^\infty(-1,1)} &+ \|wE_N''(x)\|_{L^\infty(-1,1)} \\
&= 2\left\|\sum_{n=1}^{\infty}\widehat{(E_N)_c}(n)w(p_n^\alpha)\right\|_{L^\infty(-1,1)} + 2\left\|\sum_{n=1}^{\infty}\widehat{(E_N)_c}(n)wT_n''\right\|_{L^\infty(-1,1)} \\
&\leq 2\sum_{n=1}^{\infty}\left|\widehat{(E_N)_c}(n)\right|\left(\|wp_n^\alpha\|_{L^\infty(-1,1)} + \|wT_n''\|_{L^\infty(-1,1)}\right) \\
&\leq 2\sum_{n=1}^{\infty}\left|\widehat{(E_N)_c}(n)\right|(2Dn^3) \quad \text{where } D \text{ is constant} \\
&\leq 2D\sum_{n=1}^{\infty}\left|\widehat{(E_N)_c}(n)\right|(2n^{7/2+\epsilon})(n^{-1/2+\epsilon}) \\
&\leq 2D\left[\sum_{n=1}^{\infty}n^{-1-2\epsilon}\right]^{1/2}\left[2\sum_{n=1}^{\infty}\left|\widehat{(E_N)_c}(n)\right|^2 n^{7+2\epsilon}\right]^{1/2} \\
&= 2D\left[\sum_{n=1}^{\infty}n^{-1-2\epsilon}\right]^{1/2}\|(E_N)_c\|_{H^{7/2+\epsilon}} = C_\varepsilon\|(E_N)_c\|_{H^{7/2+\epsilon}}
\end{aligned}
\tag{5.35}
$$

Therefore the second inequality is established. □

Lemma 5.3. *There exists $C > 0$ such that for any $\alpha \in [-1,1]$ and for all $g \in H^1(-1,1)$ with $g(\alpha) = 0$*

$$|g(x)| \leq C|x-\alpha|^{1/4}\|g'\|_{1,w}, \quad x \in [-1,1] \tag{5.36}$$

$$\int_{-1}^{1}\left|\frac{g(x)}{x-\alpha}\right|dx \leq C\|g'\|_{1,w}. \tag{5.37}$$

Proof. Set

$$C = \max_{\alpha\in[-1,1]}\left\|\frac{\arcsin(x)-\arcsin\alpha}{|x-\alpha|^{1/2}}\right\|_{L_\infty(-1,1)} < \infty. \tag{5.38}$$

Since $g(\alpha) = 0$, by the fundamental theorem of calculus and Hölder's inequality, we obtain

$$\begin{aligned}
|g(x)| = \left| \int_\alpha^x g'(s)ds \right| &\leq \left[\int_\alpha^x \frac{ds}{\sqrt{1-s^2}} \right]^{1/2} \left[\int_\alpha^x |g'(s)|^2 \left(\sqrt{1-s^2} \right) ds \right]^{1/2} \\
&\leq |\arcsin x - \arcsin \alpha|^{1/2} \left[\int_{-1}^1 |g'(s)|^2 w(s)ds \right]^{1/2} \\
&= \left| \frac{\arcsin x - \arcsin \alpha}{(x-\alpha)^{1/2}} \right|^{1/2} |x-\alpha|^{1/4} \left[\int_{-1}^1 |g'(s)|^2 w(s)ds \right] \\
&\leq C|x-\alpha|^{1/4} \|g'\|_{1,w} \, . \qquad (5.39)
\end{aligned}$$

Therefore the first inequality follows. We now use the first inequality to estimate

$$\int_{-1}^1 \left| \frac{g(x)}{x-\alpha} \right| dx \leq C \int_{-1}^1 \frac{1}{|x-\alpha|^{3/4}} dx \, \|g'\|_{1,w} \leq C' \|g'\|_{1,w} \, , \qquad (5.40)$$

and the second inequality follows. □

The next theorem concerns the convergence properties of $I_{k,N}(f)$.

Theorem 5.4. *For all $\alpha \in [-1,1]$ there exists $C_\alpha > 0$ so that for $\delta \in \{0,1\}$*

$$\left| I_k^\alpha(f) - I_{k,N}^\alpha(f) \right| \leq C_\alpha (1+k)^{-\delta} N^{\delta - r} \|f_c\|_{H^r} \qquad (5.41)$$

for all $r \geq 1$ and $k \geq 0$. Furthermore, for all $\varepsilon > 0$ there exits $C_\varepsilon > 0$ such that if $\alpha = \pm 1$ or $\alpha = 0$ and N is even it holds

$$\left| I_k^\alpha(f) - I_{k,N}^\alpha(f) \right| \leq C_\varepsilon (1+k)^{-2} \left(1 + \alpha^2 \log k \right) N^{7/2+\varepsilon-r} \|f_c\|_{H^r} \qquad (5.42)$$

for all $f \in H^r$ with $r > 7/2 + \varepsilon$.

Proof. For $\delta = 0$,

$$\begin{aligned}
\left|I_k^\alpha(f) - I_{k,N}^\alpha(f)\right| &= \left|\int_{-1}^1 E_N(x)\log\left((x-\alpha)^2\right)\exp(ikx)dx\right| \\
&\le \left\|\log(x-\alpha)^2\right\|_{1,w}\|E_N\|_{-1,w} \\
&\le CN^{-r}\|f_c\|_{H^r} \qquad (5.43)
\end{aligned}$$

where we have used (5.12). Now, assuming that $k \ge 1$, we have, for the case $\delta = 1$,

$$\begin{aligned}
&\left|I_k^\alpha(f) - I_{k,N}^\alpha(f)\right| = \left|\int_{-1}^1 E_N(x)\log\left((x-\alpha)^2\right)\exp(ikx)dx\right| \\
&= \left|\int_{-1}^1 E_N(x)\log\left((x-\alpha)^2\right)\exp(ikx)dx - E_N(\alpha)\xi_0^\alpha(k) + E_N(\alpha)\xi_0^\alpha(k)\right| \\
&= \left|\int_{-1}^1 (E_N(x)-E_N(\alpha))\log\left((x-\alpha)^2\right)\exp(ikx)dx + E_N(\alpha)\xi_0^\alpha(k)\right| \\
&\le \frac{1}{k}\Bigg[\left|(E_N(x)-E_N(\alpha))\log\left((x-\alpha)^2\right)\exp(ikx)\right|_{x=-1}^{x=1} \\
&+ \left|\int_{-1}^1 E_N'(x)\log\left((x-\alpha)^2\right)\exp(ikx)dx\right| \\
&+ 2\left|\int_{-1}^1 \frac{E_N(x)-E_N(\alpha)}{x-\alpha}\exp(ikx)dx\right| \\
&+ |E_N(\alpha)|\,|k\xi_0^\alpha(k)|\Bigg] \\
&:= \frac{1}{k}\left[R_N^{(1)}(k) + R_N^{(2)}(k) + R_N^{(3)}(k) + R_N^{(4)}(k)\right]. \qquad (5.44)
\end{aligned}$$

Choosing $g = E_N - E_N(\alpha)$ and $h_\alpha(x) = |x-\alpha|^{1/4}\log\left((x-\alpha)^2\right)$ in Lemma 5.3, we have the estimate

$$\begin{aligned}
R_N^{(1)}(k) &= \left|(E_N(x)-E_N(\alpha))\log\left((x-\alpha)^2\right)\exp(ikx)\right|_{x=-1}^{x=1} \\
&= \left|\left(\frac{E_N(x)-E_N(\alpha)}{|x-\alpha|^{1/4}}\right)\log\left((x-\alpha)^2\right)|x-\alpha|^{1/4}\exp(ikx)\right|_{x=-1}^{x=1} \\
&\le |C\|E_N'\|_{1,w}\log\left((x-\alpha)^2\right)|x-\alpha|^{1/4}\exp(ikx)|_{x=-1}^{x=1} \\
&\le C\|E_N'\|_{1,w}\left(|h_\alpha(-1)| + |h_\alpha(1)|\right) \qquad (5.45)
\end{aligned}$$

since, $h_\alpha \in \mathcal{C}^0[-1,1]$, for $\alpha \in [-1,1]$. To bound $R_N^{(3)}$. Using the second inequality in Lemma 5.3

$$R_N^{(3)}(k) \leq 2\int_{-1}^{1} \left| \frac{E_N(x) - E_N(\alpha)}{x-\alpha} \exp(ikx) \right| dx \leq C \left\| E_N' \right\|_{1,w}. \tag{5.46}$$

Now, to bound $R_N^{(2)}$

$$\begin{aligned}
R_N^{(2)}(k) &\leq \int_{-1}^{1} \left| E_N'(x) \log\left((x-\alpha)^2\right) \exp(ikx) \right| dx \\
&\leq \int_{-1}^{1} \left| E_N'(x)(1-x^2)^{1/4} \log\left((x-\alpha)^2\right)(1-x^2)^{-1/4} \exp(ikx) \right| dx \\
&\leq \left\| \log\left(x-\alpha^2\right) \right\|_{-1,w} \left\| E_N' \right\|_{1,w}. \qquad (5.47)
\end{aligned}$$

Considering $R_N^{(4)}(k)$, we observe that since $\mathcal{Q}_N f$ and f match at the Chebyshev nodes, which includes ± 1, and therefore $E_N(\pm 1) = 0$. So, $R_N^{(4)}(k)$ vanishes for $\alpha = \pm 1$. For $\alpha \neq \pm 1$, by Lemma 5.1, we have

$$\begin{aligned}
R_N^{(4)}(\alpha) &\leq kC_\alpha \left(\frac{1 + |\log(1-\alpha^2)|}{k} \right) |E_N(\alpha)| \\
&\leq C_\alpha' \, |E_N(\alpha)| \\
&\leq C_\alpha' \left\| E_N \right\|_{L^\infty(-1,1)}. \qquad (5.48)
\end{aligned}$$

Combining these estimates, by inequality (5.12), we have

$$\begin{aligned}
&\frac{1}{k}\left[R_N^{(1)}(k) + R_N^{(2)}(k) + R_N^{(3)}(k) + R_N^{(4)}(k) \right] \\
&\quad \leq \frac{1}{k} C \left\| E_N' \right\|_{1,w} + C_\alpha' \left\| E_N \right\|_{L^\infty(-1,1)} \\
&\quad \leq \frac{\widetilde{C_\alpha}}{(1+k)} \left[N \left\| E_N \right\|_{-1,w} + \left\| E_N' \right\|_{1,w} + N^{+1/2-\varepsilon} \left\| E_N \right\|_{L_\infty(-1,1)} \right] \\
&\quad \leq \widetilde{C_\alpha} \left(\frac{1}{1+k} \right) N^{1-r} \left\| f_c \right\|_{H^r}. \qquad (5.49)
\end{aligned}$$

Hence we established the inequality (5.41) for $\delta = 1$.

Now, for the second inequality (5.42) in the lemma, we again point out that points $\{-1, 0, 1\}$ are among the Chebyshev nodes. Therefore $E_N(\alpha) = 0$ for $\alpha \in \{-1, 0, 1\}$, and it follows that

$$R_N^{(1)}(k) = R_N^{(4)}(k) = 0. \tag{5.50}$$

Hence we only need to estimate $R_N^{(2)}(k)$ and $R_N^{(3)}(k)$. Considering $R_N^{(2)}(k)$, we have

$$\begin{aligned}
R_N^{(2)}(k) &= \left| \int_{-1}^{1} (E_N'(x) - E_N'(\alpha)) \log\left((x-\alpha)^2\right) \exp(ikx)dx + E_N'(\alpha)\xi_0^\alpha(k) \right| \\
&\leq \frac{1}{k}\Bigg[\left| (E_N'(x) - E_N'(\alpha)) \log\left((x-\alpha)^2\right) \exp(ikx)|_{x=-1}^{x=1} \right| \\
&+ \left| \int_{-1}^{1} E_N''(x) \log\left((x-\alpha)^2\right) \exp(ikx)dx \right| \\
&+ 2 \left| \int_{-1}^{1} \frac{E_N'(x) - E_N'(\alpha)}{x-\alpha} \exp(ikx)dx \right| + \left(|E_N'(\alpha)|\, |k\xi_0^\alpha(k)|\right) \Bigg] \\
&:= \frac{1}{k}\left[S_N^{(1)}(k) + S_N^{(2)}(k) + S_N^{(3)}(k) + S_N^{(4)}(k) \right].
\end{aligned} \tag{5.51}$$

With the same argument we used for $R_N^{(2)}$ and $R_N^{(3)}$ as in inequalities (5.47) and (5.46), and using inequality (5.9) we obtain

$$S_N^{(2)}(k) + S_N^{(3)}(k) \leq C \left\| E_N'' \right\|_{1,w} \leq C\sqrt{\pi} \left\| wE_N'' \right\|_{L^\infty(-1,1)} \tag{5.52}$$

so that Lemma 5.2 entails

$$C\sqrt{\pi} \left\| wE_N'' \right\|_{L^\infty(-1,1)} \leq C_\varepsilon N^{7/2+\varepsilon-r} \left\| f_c \right\|_{H^r}. \tag{5.53}$$

Estimation of $S_N^{(4)}(k)$ is similar to that of $R_N^{(4)}(k)$, and Lemma 5.1 gives

$$\begin{aligned} S_N^{(4)}(k) &\leq kC_1\Big|\frac{\log(k)}{k}\Big|\,|E_N'(\alpha)| \\ &= C_1'|\log(k)|\,|E_N'(\alpha)| \\ &\leq C_1'|\log(k)|\,\|E_N'\|_{L^\infty(-1,1)}\,. \end{aligned} \tag{5.54}$$

As for $S_N^{(1)}(k)$, clearly it vanishes for $\alpha = 0$. For $\alpha = 1$, we have

$$S_N^{(1)}(k) \leq 4\log 2\,\|E_N'\|_{L^\infty(-1,1)} + \lim_{x\to 1}\left|(E_N'(x) - E_N'(1))\log(x-1)^2\right|, \tag{5.55}$$

and Lemma 5.3 implies

$$\begin{aligned} &\left|\,(E_N'(x - E_N'(1))\log(x-1)^2\right| \\ &\qquad\qquad \leq \|E_N''w\|_{L^\infty(-1,1)}\left|(1-x)^{1/4}\log\left((x-1)^2\right)\right| \to 0 \end{aligned} \tag{5.56}$$

as $x \to 1$. Therefore, by the first inequality in Lemma 5.2,

$$S_N^{(1)}(k) \leq 4\log 2\,\|E_N'\|_{L_\infty(-1,1)} \leq CN^{7/2-r}\,\|f_c\|_{H^r}\,. \tag{5.57}$$

The case $\alpha = -1$ is treated similarly to $\alpha = 1$ case, and has the same bound.

Combining these estimates for $\alpha = \pm 1$, or for $\alpha = 0$, and using that N is even, we find

$$R_N^{(2)}(k) \leq C_\varepsilon k^{-1}N^{7/2+\varepsilon-r}\left(1 + \alpha^2\log k\right)\|f_c\|_{H^r_\#} \tag{5.58}$$

for $\varepsilon > 0, r \geq 7/2 + \varepsilon$ with C_ε independent of N, k and f. Similarly, one can show

$$R_N^{(3)}(k) \leq \frac{1}{k}\left[|e_N^\alpha(-1)| + |e_N^\alpha(1)| + C\left\|(e_N^\alpha)'\,w\right\|_{L_\infty(-1,1)}\right] \leq C_\varepsilon k^{-1}N^{7/2+\varepsilon-r}\,\|f_c\|_{H^r} \tag{5.59}$$

for all $r \geq 7/2+\varepsilon$, and C_ε depending only on $\varepsilon > 0$.

Using the inequalities we established for $R_N^{(3)}(k)$ and $R_N^{(2)}(k)$ in the last inequality in (5.44) delivers (5.42). □

5.2. Computation of The Weights

As mentioned previously, in order to apply the FCC rule with high accuracy, it is crucial to compute weights accurately, . Also, depending on N, we would have to calculate weights many times, hence we need to obtain a fast algorithm. In this section we introduce a bundle of algorithms consisting of three parts.

We introduce the of auxiliary function η_n^α as

$$\eta_n^\alpha(k) = \int_{-1}^{1} \log\left((x-\alpha)^2\right) U_n(x) \exp(ikx) dx, \tag{5.60}$$

for $n = -1$, and observe that Eq. (A.4) implies

$$\eta_{-1}^\alpha(k) = 0 \tag{5.61}$$

Now, by recursive properties of the Chebyshev polynomials, we have

$$\xi_0^\alpha(k) = \eta_0^\alpha(k), \quad \xi_n^\alpha(k) = \frac{1}{2}\left(\eta_n^\alpha(k) - \eta_{n-2}^\alpha(k)\right), \quad n \in \mathbb{N}. \tag{5.62}$$

Observe that, by equations (A.12), (A.13), (A.14) and (A.15), there exists $C > 0$ such that, for any $\alpha \in [-1,1]$ and $n \in \mathbb{N}$, we have the following

$$|\xi_n^\alpha(k)| \leq \int_{-1}^{1} \left|\log\left((x-\alpha)^2\right)\right| dx \leq C \tag{5.63}$$

$$\begin{aligned}|\eta_n^\alpha(k)| &\leq \left[\int_{-1}^{1} |U_n(x)|^2 \sqrt{1-x^2}dx\right]^{1/2} \left[\int_{-1}^{1} \left(\log\left((x-\alpha)^2\right)\right)^2 \frac{dx}{\sqrt{1-x^2}}\right]^{1/2} \\ &\leq C. \end{aligned} \tag{5.64}$$

Therefore our weights and auxiliary functions are bounded, independent of n, α and k.

5.2.1. The Nonoscillatory Case

First, we study the nonoscillatory case, $k = 0$, which produces the coefficients that help us to calculate the coefficients in the oscillatory case. We denote $\xi_n^\alpha(0)$ and $\eta_n^\alpha(0)$ as ξ_n^α and η_n^α for convenience.

Assume that $\alpha \neq \pm 1$. By the recurrence relation for Chebyshev polynomials, for $n \geq 1$ we have

$$\begin{aligned}\eta_n^\alpha &= \int_{-1}^{1} U_n(x) \log\left((x-\alpha)^2\right) dx \\ &= \int_{-1}^{1} 2xU_{n-1}(x) \log\left((x-\alpha)^2\right) dx - \int_{-1}^{1} U_{n-2}(x) \log\left((x-\alpha)^2\right) dx \\ &= \int_{-1}^{1} 2(x-a)U_{n-1}(x) \log\left((x-\alpha)^2\right) dx - \int_{-1}^{1} U_{n-2}(x) \log\left((x-\alpha)^2\right) dx \\ &+ 2\int_{-1}^{1} 2aU_{n-1}(x) \log\left((x-\alpha)^2\right) dx \\ &= \frac{2}{n}\int_{-1}^{1} (x-\alpha)T_n'(x) \log\left((x-\alpha)^2\right) dx + 2\alpha\eta_{n-1}^\alpha - \eta_{n-2}^\alpha. \end{aligned} \tag{5.65}$$

Applying integration by parts to the first integral, we observe that

$$\begin{aligned}
&\int_{-1}^{1}(x-\alpha)T_n'(x)\log\left((x-\alpha)^2\right)dx \\
&\qquad = (x-\alpha)T_n(x)\log\left((x-\alpha)^2\right)\Big|_{x=-1}^{x=1} \\
&\qquad - \int_{-1}^{1}T_n(x)\log\left((x-\alpha)^2\right)dx - 2\int_{-1}^{1}T_n(x)dx \\
&\qquad = (1-\alpha)\log\left((1-\alpha)^2\right) + (-1)^n(1+\alpha)\log\left((1+\alpha)^2\right) \\
&\qquad - \frac{1}{2}\left[\eta_n^\alpha - \eta_{n-2}^\alpha\right] + \begin{cases} \frac{4}{n^2-1} & \text{if } n \text{ is even,} \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{5.66}$$

where we used the relation in (A.6). So, inserting this into the first integral above, we have the recursive relation

$$\eta_n^\alpha = \frac{2\alpha n}{n+1}\eta_{n-1}^\alpha - \frac{n-1}{n+1}\eta_{n-2}^\alpha + \gamma_n^\alpha, \quad n \in \mathbb{N} \tag{5.67}$$

where γ_n^α is defined as,

$$\gamma_n^\alpha = \frac{4}{n+1}\begin{cases} (1-\alpha)\log(1-\alpha) + (1+\alpha)\log(1+\alpha) + \frac{2}{n^2-1}, & \text{for even, } n \\ (1-\alpha)\log(1-\alpha) - (1+\alpha)\log(1+\alpha), & \text{for odd. } n \end{cases} \tag{5.68}$$

Taking limits as $\alpha \to \pm 1$, we also have

$$\gamma_n^{\pm 1} = \frac{8}{n+1}\begin{cases} \log 2 + \frac{1}{n^2-1}, & \text{for even } n \\ \mp \log 2, & \text{for odd.} n \end{cases} \tag{5.69}$$

Moreover, straightforward calculations show

$$\eta_0^\alpha = \begin{cases} -(\alpha-1)\log\left((\alpha-1)^2\right) + (\alpha+1)\log\left((\alpha+1)^2\right) - 4, & \text{if } \alpha \neq \pm 1 \\ 4\log 2 - 4, & \text{if } \alpha = \pm 1. \end{cases} \tag{5.70}$$

With these recursive relations and identities, we introduce Algorithm I.

5.2.2. Algorithm I: Computation of ξ_n^α for $n = 0, 1, \ldots, N$

In this section we present the first algorithm.

Algorithm I: Computation of ξ_n^α for $n = 0, 1, \ldots, N$

Set: Let $\eta_{-1}^\alpha = 0$

Result: Nonoscillating weights, ξ_n^α defined in Eq. (5.3)

Initial: Compute η_n^α defined in Eq. (5.60), beginning with η_0^α defined in Eq. (5.70)

for $n = 1$ **to** N **do**

  $\eta_n^\alpha = \frac{2\alpha n}{n+1}\eta_{n-1}^\alpha - \frac{n-1}{n+1}\eta_{n-2}^\alpha + \gamma_n^\alpha$ `/*` γ_n^α `defined as Eq.` (5.68) `*/`

end

Set: Let $\eta_0^\alpha = \xi_0^\alpha$

for $n = 1$ **to** N **do**

  $\xi_n^\alpha = \frac{1}{2}\left[\eta_n^\alpha - \eta_{n-2}^\alpha\right]$ `/* using Eq.` (5.62) `*/`

end

Algorithm 3: Algorithm for nonoscillatory logarithmic weights

5.2.3. The Oscillatory Case

here, we return to the case $k > 0$. By the recursive definition of weights, we have

$$\begin{aligned} \frac{1}{2}\left(\eta_n^\alpha(k) - \eta_{n-2}^\alpha(k)\right) &= \xi_n^\alpha(k) \\ &= \int_{-1}^{1} (T_n(x) - T_n(\alpha)) \log\left((x-\alpha)^2\right) \exp(ikx)dx \\ &+ T_n(\alpha) \int_{-1}^{1} \log\left((x-\alpha)^2\right) \exp(ikx)dx. \end{aligned} \tag{5.71}$$

Assume now that $\alpha \neq \pm 1$. Integrating by parts we derive

$$\begin{aligned}
\int_{-1}^{1} (T_n(x) & -T_n(\alpha)) \log\left((x-\alpha)^2\right) \exp(ikx)dx \\
&= \frac{1}{ik}\Big[(T_n(x)-T_n(\alpha)) \log\left((x-\alpha)^2\right) \exp(ikx)\Big|_{x=-1}^{x=1} \\
&- \int_{-1}^{1} T_n'(x) \log\left((x-\alpha)^2\right) \exp(ikx)dx - 2\int_{-1}^{1} \frac{T_n(x)-T_n(\alpha)}{x-\alpha} \exp(ikx)dx\Big] \\
&= \frac{1}{ik}\big[(1-T_n(\alpha)) \log\left((1-\alpha)^2\right) \exp(ik) \\
&+ \left((-1)^{n+1}+T_n(\alpha)\right) \log\left((1+\alpha)^2\right) \exp(-ik) \\
&- n\int_{-1}^{1} U_{n-1}(x) \log\left((x-\alpha)^2\right) \exp(ikx)dx \\
&-2\int_{-1}^{1} U_{n-1}(x) \exp(ikx)dx - 4\sum_{j=0}^{n-2} T_{n-1-j}(\alpha) \int_{-1}^{1} U_j(x) \exp(ikx)dx\Bigg].
\end{aligned} \tag{5.72}$$

Inserting Eq. (5.72) into Eq. (5.71) and using Eq. (A.2), we obtain

$$\eta_n^\alpha(k) + \frac{2n}{ik}\eta_{n-1}^\alpha(k) - \eta_{n-2}^\alpha(k) = \gamma_n^\alpha(k) \tag{5.73}$$

with

$$\begin{aligned}
\gamma_n^\alpha(k) = \frac{2}{ik} & \big[(1-T_n(\alpha)) \log\left((1-\alpha)^2\right) \exp(ik) \\
& + \left((-1)^{n+1}+T_n(\alpha)\right) \log\left((1+\alpha)^2\right) \exp(-ik)\big] \\
& - \frac{4}{ik}\left[2\sum_{j=0}^{n-2} T_{n-1-j}(\alpha)\rho_j(k) + \rho_{n-1}(k)\right] + 2T_n(\alpha)\eta_0^\alpha(k).
\end{aligned} \tag{5.74}$$

where

$$\rho_j(k) = \int_{-1}^{1} U_j(x) \exp(ikx)dx, \quad j = 0, \ldots, n-1. \tag{5.75}$$

We have computed $(\rho_j(k))_{j=0}^N$ with $\mathcal{O}(N)$ operations previously in Section 3.4 with the Algorithms 1 and 2.

For $\alpha = \pm 1$ we have,

$$\begin{aligned}
\gamma_n^{\pm 1}(k) =& \frac{4}{ik}\left[\begin{array}{ll} \log(4)\exp(\mp ik), & \text{if } n \text{ is odd} \\ 0, & \text{otherwise} \end{array} \right. \\
& \left. -2\sum_{j=0}^{n-2}(\pm 1)^{n-j+1}\int_{-1}^{1} U_j(x)\exp(ikx)dx - \int_{-1}^{1} U_{n-1}(x)\exp(ikx)dx \right] \\
& + 2T_n(\pm 1)\eta_0^{\pm 1}(k).
\end{aligned} \tag{5.76}$$

Now we have defined the recursive relation to calculate the weights in the oscillatory case. For setting up the algorithm we need to calculate $\eta_0^\alpha(k)$. To this end, we recall the sine and cosine integrals

$$\mathrm{Si}(t) = \int_0^t \frac{\sin x}{x}dx, \quad \mathrm{Ci}(t) = \gamma + \log(t) + \int_0^t \frac{\cos x - 1}{x}dx \tag{5.77}$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. It follows that

$$\begin{aligned}
\eta_0^\alpha(k) = \xi_0^\alpha(k) =& \frac{2}{k}\left[\log\left(1-\alpha^2\right)\sin k + \sin(\alpha k)(\mathrm{Ci}((\alpha+1)k)) - \mathrm{Ci}((1-\alpha)k)\right) \\
& - \cos(\alpha k)(\mathrm{Si}((\alpha+1)k) + \mathrm{Si}((1-\alpha)k))] \\
& + \frac{2i}{k}\left[\log\left(\frac{1+\alpha}{1-\alpha}\right)\cos k + \cos(\alpha k)(\mathrm{Ci}((1-\alpha)k) - \mathrm{Ci}((1+\alpha)k))\right. \\
& - \sin(\alpha k)(\mathrm{Si}((1-\alpha)k) + \mathrm{Si}((1+\alpha)k)]
\end{aligned} \tag{5.78}$$

for $\alpha \neq \pm 1$, and

$$\begin{aligned}
\eta_0^{\pm 1}(k) = \xi_0^{\pm 1}(k) =& \frac{2}{k}[-(\gamma - \mathrm{Ci}(2k) + \log(k/2))\sin k - \mathrm{Si}(2k)\cos k] \\
& \pm \frac{2i}{k}[(\gamma - \mathrm{Ci}(2k) + \log(2k))\cos k - \mathrm{Si}(2k)\sin k].
\end{aligned} \tag{5.79}$$

5.2.4. Algorithm II: Computation of $\xi_n^\alpha(k)$ for $n = 0, \ldots, N$ with $N \leq \lfloor k \rfloor - 1$

In this section we present the second algorithm for first phase of oscillatory case.

Algorithm II: Computation of $\xi_n^\alpha(k)$ for $n = 0, \ldots, N$ with $N \leq \lfloor k \rfloor - 1$

Set: Let $\eta_{-1}^\alpha(k) = 0$ and evaluate $\eta_k^\alpha(k)$ using Eq. (5.78) and Eq. (5.79)

Result: Weights ξ_n^α defined in Eq. (5.60)

Initial: Compute γ_n^α defined in Eq. (5.74) and Eq. (5.76)

for $n = 1$ **to** N **do**

  $\eta_n^\alpha(k) = \gamma_n^\alpha(k) - \frac{2n}{ik}\eta_{n-1}^\alpha(k) + \eta_{n-2}^\alpha(k)$

end

Set: Let $\eta_0^\alpha = \xi_0^\alpha$

for $n = 1$ **to** N **do**

  $\xi_n^\alpha(k) = \frac{1}{2}\left[\eta_n^\alpha(k) - \eta_{n-2}^\alpha(k)\right]$

end

Algorithm 4: Algorithm for oscillatory logarithmic weights

We restricted Algorithm II to case $N \leq \lfloor k \rfloor - 1$ because the forward recurrence relation is not stable for $N \geq \lfloor k \rfloor - 1$. For the latter case, we introduce Algorithm III. Before starting Algorithm III we will study the tridiagonal system.

5.2.5. Tridiagonal System $\xi_n^\alpha(k)$ for $n = \lfloor k \rfloor, \ldots, N$

Assume that $N > \lfloor k \rfloor - 1$. From Algorithm II we obtain $\eta_0^\alpha(k), \ldots, \eta_{\lfloor k \rfloor - 1}^\alpha$. In the computation of the weights $\eta_{\lfloor k \rfloor}^\alpha(k), \ldots, \eta_N^\alpha$ we use the recurrence relation in Algorithm II but, for stable a computation, we introduce the tridiagonal system [27]

$$A_N^\alpha(k)\boldsymbol{\eta} = \mathbf{b}_N^\alpha(k) \tag{5.80}$$

with the unique solution

$$\boldsymbol{\eta} = \begin{bmatrix} \eta^\alpha_{\lfloor k \rfloor}(k) & \eta^\alpha_{\lfloor k \rfloor+1}(k) & \cdots & \eta^\alpha_{N-1}(k) \end{bmatrix}^\top \tag{5.81}$$

where

$$A^\alpha_N(k) = \begin{bmatrix} \frac{2(\lfloor k \rfloor+1)}{ik} & 1 & & \\ -1 & \frac{2(\lfloor k \rfloor+2)}{ik} & 1 & \\ & \ddots & \ddots & \\ & & -1 & \frac{N}{ik} \end{bmatrix}, \quad \mathbf{b}^\alpha_N(k) = \begin{bmatrix} \eta^\alpha_{\lfloor k \rfloor-1} + \gamma^\alpha_{\lfloor k \rfloor+1}(k) \\ \gamma^\alpha_{\lfloor k \rfloor+2}(k) \\ \vdots \\ \gamma^\alpha_N(k) - \eta^\alpha_N(k) \end{bmatrix} \tag{5.82}$$

This is the same approach that we used in (3.80). In $\mathbf{b}^\alpha_N(k)$, the γ^α_t values can be computed straightforwardly. Also, $\eta^\alpha_{\lfloor k \rfloor-1}$ is computed in Algorithm II, hence the crucial part is approximating $\eta^\alpha_N(k)$. To calculate $\eta^\alpha_N(k)$, we use the Jacobi-Anger expansion (see Section 2.2 in [22]),

$$\exp(ikx) = J_0(k) + 2\sum_{n=1}^{\infty} i^n J_n(k) T_n(x) \tag{5.83}$$

where J_n is the Bessel function of the first kind and order n. Hence, using equations (A.7), we derive

$$\begin{aligned} \eta^\alpha_N(k) &= \int_{-1}^{1} U_N(x) \exp(ikx) \log\left((x-\alpha)^2\right) dx \\ &= J_0(k) \int_{-1}^{1} U_N(x) \log\left((x-\alpha)^2\right) dx \\ &+ 2\sum_{m=1}^{\infty} i^n J_m(k) \int_{-1}^{1} U_N(x) T_m(x) \log\left((x-\alpha)^2\right) dx \\ &= J_0(k)\eta^\alpha_N \\ &+ \sum_{m=1}^{N} i^m J_m(k) \left(\eta^\alpha_{N+m} + \eta^\alpha_{N-m}\right) + \sum_{m=N+1}^{\infty} i^m J_m(k) \left(\eta^\alpha_{N+m} - \eta^\alpha_{m-N-2}\right). \end{aligned} \tag{5.84}$$

The coefficients η_n^α can be computed as we did in the nonoscillatory case, namely as in Algorithm I. To determine how many terms are needed for a satisfactory approximation of $\eta_N^\alpha(k)$ we must check the decay rate of Bessel function $J_M(k)$. Indeed (see Eq. (9.1.10) in [21]),

$$J_M(k) \approx \frac{1}{M!}\left(\frac{k}{2}\right)^M \approx \frac{1}{\sqrt{2\pi M}}\left(\frac{ek}{2M}\right)^M \tag{5.85}$$

as $M \to \infty$, which shows that $J_M(k)$ decreases very fast as $M \to \infty$. Taking $M = M(k) \gtrapprox ek/2$ is therefore sufficient to approximate the coefficient $\eta_N^\alpha(k)$ with high precision. So we use the approximation

$$\eta_N^\alpha(k) \approx J_0(k)\eta_N^\alpha(k) + \sum_{n=1}^{N} i^n J_n(k)\left(\eta_{N+n}^\alpha - \eta_{N-n}^\alpha\right) + \sum_{n=N+1}^{M(k)} i^n J_n(k)\left(\eta_{N+n}^\alpha - \eta_{n-N-2}^\alpha\right). \tag{5.86}$$

Indeed, our implementations show tat any choice $M(k) = 25 + \lceil ek/2 \rceil$ yields a sufficient approximation.

5.2.6. Algorithm III: Computation of $\xi_n^\alpha(k)$ for $n = \lfloor k \rfloor, \dots, N$

In this section we present the third algorithm for second phase of oscillatory case. The third algorithm utilizes the same steps of previous the algorithms.

Algorithm III: Computation of $\xi_n^\alpha(k)$ for $n = \lfloor k \rfloor, \ldots, N$

Initial: Construct $\mathbf{b}_N^\alpha(k)$ using $\eta_{\lfloor k \rfloor -1}^\alpha(k)$ obtained from Algorithm II, $\gamma_n^\alpha(k)$ for $n = \lfloor k \rfloor + 1, \ldots, N$ defined in Eq. (5.74) and Eq. (5.76), and approximation of $\eta_N^\alpha(k)$ using (5.86)

Initial: Construct matrix $A_N^\alpha(k)$ using definition (5.82)

Result: Weights ξ_n^α defined as Eq. (5.60)

Solve: Solve the tridiagonal system as in Eq. (5.80)

Set: Set $\eta_{\lfloor k \rfloor -1+\ell}^\alpha(k) = \boldsymbol{\eta}_\ell, \quad \ell = 1, \ldots, N - \lfloor k \rfloor$

for $n = \lfloor k \rfloor$ **to** N **do**

$\quad \xi_n^\alpha(k) = \frac{1}{2}\left[\eta_n^\alpha(k) - \eta_{n-2}^\alpha(k)\right]$

end

Algorithm 5: Algorithm for oscillatory logarithmic weights

5.3. Numerical Stability

In this section we analyze the stability of Algorithms I, II, and III. The parameter ϵ_j represents any round-off error.

In the first theorem, we consider the stability of Algorithm I (the nonoscillatory case).

Theorem 5.5. *Let $\varepsilon_N = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N) \in \mathbb{R}^{N+1}$ with $\|\varepsilon\|_\infty \leq \varepsilon$ and define the sequence*

$$\begin{aligned} &\eta_{-1}^{\alpha,\varepsilon} = \eta_{-1}^\alpha = 0, \quad \eta_0^{\alpha,\varepsilon} = \eta_0^\alpha + \varepsilon_0 \\ &\eta_n^{\alpha,\varepsilon} = \gamma_n^\alpha + \frac{2\alpha n}{n+1}\eta_{n-1}^{\alpha,\varepsilon} - \frac{n-1}{n+1}\eta_{n-2}^{\alpha,\varepsilon} + \varepsilon_n, \quad \textit{for } n \in \mathbb{N} \end{aligned} \tag{5.87}$$

Then for all $N > 0$

$$|\eta_N^\alpha - \eta_N^{\alpha,\varepsilon}| \le \left[\frac{1}{N+1}\sum_{j=0}^{N}(j+1)\,|U_{N-j}(\alpha)|\right]\varepsilon \le \frac{1}{6}(N+2)(N+3)\varepsilon \tag{5.88}$$

Proof. As can be readily verified

$$\varepsilon_N = \eta_N^\alpha - \eta_N^{\alpha,\varepsilon} = \sum_{j=0}^{N}\delta_N^{(j)} \tag{5.89}$$

where

$$\begin{aligned} &\delta_{j-1}^{(j)} = 0, \quad \delta_j^{(j)} = \varepsilon_j \\ &\delta_n^{(j)} = \frac{2\alpha n}{n+1}\delta_{n-1}^{(j)} - \frac{n-1}{n+1}\delta_{n-2}^{(j)}, \quad n = j+1, j+2, \dots \end{aligned} \tag{5.90}$$

Any easy calculation using (A.1) shows that

$$\delta_n^{(j)} = \frac{j+1}{n+1}U_{n-j}(\alpha)\varepsilon_j. \tag{5.91}$$

Therefore the identity (A.13) implies

$$\begin{aligned} |\eta_N^\alpha - \eta_N^{\alpha,\varepsilon}| &\le \sum_{j=0}^{N}\left|\frac{j+1}{n+1}U_{n-j}(\alpha)\varepsilon_j\right| \\ &\le \frac{1}{N+1}\left[\sum_{j=0}^{N}(j+1)\,|U_{N-j}(\alpha)|\right]\varepsilon \\ &\le \frac{1}{N+1}\left[\sum_{j=0}^{N}(j+1)(N-j+1)\right]\varepsilon \\ &= \frac{1}{6}(N+2)(N+3)\varepsilon \end{aligned} \tag{5.92}$$

which gives the desired result. $\square$

Next we turn to Algorithm II which corresponds to the oscillatory case where $N \leq \lfloor k \rfloor - 1$.

Proposition 5.6. *Let $N \leq k-1$ and set $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_N) \in \mathbb{C}^{N+1}$ with $\|\varepsilon\|_\infty \leq \varepsilon$ and consider the sequence*

$$\begin{aligned} \eta_{-1}^{\alpha,\varepsilon}(k) &= \eta_{-1}^{\alpha}(k) = 0 \\ \eta_{0}^{\alpha,\varepsilon}(k) &= \eta_{0}^{\alpha}(k) + \varepsilon_0 \\ \eta_{n}^{\alpha,\varepsilon}(k) &= \gamma_n^\alpha(k) - \frac{2n}{ik}\eta_{n-1}^{\alpha,\varepsilon}(k) + \eta_{n-2}^{\alpha,\varepsilon}(k) + \varepsilon_n, \quad n = 1, 2, \ldots, N. \end{aligned} \tag{5.93}$$

Then, for all $0 \leq n < k-1$

$$|\eta_n^{\alpha,\varepsilon}(k) - \eta_n^\alpha(k)| \leq \left[1 + \frac{4}{3}\frac{(n+1)k^{1/2}}{(k^2-(n+1)^2)^{1/4}}\right]\varepsilon \tag{5.94}$$

In particular, for all $n \leq k-2$.

$$|\eta_n^{\alpha,\varepsilon}(k) - \eta_n^\alpha(k)| \leq \left[1 + \frac{2^{3/4}}{3}(n+1)^{3/4}k^{1/2}\right]\varepsilon \tag{5.95}$$

whereas for $k-2 < n \leq k-1$, i.e., for $n = \lfloor k \rfloor - 1$,

$$\left|\eta_{\lfloor k \rfloor -1}^{\alpha,\varepsilon}(k) - \eta_{\lfloor k \rfloor -1}^{\alpha}(k)\right| \leq \left[4 + 2^{7/4}k^{5/4}\right]\varepsilon \tag{5.96}$$

Proof. Inequality (5.94) follows similar to the proof of Theorem 5.5. Inequality (5.95) then follows from (5.94), and (5.96) from (5.95). $\square$

Concerning the stability of Algorithm III, we first note the following.

Proposition 5.7. *Let $N > k$ and consider the solutions of the original and perturbed systems*

$$A_N^\alpha(k)\boldsymbol{\eta} = \mathbf{b}_N^\alpha(k), \quad \left(A_N^\alpha(k) + \varDelta A_N^\alpha(k)\right)\boldsymbol{\eta}^\varepsilon = \mathbf{b}_N^\alpha(k) + \varDelta\mathbf{b}_N^\alpha(k). \tag{5.97}$$

Then, if $(k+2)\left\|\varDelta A_N^\alpha(k)\right\|_\infty < 2$, it holds

$$\|\boldsymbol{\eta}^\varepsilon - \boldsymbol{\eta}\|_\infty \le \frac{k+2}{2-(k+2)\left\|\varDelta A_N^\alpha(k)\right\|_\infty}\left[\left\|\varDelta\mathbf{b}_N^\alpha(k)\right\|_\infty + \left\|\varDelta A_N^\alpha(k)\right\|_\infty\|\boldsymbol{\eta}\|_\infty\right]. \tag{5.98}$$

Proof. In this proof we begin with a result that is established in [23] in Theorem 8.4, which gives the inequality

$$\|\boldsymbol{\eta}^\varepsilon - \boldsymbol{\eta}\|_\infty \le \frac{\left\|\left(A_N^\alpha(k)\right)^{-1}\right\|_\infty}{1-\left\|\varDelta A_N^\alpha(k)\right\|_\infty\left\|\left(A_N^\alpha(k)\right)^{-1}\right\|_\infty}\left[\left\|\varDelta\mathbf{b}_N^\alpha(k)\right\|_\infty + \left\|\varDelta A_N^\alpha(k)\right\|_\infty\|\boldsymbol{\eta}\|_\infty\right]. \tag{5.99}$$

Let I denote the identity matrix. By the definition (5.82), we see that

$$A_N^\alpha(k) = \left(I + K_N(k)\right)D_N(k) \tag{5.100}$$

where

$$D_N(k) = \frac{2}{ik}\begin{bmatrix} \lfloor k\rfloor + 1 & & & \\ & \lfloor k\rfloor + 2 & & \\ & & \ddots & \\ & & & N-1 \end{bmatrix},$$

$$K_N(k) = \begin{bmatrix} 0 & \frac{ik}{2(\lfloor k \rfloor+2)} & & & & \\ -\frac{ik}{2(\lfloor k \rfloor+1)} & 0 & \frac{ik}{2(\lfloor k \rfloor+3)} & & & \\ & -\frac{ik}{2(\lfloor k \rfloor+2)} & 0 & \frac{ik}{2(\lfloor k \rfloor+4)} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & -\frac{ik}{2(N-2)} & 0 \end{bmatrix}. \tag{5.101}$$

From (5.100), we have

$$\left(A_N^\alpha(k)\right)^{-1} = \left(D_N(k)\right)^{-1}\left(I + K_N(k)\right)^{-1}, \tag{5.102}$$

and we estimate

$$\left\|\left(D_N(k)\right)^{-1}\right\|_\infty = \frac{k}{2(\lfloor k \rfloor + 1)} < \frac{1}{2}, \tag{5.103}$$

$$\left\|K_N(k)\right\|_\infty = \frac{k}{2(\lfloor k \rfloor + 1)} + \frac{k}{2(\lfloor k \rfloor + 3)} < \frac{1}{2} + \frac{k}{2(k+2)} = \frac{k+1}{k+2}. \tag{5.104}$$

Therefore

$$\left\|\left(A_N^\alpha(k)\right)^{-1}\right\| \le \frac{\left\|\left(D_N(k)\right)^{-1}\right\|_\infty}{1 - \left\|K_N(k)\right\|_\infty} < \frac{k+2}{2}. \tag{5.105}$$

Using this estimate, we obtain

$$\begin{aligned} \left\|\boldsymbol{\eta}^\varepsilon - \boldsymbol{\eta}\right\|_\infty &\le \frac{\left\|\left(A_N^\alpha(k)\right)^{-1}\right\|_\infty}{1 - \left\|\Delta A_N^\alpha(k)\right\|_\infty \left\|\left(A_N^\alpha(k)\right)^{-1}\right\|_\infty}\left[\left\|\Delta \mathbf{b}_N^\alpha(k)\right\|_\infty + \left\|\Delta A_N^\alpha(k)\right\|_\infty \left\|\boldsymbol{\eta}\right\|_\infty\right] \\ &\le \frac{((k+2)/2)}{1 - \left\|\Delta A_N^\alpha(k)\right\|_\infty ((k+2)/2)}\left[\left\|\Delta \mathbf{b}_N^\alpha(k)\right\|_\infty + \left\|\Delta A_N^\alpha(k)\right\|_\infty \left\|\boldsymbol{\eta}\right\|_\infty\right] \\ &= \frac{k+2}{2 - (k+2)\left\|\Delta A_N^\alpha(k)\right\|_\infty}\left[\left\|\Delta \mathbf{b}_N^\alpha(k)\right\|_\infty + \left\|\Delta A_N^\alpha(k)\right\|_\infty \left\|\boldsymbol{\eta}\right\|_\infty\right] \end{aligned} \tag{5.106}$$

and the proof is complete. □

In the last theorem, we combine the stability results in the oscillatory case.

Theorem 5.8. *With the notation of Proposition 5.7 and Proposition 5.6 we have the followings.*

(a) For any $\nu \in (0,1)$ there exists C_ν, depending only on ν, so that for $N < \nu k$,

$$\max_{n=0,\dots,N} |\eta_n^{\alpha,\varepsilon}(k) - \eta_n^\alpha(k)| \le C_\nu N \varepsilon \tag{5.107}$$

with C_ν independent of k and N.

(b) There exists $C > 0$ independent of k and α so that

$$\max_{n=0,\dots,\lfloor k \rfloor -1} |\eta_n^{\alpha,\varepsilon}(k) - \eta_n^\alpha(k)| \le C k^{5/4} \varepsilon \tag{5.108}$$

(c) For $N > k$, if the following conditions are, in addition, satisfied

$$\|\Delta A_N^\alpha(k)\|_\infty \le \varepsilon, \quad \|\Delta \mathbf{b}_N^\alpha(k)\|_\infty \le \varepsilon + \left|\eta_{\lfloor k \rfloor -1}^\alpha(k) - \eta_{\lfloor k \rfloor -1}^{\alpha,\varepsilon}(k)\right| \tag{5.109}$$

and $\varepsilon < (k+1)^{-1}$, then

$$\begin{aligned}\max_{j=\lfloor k \rfloor -1,\dots,N} \left|\eta_j^{\alpha,\varepsilon}(k) - \eta_j^\alpha(k)\right| &\le (k+2)\left(1 + \left|\eta_{\lfloor k \rfloor -1}^\alpha(k) - \eta_{\lfloor k \rfloor -1}^{\alpha,\varepsilon}(k)\right| + \|\boldsymbol{\eta}\|_\infty\right)\varepsilon \\ &\le C' k^{9/4} \varepsilon \end{aligned} \tag{5.110}$$

where C is independent of k and N.

Proof. Inequalities (5.107) and (5.108) follow from equations (5.95) and (5.96) in Proposition 5.6. For the last inequality, we use Theorem 5.7 observing that $\|\boldsymbol{\eta}\|_\infty$ is uniformly bounded independently of N and α as established in inequality (5.64). □

6. NUMERICAL EXPERIMENTS

In this section we present some numerical experiments to depicting the theoretical results presented in this thesis.

6.1. Some Generic Examples

6.1.1. Experiment 1

Here we consider the decay rates the error in the FCC rule for fixed $N = 24$, as $k \to \infty$, for the function

$$f_\beta(s) = \frac{(1+s)^\beta}{1+s^2}, \quad s \in [-1,1]. \tag{6.1}$$

In particular we note its dependence on the Sobolev norm on the right-hand side of the inequality in Theorem 3.1. For definiteness we choose $\beta \in \{1/4, 7/8, 3/2, 3\}$. While, for $\beta = 1/4$, there is an algebraic singularity at -1, for this experiment, we do not use the composite algorithm. The composite algorithm would yield a more accurate result.

We display the error rate

$$E_k\left(f_\beta\right) = \left| \int_{-1}^{1} f_\beta(s) \exp(iks) ds - I_{k,24}\left(f_\beta\right) \right| \tag{6.2}$$

along with the convergence rate

$$e_k(\beta) = \log_2\left(E_{k/2}\left(f_\beta\right)/E_k\left(f_\beta\right)\right) \tag{6.3}$$

in Tables 6.1 and 6.2 for $k = k_i = 100 \times 2^i, i = 0, \ldots, 9$. Observe that for noninteger $\beta > 0$, $\left(f_\beta\right)_{\mathrm{c}} \in H^{2\beta+1/2-\varepsilon}$ for all $\varepsilon > 0$, and Theorem 3.1 ensures a decay of $\mathcal{O}\left(k^{-1}\right)$ the error for $E_k\left(f_\beta\right)$, as $k \to \infty$, provided $\beta > 1/4$; and convergence of order $\mathcal{O}\left(k^{-2}\right)$

provided $\beta > 3/2$. For $\beta = 3$, f_c is actually smooth. In Table 6.2 we clearly observe $\mathcal{O}\left(k^{-2}\right)$ decay in the error for $\beta = 3$, and for $\beta = 3/2$ the convergence is close to $\mathcal{O}\left(k^{-2}\right)$ in agreement with Theorem 3.1. For $\beta = 1/4$ the observed rate of decay is approximately $\mathcal{O}\left(k^{-5/4}\right)$ which is a little better than the error analysis provides. For $\beta = 7/8$ the rate is between $\mathcal{O}\left(k^{-5/4}\right)$ and $\mathcal{O}\left(k^{-2}\right)$.

Table 6.1. Error and convergence rates of Experiment 1.

k_i	$E_{k_i}(f_{1/4})$	$e_{k_i}(1/4)$	$E_{k_i}(f_{7/8})$	$e_{k_i}(7/8)$
100	6.64×10^{-4}		3.81×10^{-6}	
200	4.12×10^{-4}	0.69	1.93×10^{-6}	0.98
400	2.03×10^{-4}	1.02	8.03×10^{-7}	1.26
800	9.30×10^{-5}	1.13	3.04×10^{-7}	1.40
1600	4.12×10^{-5}	1.17	1.08×10^{-7}	1.49
3200	1.79×10^{-5}	1.20	3.62×10^{-8}	1.58
6400	7.68×10^{-6}	1.22	1.17×10^{-8}	1.63
12800	3.27×10^{-6}	1.23	3.66×10^{-9}	1.67
25600	5.06×10^{-7}	2.70	1.23×10^{-9}	1.57
51200	3.87×10^{-7}	0.38	3.25×10^{-10}	1.93

Table 6.1. Error and convergence rates of Experiment 1. (cont.)

k_i	$E_{k_i}(f_{3/2})$	$e_{k_i}(3/2)$	$E_{k_i}(f_3)$	$e_{k_i}(3)$
100	3.42×10^{-7}		1.36×10^{-11}	
200	1.46×10^{-7}	1.22	2.58×10^{-12}	2.40
400	5.34×10^{-8}	1.45	5.79×10^{-13}	2.15
800	1.76×10^{-8}	1.60	1.40×10^{-13}	2.05
1600	5.76×10^{-9}	1.61	3.46×10^{-14}	2.01
3200	1.64×10^{-9}	1.82	8.63×10^{-15}	2.00
6400	3.95×10^{-10}	2.05	2.16×10^{-15}	2.00
12800	9.24×10^{-11}	2.10	5.39×10^{-16}	2.00
25600	2.65×10^{-11}	1.80	1.34×10^{-16}	2.00
51200	7.17×10^{-12}	1.89	3.39×10^{-17}	1.99

6.1.2. Experiment 2

Now we turn our interest to behavior of the Filon-Clenshaw-Curtis rule when f has an algebraic singularity in $(-1,1)$ where

$$f(s) = \frac{|s+0.25|^{3/2}}{1+s^2}. \tag{6.4}$$

The FCC quadrature may perform rather badly if we apply the FCC directly on $[-1,1]$. Error shows disorderly behaviour because of the singularity at $s = -0.25$. For that matter, we apply FCC quadrature on $[-1,-0.25]$ and $[-0.25,1]$ separately and set $N_{new} = (N/2)+1$. It can be seen that for large k the method is already highly accurate even for small N. The results are shown in Tables 6.2, 6.3, 6.4 and 6.5 with e.c.r. denoting the estimated convergence rate stated in Experiment 1. The exact value of the integral computed with Mathematica.

Table 6.2. Errors when FCC rule directly applied on $[1, 1]$.

N	$k = 100$	$e.c.r.$	$k = 400$	$e.c.r.$
24	2.39×10^{-5}		4.33×10^{-7}	
48	1.39×10^{-5}	0.78	5.50×10^{-7}	-0.34
96	1.13×10^{-5}	0.29	5.83×10^{-7}	-0.08
192	1.29×10^{-6}	3.14	5.50×10^{-7}	0.08
384	1.58×10^{-7}	3.03	2.35×10^{-7}	1.23
768	5.73×10^{-9}	4.79	2.63×10^{-8}	3.16

Table 6.2. Errors when FCC rule directly applied on $[1, 1]$ (cont.)

N	$k = 1600$	$e.c.r.$	$k = 6400$	$e.c.r.$
24	1.11×10^{-8}		5.35×10^{-10}	
48	1.71×10^{-8}	-0.62	3.89×10^{-10}	0.46
96	1.79×10^{-8}	-0.07	5.21×10^{-10}	-0.42
192	1.74×10^{-8}	0.04	5.35×10^{-10}	-0.04
384	1.66×10^{-8}	0.07	5.68×10^{-10}	-0.09
768	1.70×10^{-8}	-0.03	5.54×10^{-10}	0.03

Next, we divide the integral into two parts. $[-1, -0.25]$ and $[-0.25, 1]$.

Table 6.3. Errors for separately evaluated intervals $[-1, -0.25]$ and $[-0.25, 1]$ where $N_{new} = (N/2) + 1$.

N	$k = 100$	$e.c.r.$	$k = 400$	$e.c.r.$
24	2.35×10^{-6}		1.75×10^{-7}	
48	1.65×10^{-7}	3.84	7.32×10^{-8}	1.26
96	1.93×10^{-8}	3.10	8.84×10^{-9}	3.05
192	4.88×10^{-11}	8.62	5.19×10^{-10}	4.09
384	4.18×10^{-11}	0.22	7.96×10^{-11}	2.70
768	4.16×10^{-11}	0.01	4.55×10^{-11}	0.81

Table 6.3. Errors for separately evaluated intervals $[-1, -0.25]$ and $[-0.25, 1]$ where $N_{new} = (N/2) + 1$ (cont.)

N	$k = 1600$	$e.c.r.$	$k = 6400$	$e.c.r.$
24	2.69×10^{-8}		2.22×10^{-9}	
48	6.27×10^{-9}	2.10	8.96×10^{-10}	1.31
96	2.28×10^{-9}	1.46	2.10×10^{-10}	2.09
192	3.28×10^{-10}	2.80	7.05×10^{-11}	1.57
384	2.45×10^{-11}	3.74	1.09×10^{-11}	2.70
768	2.98×10^{-12}	3.04	1.18×10^{-12}	3.20

6.1.3. Experiment 3

In this experiment, we will consider the integral

$$\int_0^{2\pi} \frac{1}{\sqrt{x}} \exp(ikg(x))dx \tag{6.5}$$

where $g(x) = 2\left|\sin(\frac{3\pi/4-x}{2})\right| - \cos(3\pi/4) + \cos(x)$. The integral (6.5) has stationary point order 1 at $23\pi/12$ and singularity at 0. Also, g' has discontinuity at point $3\pi/4$. We set initial conditions as $k = 100$, $\beta = -1/2$ and $q = 12$ for the subintervals of graded mesh and we consider $N_{mesh} \in \{2, 3, 4, 5, 6\}$. For the intervals without singularity or stationary point we fix $N = 24$.

Table 6.4. Error estimates for $N_{mesh} = 4$, intervals with stationary and singularity points.

M	24	48	72
Error	5.82×10^{-6}	4.38×10^{-7}	9.57×10^{-8}

Table 6.4. Error estimates for $N_{mesh} = 4$, intervals with stationary and singularity points (cont.)

M	96	120	144
Error	5.82×10^{-6}	4.38×10^{-7}	9.57×10^{-8}

Indeed, for fixed N_{mesh} accuracy increases as M grows larger. However for $M = 72$ and $M = 120$ this increasing accuracy trend does not follow. We observe that this increase in accuracy might fluctuate as M increases.

Now, another question arises, does increasing N_{mesh} increase the accuracy. For that, we fix $M = 72$, and the other parts of the setup remain similar.

Table 6.5. Error estimates for $M = 72$, for the intervals with stationary and singularity points.

N_{mesh}	2	3	4
Error	1.20×10^{-5}	1.11×10^{-6}	9.57×10^{-8}

Table 6.5. Error estimates for $M = 72$, for the intervals with stationary and singularity points (cont.)

N_{mesh}	5	6	7
Error	2.87×10^{-7}	6.82×10^{-6}	6.35×10^{-8}

We observe that N_{mesh} may have an optimal value, for this case $N_{mesh} = 4$. For smaller M, how does increasing N_{mesh} changes the error? For that matter, we fix $M = 24$ and the other parts of the setup stay similar.

Table 6.6. Error estimates for $M = 24$, for the intervals with stationary and singularity points.

N_{mesh}	2	3	4	5
Error	2.79×10^{-4}	7.67×10^{-5}	5.83×10^{-6}	2.47×10^{-6}

Table 6.6. Error estimates for $M = 24$, for the intervals with stationary and singularity points (cont.)

N_{mesh}	6	7	8	9	10
Error	9.62×10^{-7}	7.71×10^{-7}	7.01×10^{-7}	3.07×10^{-5}	1.31×10^{-4}

Hence, we can observe that for $M = 24$ and $N_{mesh} = 8$ is the optimal solution.

Thus, for lower M values we can take greater N_{mesh} value for higher accuracy.

6.1.4. Experiment 4

In this experiment we consider the approximation of a high frequency scattering problem that investigated in Example in [4]. Let Ω be the obstacle and Γ its boundary and consider incident plane $\exp(ik\mathbf{x}.\widehat{\mathbf{d}})$, where $\widehat{\mathbf{d}}$ is incident direction, scattered by a smooth convex sound-soft obstacle with boundary Γ. The following integral must be computed for the computation of scattered field,

$$\int_{\Gamma} \frac{i}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)v(\mathbf{y})ds(\mathbf{y}) = \exp(ik\mathbf{x}\cdot\widehat{\mathbf{d}}) \quad \mathbf{x} \in \Gamma \tag{6.6}$$

We consider the function $v(\mathbf{y}) = V(\mathbf{y})\exp(ik\mathbf{y}\widehat{\mathbf{d}})$ where V is nonoscillatory (or less oscillatory than v). Drawing upon the observation that $H_0^{(1)}(kr)\exp(-ikr)$ is nonoscillatory and while $r > 0$ it is smooth. Now, introduce the parametrization $\mathbf{x} : [0, 2\pi] \to \Gamma$, we rewrite the integral for $s \in [0, 2\pi]$,

$$\int_0^{2\pi} M_k(s,t)\exp\left(ik\Psi_{[s]}(t)\right)V(t)dt = 1 \tag{6.7}$$

where, $\Psi_{[s]}(t) := |\mathbf{x}(s) - \mathbf{x}(t)| - (\mathbf{x}(s) - \mathbf{x}(t))\cdot\widehat{\mathbf{d}}$. $M_k(s,t)$ is nonoscillatory, smooth and has a logarithmic singularity at $t = s$. If we choose s such that $x(s)$ lies in the region where incident waves hit the plane, there is only one stationary point. For our experiment, let Γ be unit circle, $\mathbf{x}(s) = (\cos(s), \sin(s))$ and $\widehat{\mathbf{d}} = (1, 0)$. In this setup, we have $\Psi_{[s]}(t) = 2\left|\sin\left(\frac{s-t}{2}\right)\right| - \cos(s) + \cos(t)$. Taking $V(t) = 1$ for all t and $s = 3\pi/4$ and $N = \min\{129, M+1\}$. We obtained the following results and compared the results in [4],

Table 6.7. Results for $N_{mesh} = 6$, and $q = (N_{mesh} + 1)$ for intervals with log singularities and $q = (N_{mesh} + 1)/(1 - 1/2)$ for integrals having a stationary point.

	$k = 10$		$k = 100$		$k = 1000$	
M	Error	Ratio	Error	Ratio	Error	Ratio
12	1.7×10^{-7}		1.1×10^{-6}		3.9×10^{-8}	
24	2.3×10^{-9}	6.21	2.1×10^{-10}	12.31	4.6×10^{-10}	6.39
48	7.6×10^{-9}	-1.74	4.4×10^{-10}	-1.07	4.4×10^{-10}	0.08
96	1.1×10^{-6}	-7.13	2.8×10^{-6}	-12.66	4.7×10^{-8}	-6.76
192	1.6×10^{-3}	-10.54	7.5×10^{-5}	-4.73	6.0×10^{-8}	-0.35

Table 6.7. Results for $N_{mesh} = 6$, and $q = (N_{mesh} + 1)$ for intervals with log singularities and $q = (N_{mesh} + 1)/(1 - 1/2)$ for integrals having a stationary point (cont.)

	$k = 10000$		$k = 10000$	
M	Error	Ratio	Error	Ratio
12	3.3×10^{-8}		7.1×10^{-9}	
24	1.8×10^{-10}	7.54	2.4×10^{-10}	4.88
48	2.7×10^{-11}	2.71	2.1×10^{-12}	6.84
96	4.5×10^{-9}	-7.37	5.7×10^{-14}	5.20
192	1.1×10^{-7}	-4.66	3.3×10^{-12}	-5.83

With the same constraints and values, in [4] it is obtained,

Table 6.8. Results in [4] with same constraints as in Table 6.7.

	$k = 10$		$k = 100$		$k = 1000$	
M	Error	Ratio	Error	Ratio	Error	Ratio
12	4.5×10^{-7}		2.4×10^{-7}		1.2×10^{-7}	
24	5.0×10^{-9}	6.5	1.0×10^{-9}	7.9	1.9×10^{-9}	6.0
48	4.6×10^{-11}	6.8	5.1×10^{-12}	7.6	1.4×10^{-11}	7.1
96	2.3×10^{-13}	7.6	1.1×10^{-13}	5.5	1.3×10^{-13}	6.8
192	8.3×10^{-15}	4.8	6.4×10^{-15}	4.1	5.1×10^{-15}	4.6

Table 6.8. Results in [4] with same constraints as in Table 6.7 (cont.)

	$k = 10000$		$k = 10000$	
M	Error	Ratio	Error	Ratio
12	1.6×10^{-8}		7.9×10^{-8}	
24	1.3×10^{-9}	3.6	1.0×10^{-9}	6.2
48	8.1×10^{-12}	7.4	9.0×10^{-12}	6.9
96	1.0×10^{-13}	6.3	4.4×10^{-13}	4.4
192	1.1×10^{-15}	6.6	1.1×10^{-15}	8.6

In the paper [4] same experiment is carried out with different constraints,

Table 6.9. Results for $N_{mesh} = 4$, and $q = (N_{mesh} + 1)/(1 - 1/4)$ for intervals with log singularities and $q = (N_{mesh} + 1)/(1 - 1/2 - 1/4)$ for integrals having a stationary point.

	$k = 10$		$k = 100$		$k = 1000$	
M	Error	Ratio	Error	Ratio	Error	Ratio
12	8.5×10^{-6}		1.8×10^{-6}		6.4×10^{-7}	
24	1.2×10^{-7}	6,2	2.0×10^{-7}	3.2	$4,0 \times 10^{-8}$	4.0
48	7.3×10^{-5}	-9.3	1.9×10^{-7}	0.0	1.9×10^{-7}	-2.3
96	3.6×10^{-5}	1.1	2.2×10^{-8}	3.2	8.2×10^{-7}	-2.1
192	1.6×10^{-3}	-5.5	2.9×10^{-6}	-7.1	1.7×10^{-6}	-1.1

Table 6.9. Results for $N_{mesh} = 4$, and $q = (N_{mesh} + 1)/(1 - 1/4)$ for intervals with log singularities and $q = (N_{mesh} + 1)/(1 - 1/2 - 1/4)$ for integrals having a stationary point (cont.).

	$k = 10000$		$k = 10000$	
M	Error	Ratio	Error	Ratio
12	4.6×10^{-7}		9.4×10^{-8}	
24	1.9×10^{-8}	4.6	2.4×10^{-9}	5.3
48	1.4×10^{-8}	0.5	1.2×10^{-10}	4.3
96	2.3×10^{-8}	-0.7	6.0×10^{-10}	$-2,34$
192	2.8×10^{-11}	9.7	5.3×10^{-11}	3.5

Table 6.10. Results in [4] with same constraints as in Table 6.9.

	$k = 10$		$k = 100$		$k = 1000$	
M	Error	Ratio	Error	Ratio	Error	Ratio
12	4.7×10^{-5}		1.9×10^{-5}		2.4×10^{-6}	
24	6.0×10^{-7}	6.3	1.7×10^{-7}	6.8	1.2×10^{-7}	4.3
48	1.2×10^{-8}	5.7	1.4×10^{-8}	3.6	3.8×10^{-9}	5.0
96	3.1×10^{-10}	5.2	3.9×10^{-10}	5.1	5.3×10^{-11}	6.2
192	1.4×10^{-11}	4.5	3.3×10^{-11}	3.6	2.8×10^{-11}	0.9

Table 6.10. Results in [4] with same constraints as in Table 6.9 (cont.).

	$k = 10000$		$k = 10000$	
M	Error	Ratio	Error	Ratio
12	8.2×10^{-7}		1.8×10^{-7}	
24	1.4×10^{-8}	5.9	1.0×10^{-8}	4.1
48	2.4×10^{-10}	5.9	2.0×10^{-10}	5.7
96	7.0×10^{-12}	5.1	1.9×10^{-12}	6.7
192	3.7×10^{-12}	0.9	3.0×10^{-13}	2.7

In our implementation, we did not achieve the results that are obtained in [4] as can be seen in the tables above. As the observation of previous experiment suggested we choose smaller q for larger M.

Table 6.11. Result for $N_{mesh} = 6$ and $q = 7$ for intervals with log singularities and stationary points.

	$k = 10$		$k = 100$		$k = 1000$	
M	Error	Ratio	Error	Ratio	Error	Ratio
12	3.1×10^{-6}		1.9×10^{-6}		3.2×10^{-7}	
24	2.7×10^{-7}	3.52	8.4×10^{-8}	4.51	2.7×10^{-8}	3.56
48	2.4×10^{-8}	3.48	7.4×10^{-9}	3.50	2.3×10^{-9}	3.51
96	2.6×10^{-9}	3.20	6.1×10^{-10}	3.59	2.1×10^{-10}	3.50
192	9.4×10^{-10}	1.47	1.3×10^{-11}	5.53	1.9×10^{-11}	3.45

Table 6.11. Result for $N_{mesh} = 6$ and $q = 7$ for intervals with log singularities and stationary points (cont.).

	$k = 10000$		$k = 10000$	
M	Error	Ratio	Error	Ratio
12	8.5×10^{-8}		2.1×10^{-8}	
24	8.2×10^{-9}	3.37	2.4×10^{-9}	3.14
48	7.4×10^{-10}	3.47	2.3×10^{-10}	3.37
96	6.5×10^{-11}	3.50	2.1×10^{-11}	3.49
192	5.5×10^{-12}	3.57	1.8×10^{-12}	3.55

Table 6.12. Result for $N_{mesh} = 4$ and $q = 7$ for intervals with log singularities and stationary points.

	$k = 10$		$k = 100$		$k = 1000$	
M	Error	Ratio	Error	Ratio	Error	Ratio
12	2.0×10^{-6}		2.00×10^{-6}		2.2×10^{-7}	
24	1.7×10^{-7}	3.60	4.3×10^{-8}	5.52	1.3×10^{-8}	4.02
48	1.5×10^{-8}	3.47	4.6×10^{-9}	3.23	1.6×10^{-9}	3.45
96	1.8×10^{-9}	3.04	3.6×10^{-10}	3.67	1.3×10^{-10}	3.68
192	9.4×10^{-10}	0.97	8.8×10^{-12}	5.37	1.2×10^{-11}	3.39

Table 6.12. Result for $N_{mesh} = 4$ and $q = 7$ for intervals with log singularities and stationary points (cont.).

	$k = 10000$		$k = 10000$	
M	Error	Ratio	Error	Ratio
12	2.7×10^{-7}		1.8×10^{-7}	
24	5.2×10^{-9}	5.70	1.2×10^{-9}	7.27
48	4.8×10^{-10}	3.59	9.70×10^{-10}	3,37
96	4.0×10^{-11}	3.59	1.3×10^{-11}	2.94
192	3.3×10^{-12}	3.59	1.1×10^{-12}	3.56

6.1.5. Experiment 5

In this experiment we will consider a family of functions

$$f_\beta = \begin{cases} x^\beta & \text{for } \beta \in (-1,0) \cup (0,1), \\ \log(x) & \text{for } \beta = 0. \end{cases} \tag{6.8}$$

and approximate integrals with linear oscillator

$$\int_0^1 f_\beta(x)\exp(ikx)dx. \tag{6.9}$$

which has singularity at 0. We fix $k = 1000$ and increase M for different N_{mesh} and check the convergence rate as $M \to \infty$. We choose $q = (N_{mesh}+1)/(\beta+1)+0.1$.

Table 6.13. Numerical results for $\beta = 1/2$ and various M and $N_{mesh} = N$ obtained from our implementation.

$\beta = 1/2$ Expected ratio	$N = 4$ 5		$N = 6$ 7		$N = 8$ 9	
M	Error	Ratio	Error	Ratio	Error	Ratio
8	1.5×10^{-5}		2.2×10^{-7}		2.5×10^{-9}	
16	4.1×10^{-7}	5.21	1.8×10^{-9}	6.97	9.3×10^{-12}	8.09
32	1.2×10^{-8}	5.12	1.0×10^{-11}	7.42	1.8×10^{-14}	8.99
64	3.3×10^{-10}	5.15	8.3×10^{-14}	9.98	5.5×10^{-16}	5.05

Table 6.14. Numerical results for $\beta = 0$ and various M and $N_{mesh} = N$ obtained from our implementation.

$\beta = 0$ Expected ratio	$N = 4$ 5		$N = 6$ 7		$N = 8$ 9	
M	Error	Ratio	Error	Ratio	Error	Ratio
8	2.2×10^{-7}		1.4×10^{-8}		6.9×10^{-9}	
16	1.3×10^{-8}	4.08	4.0×10^{-10}	5.12	4.0×10^{-11}	7.44
32	9.6×10^{-11}	7.07	3.7×10^{-12}	6.74	6.2×10^{-14}	9.33
64	6.4×10^{-12}	3.91	4.4×10^{-14}	6.38	8.4×10^{-16}	6.20

Table 6.15. Numerical results for $\beta = -1/4$ and various M and $N_{mesh} = N$ obtained from our implementation.

$\beta = -1/4$ Expected ratio	$N = 4$ 5		$N = 6$ 7		$N = 8$ 9	
M	Error	Ratio	Error	Ratio	Error	Ratio
8	3.4×10^{-7}		6.5×10^{-8}		2.8×10^{-8}	
16	1.7×10^{-8}	4.37	1.3×10^{-9}	5.58	1.6×10^{-10}	7.87
32	6.8×10^{-10}	4.61	1.2×10^{-11}	6.77	6.2×10^{-14}	11.35
64	8.6×10^{-12}	6.31	5.5×10^{-14}	7.81	1.8×10^{-13}	-1.52

In [4] following result are achieved.

Table 6.16. Numerical results for various M and $N_{mesh} = N$ obtained in [4].

$\beta = 1/2$ Expected ratio	$N = 4$ 5		$N = 6$ 7		$N = 8$ 9	
M	Error	Ratio	Error	Ratio	Error	Ratio
8	4.3×10^{-6}		5.2×10^{-8}		1.7×10^{-9}	
16	9.5×10^{-8}	5.49	5.7×10^{-10}	6.50	6.6×10^{-12}	8.06
32	2.9×10^{-9}	5.03	2.0×10^{-12}	8.13	1.0×10^{-14}	9.30
64	8.1×10^{-11}	5.17	2.3×10^{-14}	6.50	1.3×10^{-16}	6.37

Table 6.17. Numerical results for $\beta = 1/2$ and various M and $N_{mesh} = N$ obtained in [4].

$\beta = 0$ Expected ratio	$N = 4$ 5		$N = 6$ 7		$N = 8$ 9	
M	Error	Ratio	Error	Ratio	Error	Ratio
8	2.7×10^{-4}		7.9×10^{-6}		1.0×10^{-6}	
16	1.0×10^{-5}	4.70	7.3×10^{-8}	6.77	2.2×10^{-9}	8.82
32	4.0×10^{-7}	4.67	7.4×10^{-10}	6.62	3.0×10^{-12}	9.53
64	1.4×10^{-8}	4.87	3.8×10^{-12}	7.59	1.9×10^{-15}	10.59

Table 6.18. Numerical results for $\beta = -1/4$ and various M and $N_{mesh} = N$ obtained in [4].

$\beta = -1/4$ Expected ratio	$N = 4$ 5		$N = 6$ 7		$N = 8$ 9	
M	Error	Ratio	Error	Ratio	Error	Ratio
8	4.5×10^{-5}		1.6×10^{-5}		6.0×10^{-6}	
16	2.6×10^{-6}	4.10	8.0×10^{-8}	7.62	2.0×10^{-8}	8.22
32	1.9×10^{-8}	7.15	9.3×10^{-10}	6.43	1.1×10^{-11}	10.91
64	1.9×10^{-9}	3.33	3.9×10^{-12}	7.88	2.9×10^{-14}	8.51

6.1.6. Experiment 6

Here we approximate the integral

$$\int_0^1 \frac{\cos(4x)}{x^2 + x + 1} \log((x - \alpha)^2) \exp(ikx) dx \tag{6.10}$$

which has logarithmic singularity. We have calculated the integral for various N, k and $\alpha \in \{0, 1\}$ values. Exact values are computed with the Filon-Clenshaw-Curtis method for $N = 1000$ value. MATLAB and Mathematica do not provide a suitable computation for the integral. We observe that even for small N values, the Filon-Clenshaw-Curtis method provides a satisfactory approximation.

Table 6.19. Absolute errors for integral with $\alpha = 0$ obtained from our implementation.

$N\backslash k$	0	10	10^2
11	1.71×10^{-3}	5.00×10^{-3}	1.85×10^{-4}
12	4.56×10^{-5}	3.28×10^{-4}	1.44×10^{-6}
23	1.65×10^{-8}	2.04×10^{-8}	3.98×10^{-9}
24	2.96×10^{-10}	8.24×10^{-9}	9.93×10^{-10}
47	2.22×10^{-16}	1.45×10^{-15}	3.06×10^{-15}
48	2.00×10^{-15}	2.49×10^{-15}	3.11×10^{-15}

Table 6.19. Absolute errors for integral with $\alpha = 0$ obtained from our implementation (cont.).

$N\backslash k$	10^3	10^4	10^5
11	1.83×10^{-5}	1.83×10^{-6}	1.83×10^{-7}
12	1.37×10^{-8}	1.37×10^{-10}	1.37×10^{-12}
23	3.80×10^{-10}	3.80×10^{-11}	3.80×10^{-12}
24	9.09×10^{-12}	9.09×10^{-14}	9.09×10^{-16}
47	1.56×10^{-14}	8.62×10^{-17}	8.71×10^{-18}
48	1.56×10^{-14}	8.72×10^{-17}	8.81×10^{-18}

Table 6.20. Relative errors for integral with $\alpha = 0$ obtained from our implementation.

$N\backslash k$	0	10	10^2
11	9.38×10^{-4}	6.92×10^{-3}	2.94×10^{-3}
12	2.50×10^{-5}	4.50×10^{-4}	2.28×10^{-5}
23	9.04×10^{-9}	2.80×10^{-8}	6.31×10^{-8}
24	1.63×10^{-10}	1.13×10^{-8}	1.58×10^{-8}
47	1.22×10^{-16}	1.99×10^{-15}	4.85×10^{-14}
48	1.10×10^{-15}	3.42×10^{-15}	4.93×10^{-14}

Table 6.20. Relative errors for integral with $\alpha = 0$ obtained from our implementation (cont.).

$N\backslash k$	10^3	10^4	10^5
11	2.91×10^{-3}	2.91×10^{-3}	2.91×10^{-3}
12	2.18×10^{-6}	2.17×10^{-7}	2.18×10^{-8}
23	6.04×10^{-8}	6.04×10^{-8}	6.04×10^{-8}
24	1.45×10^{-9}	1.45×10^{-10}	1.45×10^{-11}
47	2.48×10^{-12}	1.37×10^{-13}	1.39×10^{-13}
48	2.49×10^{-12}	1.39×10^{-13}	1.40×10^{-13}

Table 6.21. Absolute errors for integral with $\alpha = 0$ obtained in [4].

$N \backslash k$	0	10	10^2
11	1.71×10^{-3}	4.00×10^{-3}	1.75×10^{-4}
12	4.56×10^{-5}	3.28×10^{-4}	1.44×10^{-6}
23	1.65×10^{-8}	2.56×10^{-8}	4.80×10^{-9}
24	2.96×10^{-10}	8.24×10^{-9}	9.93×10^{-10}
47	6.66×10^{-16}	1.11×10^{-16}	8.97×10^{-17}
48	6.66×10^{-16}	2.73×10^{-16}	8.85×10^{-17}

Table 6.21. Absolute errors for integral with $\alpha = 0$ obtained in [4] (cont.).

$N \backslash k$	10^3	10^4	10^5
11	1.82×10^{-5}	1.83×10^{-6}	1.83×10^{-7}
12	1.37×10^{-8}	1.37×10^{-10}	1.37×10^{-12}
23	3.89×10^{-10}	3.80×10^{-11}	3.80×10^{-12}
24	9.09×10^{-12}	9.09×10^{-14}	9.08×10^{-16}
47	1.29×10^{-17}	1.08×10^{-19}	1.36×10^{-20}
48	1.26×10^{-17}	1.08×10^{-19}	2.71×10^{-20}

Table 6.22. Relative errors for integral with $\alpha = 0$ obtained in [4].

$N\backslash k$	0	10	10^2
11	9.38×10^{-4}	5.49×10^{-3}	2.78×10^{-3}
12	2.50×10^{-5}	4.50×10^{-4}	2.28×10^{-5}
23	9.04×10^{-9}	3.51×10^{-8}	7.61×10^{-8}
24	1.63×10^{-10}	1.13×10^{-8}	1.58×10^{-8}
47	3.66×10^{-16}	1.52×10^{-16}	1.42×10^{-15}
48	3.66×10^{-16}	3.75×10^{-16}	1.40×10^{-15}

Table 6.22. Relative errors for integral with $\alpha = 0$ obtained in [4] (cont.).

$N\backslash k$	10^3	10^4	10^5
11	2.89×10^{-3}	2.91×10^{-3}	2.91×10^{-3}
12	2.18×10^{-6}	2.17×10^{-7}	2.18×10^{-8}
23	6.20×10^{-8}	6.05×10^{-8}	6.04×10^{-8}
24	1.45×10^{-9}	1.45×10^{-10}	1.45×10^{-11}
47	2.05×10^{-15}	1.73×10^{-16}	2.17×10^{-16}
48	2.01×10^{-15}	1.73×10^{-16}	4.32×10^{-16}

Table 6.23. Absolute errors for integral with $\alpha = 1$ obtained from our implementation.

$N\backslash k$	0	10	10^2
11	1.81×10^{-5}	8.87×10^{-4}	3.04×10^{-5}
12	2.43×10^{-6}	7.72×10^{-5}	8.94×10^{-6}
23	4.21×10^{-11}	3.23×10^{-11}	1.50×10^{-9}
24	5.25×10^{-11}	4.91×10^{-11}	1.89×10^{-9}
47	6.25×10^{-17}	2.08×10^{-16}	1.12×10^{-15}
48	3.82×10^{-17}	2.78×10^{-17}	1.13×10^{-15}

Table 6.23. Absolute errors for integral with $\alpha = 1$ obtained from our implementation (cont.).

$N\backslash k$	10^3	10^4	10^5
11	5.04×10^{-7}	6.33×10^{-9}	7.90×10^{-11}
12	1.74×10^{-7}	1.77×10^{-9}	2.15×10^{-11}
23	5.51×10^{-12}	1.23×10^{-13}	1.42×10^{-15}
24	1.84×10^{-11}	2.81×10^{-13}	3.49×10^{-15}
47	1.01×10^{-15}	6.27×10^{-16}	7.65×10^{-17}
48	9.31×10^{-16}	6.20×10^{-16}	7.69×10^{-17}

Table 6.24. Relative errors for integral with $\alpha = 1$ obtained from our implementation.

$N\backslash k$	0	10	10^2
11	8.09×10^{-4}	3.07×10^{-3}	1.00×10^{-3}
12	1.09×10^{-4}	2.67×10^{-4}	2.94×10^{-4}
23	1.89×10^{-9}	1.12×10^{-10}	4.92×10^{-8}
24	2.35×10^{-9}	1.70×10^{-10}	6.22×10^{-8}
47	2.80×10^{-15}	7.20×10^{-16}	3.68×10^{-14}
48	1.71×10^{-15}	9.60×10^{-17}	3.73×10^{-14}

Table 6.24. Relative errors for integral with $\alpha = 1$ obtained from our implementation (cont.).

$N\backslash k$	10^3	10^4	10^5
11	1.71×10^{-4}	1.26×10^{-5}	1.27×10^{-6}
12	5.92×10^{-5}	3.53×10^{-6}	3.45×10^{-7}
23	1.87×10^{-9}	2.50×10^{-10}	2.28×10^{-11}
24	6.26×10^{-9}	5.61×10^{-10}	5.61×10^{-11}
47	3.45×10^{-13}	1.25×10^{-12}	1.23×10^{-12}
48	3.16×10^{-13}	1.24×10^{-12}	1.24×10^{-12}

Table 6.25. Absolute errors for integral with $\alpha = 1$ obtained in [4].

11	1.81×10^{-5}	8.89×10^{-4}	3.04×10^{-5}
12	2.43×10^{-6}	7.72×10^{-5}	8.94×10^{-6}
23	4.21×10^{-11}	2.60×10^{-11}	1.50×10^{-9}
24	5.25×10^{-11}	4.91×10^{-11}	1.89×10^{-9}
47	1.04×10^{-18}	7.85×10^{-17}	9.22×10^{-17}
48	7.31×10^{-17}	8.89×10^{-17}	9.17×10^{-17}

Table 6.25. Absolute errors for integral with $\alpha = 1$ obtained in [4] (cont.).

11	5.04×10^{-7}	6.33×10^{-9}	7.90×10^{-11}
12	1.74×10^{-7}	1.77×10^{-9}	2.15×10^{-11}
23	5.51×10^{-12}	1.25×10^{-13}	1.48×10^{-15}
24	1.84×10^{-11}	2.81×10^{-13}	3.55×10^{-15}
47	2.47×10^{-17}	2.09×10^{-18}	1.10×10^{-19}
48	2.17×10^{-17}	1.89×10^{-18}	1.12×10^{-19}

Table 6.26. Relative errors for integral with $\alpha = 1$ obtained in [4].

$N\backslash k$	0	10	10^2
11	8.09×10^{-4}	3.07×10^{-3}	1.00×10^{-3}
12	1.09×10^{-4}	2.67×10^{-4}	2.94×10^{-4}
23	1.89×10^{-9}	9.00×10^{-11}	4.92×10^{-8}
24	2.30×10^{-9}	1.70×10^{-10}	6.22×10^{-8}
47	4.67×10^{-17}	2.71×10^{-16}	3.03×10^{-15}
48	3.27×10^{-15}	3.07×10^{-16}	3.02×10^{-15}

Table 6.26. Relative errors for integral with $\alpha = 1$ obtained in [4] (cont.).

$N\backslash k$	10^3	10^4	10^5
11	1.71×10^{-4}	1.26×10^{-5}	1.27×10^{-6}
12	5.92×10^{-5}	3.53×10^{-6}	3.45×10^{-7}
23	1.87×10^{-9}	2.49×10^{-10}	2.39×10^{-11}
24	6.26×10^{-9}	5.61×10^{-10}	5.72×10^{-11}
47	8.38×10^{-15}	4.17×10^{-15}	1.77×10^{-15}
48	7.37×10^{-15}	3.78×10^{-15}	1.80×10^{-15}

6.2. Experiment on Scattering Problem

6.2.1. Experiment 7

Here we consider oscillatory integrals arising in high frequency scattering problems. More precisely, the integrals we consider are related to scattering by torodial surfaces of revolution, and they take on the form

$$\mathbb{I} = \left\{ I_k^{[a,b]}\left(f_1 f_4\right) - ikI_k^{[a,b]}\left(f_1 f_2\right) \right\} - \left\{ I_k^{[a,b]}\left(f_1 f_5\right) - ikI_k^{[a,b]}\left(f_1 f_3\right) \right\}, \tag{6.11}$$

where

$$
\begin{gathered}
f_1(v) = \frac{\omega(v)}{\sqrt{b^2 - v^2}}, \\
f_2(v) = \frac{a^2}{\sqrt{v^2 - a^2}v}, \quad f_3(v) = \frac{v}{\sqrt{v^2 - a^2}}, \\
f_4(v) = \frac{a^2}{\sqrt{v^2 - a^2}v^2}, \quad f_5(v) = \frac{1}{\sqrt{v^2 - a^2}},
\end{gathered} \tag{6.12}
$$

with

$$
\omega(v) = 2 \arcsin \sqrt{\frac{v^2 - a^2}{b^2 - a^2}}. \tag{6.13}
$$

For this experiment we choose $[a, b]$ as $[10^{-5}, 1]$.

Table 6.27. Error rates for $I_k^{[a,b]}(f_1, f_4)$.

	$k = 10$		$k = 100$	
	Error	Ratio	Error	Ratio
$M = 12$	8.50×10^{-10}		8.50×10^{-10}	
$M = 24$	6.78×10^{-13}	10.29	6.78×10^{-13}	10.29
$M = 48$	1.11×10^{-13}	2.60	1.12×10^{-13}	2.60
$M = 96$	1.12×10^{-13}	0.00	1.12×10^{-13}	0.00
$M = 128$	5.17×10^{-14}	1.12	5.16×10^{-14}	1.12

Table 6.27. Error rates for $I_k^{[a,b]}(f_1, f_4)$ (cont.).

	$k = 1,000$		$k = 10,000$		$k = 100,000$	
	Error	Ratio	Error	Ratio	Error	Ratio
$M = 12$	8.92×10^{-10}		8.42×10^{-10}		6.34×10^{-9}	
$M = 24$	5.15×10^{-11}	4.11	2.06×10^{-11}	5.35	8.37×10^{-11}	6.24
$M = 48$	5.08×10^{-11}	0.02	2.05×10^{-11}	0.01	9.61×10^{-11}	-0.20
$M = 96$	5.08×10^{-11}	0.00	2.05×10^{-11}	0.00	9.61×10^{-11}	0.00
$M = 128$	5.08×10^{-11}	0.00	2.05×10^{-11}	0.00	9.61×10^{-11}	0.00

Table 6.28. Error rates for $I_k^{[a,b]}(f_1, f_2)$.

	$k = 10$		$k = 100$	
	Error	Ratio	Error	Ratio
$M = 12$	1.03×10^{-12}		1.02×10^{-12}	
$M = 24$	1.15×10^{-12}	0.01	1.02×10^{-12}	0.01
$M = 48$	1.15×10^{-12}	0.00	1.02×10^{-12}	0.00
$M = 96$	1.15×10^{-12}	0.00	1.02×10^{-12}	-0.00
$M = 128$	1.15×10^{-12}	0.00	1.02×10^{-12}	0.00

Table 6.28. Error rates for $I_k^{[a,b]}(f_1, f_5)$ (cont.).

	$k = 1,000$		$k = 10,000$		$k = 100,000$	
	Error	Ratio	Error	Ratio	Error	Ratio
$M = 12$	4.23×10^{-11}		2.96×10^{-11}		4.50×10^{-11}	
$M = 24$	4.23×10^{-11}	0.00	2.96×10^{-11}	0.00	4.50×10^{-11}	0.00
$M = 48$	4.23×10^{-11}	0.00	2.96×10^{-11}	0.00	4.50×10^{-11}	0.00
$M = 96$	4.23×10^{-11}	0.00	2.96×10^{-11}	0.00	4.50×10^{-11}	0.00
$M = 128$	4.23×10^{-11}	0.00	2.96×10^{-11}	0.00	4.50×10^{-11}	0.00

Table 6.29. Error rates for $I_k^{[a,b]}(f_1, f_3)$.

	$k = 10$		$k = 100$	
	Error	Ratio	Error	Ratio
$M = 12$	7.53×10^{-6}		8.20×10^{-6}	
$M = 24$	1.09×10^{-6}	2.78	1.09×10^{-6}	2.91
$M = 48$	1.05×10^{-6}	0.05	1.05×10^{-6}	0.05
$M = 96$	1.05×10^{-6}	0.00	1.05×10^{-6}	0.00
$M = 128$	7.14×10^{-7}	0.56	7.14×10^{-7}	0.56

Table 6.29. Error rates for $I_k^{[a,b]}(f_1, f_3)$. (cont.).

	$k = 1,000$		$k = 10,000$		$k = 100,000$	
	Error	Ratio	Error	Ratio	Error	Ratio
$M = 12$	8.78×10^{-6}		7.10×10^{-6}		5.01×10^{-6}	
$M = 24$	1.09×10^{-6}	3.03	1.10×10^{-6}	2.70	1.09×10^{-6}	2.20
$M = 48$	1.05×10^{-6}	0.05	1.05×10^{-6}	0.06	1.05×10^{-6}	0.06
$M = 96$	1.05×10^{-6}	0.00	1.05×10^{-6}	0.00	1.05×10^{-6}	0.00
$M = 128$	7.14×10^{-7}	0.56	7.14×10^{-7}	0.56	7.14×10^{-7}	0.56

Table 6.30. Error rates for $I_k^{[a,b]}(f_1, f_5)$.

	$k = 10$		$k = 100$	
	Error	Ratio	Error	Ratio
$M = 12$	7.53×10^{-6}		8.21×10^{-6}	
$M = 24$	1.09×10^{-6}	2.78	1.09×10^{-6}	2.91
$M = 48$	1.05×10^{-6}	0.05	1.05×10^{-6}	0.05
$M = 96$	1.05×10^{-6}	0.00	1.05×10^{-6}	0.00
$M = 128$	7.14×10^{-7}	0.56	7.14×10^{-7}	0.56

Table 6.30. Error rates for $I_k^{[a,b]}(f_1, f_5)$. (cont.).

	$k = 1,000$		$k = 10,000$		$k = 100,000$	
	Error	Ratio	Error	Ratio	Error	Ratio
$M = 12$	8.88×10^{-6}		7.10×10^{-6}		5.01×10^{-6}	
$M = 24$	1.09×10^{-6}	3.03	1.10×10^{-6}	2.70	1.09×10^{-6}	2.20
$M = 48$	1.05×10^{-6}	0.05	1.05×10^{-6}	0.06	1.05×10^{-6}	0.06
$M = 96$	1.05×10^{-6}	0.00	1.05×10^{-6}	0.00	1.05×10^{-6}	0.00
$M = 128$	7.14×10^{-7}	0.56	7.14×10^{-7}	0.56	7.14×10^{-7}	0.56

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APPENDIX A: CHEBYSHEV POLYNOMIALS

In this section definitions and some properties of Chebyshev polynomials will be presented. Chebyshev polynomials are defines recursively as

$$\begin{aligned} &T_0(x) = 1, \\ &T_1(x) = x, \\ &T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad \text{for } n \geq 2. \end{aligned} \tag{A.1}$$

The Chebyshev polynomial of the second kind satisfies,

$$U_n(\cos\theta) = \frac{1}{n+1} T'_{n+1}(\cos\theta) \tag{A.2}$$

$$T'_{n+1}(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \tag{A.3}$$

For convenience, we take

$$U_{-1} = 0. \tag{A.4}$$

By (A.3) we have

$$\sin\theta T''_n(\cos\theta) = -n\frac{d}{d\theta}\left(\frac{\sin n\theta}{\sin\theta}\right). \tag{A.5}$$

Besides that, they also have symmetry properties as given by

$$
\begin{aligned}
T_n(-x) = (-1)^n T_n(x) &= \begin{cases} T_n(x) & \text{for } n \text{ even,} \\ -T_n(x) & \text{for } n \text{ odd,} \end{cases} \\
U_n(-x) = (-1)^n U_n(x) &= \begin{cases} U_n(x) & \text{for } n \text{ even,} \\ -U_n(x) & \text{for } n \text{ odd,} \end{cases}
\end{aligned} \tag{A.6}
$$

and T_n satisfies

$$
\int_{-1}^{1} T_n(x)dx = \begin{cases} -\frac{2}{n^2-1} & \text{if } n \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \tag{A.7}
$$

By trigonometric identities one can also show that

$$
T_n U_m = \begin{cases} \frac{1}{2}\left(U_{m+n} + U_{m-n}\right), & \text{if } m \geq n-1, \\ \frac{1}{2}\left(U_{m+n} - U_{n-m-2}\right), & \text{if } m \leq n-2. \end{cases} \tag{A.8}
$$

Furthermore, since $U_0(x) = 1$ and $T_1(x) = x$, for $n \geq 1$, we have the relations

$$
2xT_n'(x) = 2nT_1(x)U_{n-1}(x) = n\left[U_n(x) + U_{n-2}(x)\right], \tag{A.9}
$$

$$
T_n(x) = T_n(x)U_0(x) = \frac{1}{2}\left[U_n(x) - U_{n-2}(x)\right]. \tag{A.10}
$$

This entails

$$
\begin{aligned}
\frac{T_n(x) - T_n(y)}{x-y} &= 2\sum_{j=0}^{n-2} U_j(x)T_{n-1-j}(y) + U_{n-1}(x) \\
&= 2\sum_{j=0}^{n-2} U_j(y)T_{n-1-j}(x) + U_{n-1}(y).
\end{aligned} \tag{A.11}
$$

Concerning the norms of T_n and U_n, we have

$$\|T_n\|_{L^\infty(-1,1)} \le T_n(1) = 1 \tag{A.12}$$

and

$$\|U_n\|_{L^\infty(-1,1)} = \frac{1}{n+1} \left\|T'_{n+1}\right\|_{L^\infty(-1,1)} = n+1. \tag{A.13}$$

In particular, T_n is uniformly bounded on $x \in [-1,1]$, but U_n is not.

For the function $w(x) = \sqrt{1-x^2}$ used in Chapter 3, we note that

$$\left\|wT'_{n+1}\right\|_{L^\infty(-1,1)} = (n+1), \quad \left\|wT''_{n+1}\right\|_{L^\infty(-1,1)} \le C(n+1)^3. \tag{A.14}$$

Finally, we also make use of

$$\int_{-1}^{1} |U_n(x)|^2 \, w(x)dx = \int_{-1}^{1} |U_n(x)|^2 \sqrt{1-x^2}dx = \int_0^\pi \sin^2 n\theta d\theta = \frac{\pi}{2}. \tag{A.15}$$

APPENDIX B: SOBOLEV SPACES

We begin with defining the following function space.

Definition B.1. *For any $\Omega \subset \mathbb{R}^n$*

$$L^p_{loc}(\Omega) = \Big\{ f : \Omega \to \mathbb{R} \mid f \in L^p(\tilde{\Omega}) \textit{ for all } \tilde{\Omega} \subset\subset \Omega \Big\}$$

where $\tilde{\Omega} \subset\subset \Omega$ means that there exists a compact set K satisfying $\tilde{\Omega} \subset K \subset \Omega$.

Now, we define Sobolev spaces,

Definition B.2. *For $k \geq 0$ and $1 \leq p \leq \infty$, we define the Sobolev space*

$$W^{k,p}(\Omega) = \{ f \in L^p(\Omega) \mid \textit{for } |\alpha| \leq k \ , \ D^\alpha f \in L^p(\Omega) \}$$

where $D^\alpha f = \dfrac{\partial^{|\alpha|} f}{\partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_n}_{x_n}} = \dfrac{\partial^{\alpha_1}}{\partial^{\alpha_1}_{x_1}} \cdots \dfrac{\partial^{\alpha_n}}{\partial^{\alpha_n}_{x_n}} f$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is multiindex with $|\alpha| = \alpha_1 + \dots + \alpha_n$, and

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \quad \textit{for } 1 \leq p < \infty. \tag{B.1}$$

We denote $W^{k,2}$ as H^k which is a Hilbert space. Now we define H^k_{loc}, before that we need to define the following space.

Definition B.3.

$$W^{k,1}_{loc}(\Omega) = \left\{ f \in L^1_{loc} \mid \textit{all weak derivatives } f_\alpha \textit{ up to order } k \textit{ exists} \right\}$$

Now using this function space, we define the function space.

Definition B.4.

$$H^k_{loc}(\Omega) = \left\{ f \in W^{k,1}_{loc} \; | f \in H^k(\tilde{\Omega}) \textit{ for all } \tilde{\Omega} \subset\subset \Omega \textit{ and for all } |\alpha| \leq k \right\}.$$

We also define the Sobolev space of 2π-periodic functions of order r as

$$H^r_{per} = \{ \varphi \in H^r_{\mathrm{loc}}(\mathbb{R}) \mid \varphi(\cdot) = \varphi(2\pi + \cdot) \}. \tag{B.2}$$

The norms on these spaces can be characterized in terms of the Fourier coefficients of the elements in the form

$$\|\varphi\|^2_{H^r} = |\widehat{\varphi}(0)|^2 + \sum_{n \neq 0} |n|^{2r} |\widehat{\varphi}(n)|^2, \quad \widehat{\varphi}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(\theta) \exp(-in\theta) d\theta. \tag{B.3}$$

In case $r = 0$, we have the $L_2(-\pi, \pi)$ as norm, whereas for $r \in \mathbb{N}$ an equivalent norm is given by

$$\|\varphi\|^2_{H^r} = \left[\int_{-\pi}^{\pi} |\varphi(\theta)|^2 d\theta + \int_{-\pi}^{\pi} \left| \varphi^{(r)}(\theta) \right|^2 d\theta \right]^{1/2}. \tag{B.4}$$

Throughout the thesis, we use the periodic Sobolev norm and write H^r rather than H^r_{per}.

Lastly, we introduce the truncated Fourier series, which is often used in this thesis.

Definition B.5. *The truncated Fourier series of a function $\phi \in L^2[-\pi,\pi]$ is defined as*

$$(\mathcal{S}_J\phi)(\theta) = \sum_{\mu=-J}^{J} \hat{\phi}(\mu)\exp(i\mu\theta) \ \ \textit{where} \quad \hat{\phi}(\mu) = \frac{1}{\pi}\int_0^{\pi} \phi(\theta)\cos(\mu\theta)d\theta. \tag{B.5}$$

For an even function ϕ, the truncated Fourier series takes on the form

$$(\mathcal{S}_J\phi)(\theta) = \hat{\phi}(0) + 2\sum_{\mu=1}^{J} \hat{\phi}(\mu)\cos(\mu\theta). \tag{B.6}$$

APPENDIX C: HYPERGEOMETRIC FUNCTIONS

In this thesis, we use hypergeometric functions

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0, \tag{C.1}$$

and the confluent hypergeometric functions

$${}_1F_1(a;b;z) = M(a,b,z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} z^n \tag{C.2}$$

where

$$(a)_n = \begin{cases} 1 & n = 0, \\ a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} & n > 0, \end{cases} \tag{C.3}$$

are called the Pochhammer symbol.