H_∞ FILTER BASED TARGET TRACKING UNDER TIME DELAYED MEASUREMENTS

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By

Ezgi Ateş

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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September, 2015

In this thesis target tracking problem for time delayed linear systems is studied. The standard mixed sensitivity problem is considered with time delayed continuous time processes. Using the duality between control and filtering methods, the H_∞ control problem is converted to H_∞ filtering problem and a new H_∞ optimal filtering approach is proposed.

To investigate state estimation for target tracking, a typical vehicle model moving in 1-D is used but the proposed method can be expanded for movements in 3-D. Delay in both process and measurement is considered. The estimation of the states and their performances are analyzed under different scenarios including change in delay, input pattern, noise parameters etc. The results obtained by proposed H_∞ optimal filtering are compared with the results obtained by H_2 filter and simulation results are shown.

Keywords: Time Delay Systems, State Estimation, H_∞ Filtering, Target Tracking.

ÖZET

ZAMAN GECİKMELİ ÖLÇÜMLER ALTINDA H_∞ SÜZGEÇ TABANLI HEDEF İZLEME

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Bu tez kapsamında zaman gecikmesi olan doğrusal sistemler için hedef izleme problemi üzerinde çalışılmıştır. Bu çalışmada standart karışık hassasiyet problemi ile H_∞ süzgeç tasarımı arasındaki benzerlikten yararlanılarak hedef izleme problemi bir filtre uyarlama problemine dönüştürülmüştür.

Hedef izleme problemi incelenirken tek boyutta hareket eden tipik bir araç modeli örnek alınmıştır ancak çalışmalar 3 boyuttaki hareket için de uygulanabilmektedir. Ayrıca hem süreç hem de ölçüm gürültüsü hesaba katılmıştır. Durum kestirimi ve performansları zaman gecikmesi, giriş işareti, gürültüdeki farklılık vb. gibi değişik senaryolar altında incelenmiştir. Önerilen H_∞ optimal filtreleme yöntemiyle elde edilen sonuçlar, H_2 optimal filtreleme yöntemiyle elde edilen sonuçlarla karşılaştırılmış ve simülasyon sonuçları sunulmuştur.

Anahtar sözcükler: Zaman gecikmeli sistemler, Konum kestirimi, H_∞ süzgeç, Hedef izleme.

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to my nephews Cihan Berk and Kayra Deniz...

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Chapter 1

Introduction

In this thesis, target tracking problem under delayed and noisy measurements is considered. Using duality between control and estimation, a new filter structure is proposed for H_∞ estimation under delayed measurements for continuous time processes. Existing literature on estimation, filtering and H_∞ problem in control theory is examined. An overview of this literature survey is given in this chapter. The thesis work is concentrated on target tracking problem and results obtained from the classical H_2-optimal filtering and proposed H_∞-optimal filtering approaches are compared.

1.1 Aim and Scope

The aim of this work is the design of a controller such that some performance criteria and constraints are guaranteed. In order to understand system characteristics and propose a controller solution, it would be beneficial to clear concepts of time delay systems, H_∞ control, estimation, optimal filtering and H_∞ filtering.

1.1.1 Time delay systems

Time delays are often encountered in various physical systems. Time delay can occur in reporting the output when using output sensors (a sensor delay), it can be a communication delay at the controller or it can occur when transmitting the control output (an actuator delay). Whether it is present in state variables or in the input, existence of time delay in the overall system may result in poor performance, difficulty in controlling and complexity in system behaviour as oscillations, instability and bad performance [1]. Stability of a closed-loop system can be strongly effected by small delays [2], although even large time delays can stabilize the other systems [3]. Therefore controllability, observability, robustness, optimization, stability and robust stabilization of time delay systems has been an important issue. The control and stability issues with delay systems are more complex than of the finite dimensional systems. When the delays are non-negligible, a delay system model should be used and controllers should be designed on this infinite dimensional system model. The main difficulty of handling time delay in a continuous system is that time delays cannot be expressed as rational polynomials, yet there are approximation methods that model time delay as rational polynomial forms or state space representations [4]. Controlling infinite dimensional systems is more difficult than finite dimensional systems. Especially the random characteristics of a time delay makes the system behaviour time-variant, therefore conventional methods used for time-invariant systems cannot be used directly for delayed systems [4].

1.1.2 H_∞ Control

In control theory applications, H_∞ methods are used to provide good performance with guaranteed stability. The control problem is redefined as a mathematical optimization problem and then the controller that solves this optimization problem is searched for. Robustness means that stability of the system is preserved against disturbances and uncertainties in the system. First, an overview of the H_∞ control problem is given. To clearly state the problem and background, a simple control structure block is given in Figure 1.1.

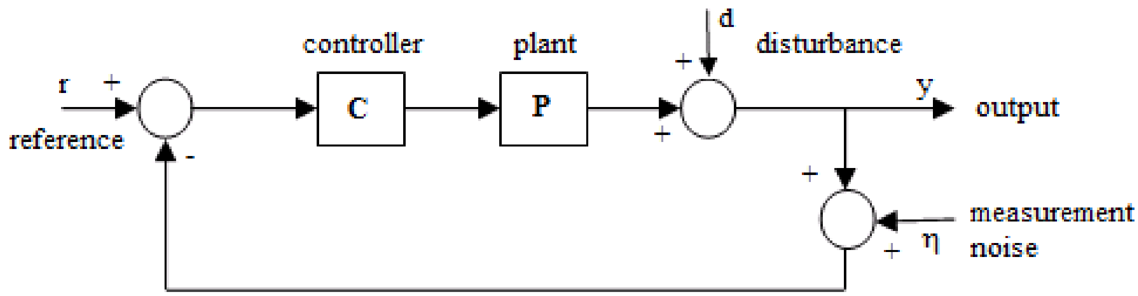

Figure 1.1: Simple Control Problem Structure

One should note that real plant transfer function also includes model uncertainty. The aim of the H_∞ control is to design a controller C such that desired performance criteria is satisfied as well as stability notwithstanding the changes in plant dynamics and modelling errors with a predefined level [5]. The desired performance criteria and constraints can be described as below:

- Closed loop system should be stable - *stability*
- The output should track the reference input signal well - *tracking*
- The sensor noise and disturbance should not affect the output – *noise attenuation , disturbance rejection*
- The stability and performance of the system should not be deteriorated by the prescribed changes in process dynamics and modelling errors - *robustness*

The changes in the process dynamics are inevitable in real systems due to aging, usage of actuators and sensors, unmodeled dynamics, time variance, limited identification etc. These control constraints as stability, robust stability, actuator saturation and control goals such as rejection, tracking, noise rejection conflict when designing the controller, what is meant by this notion will be made clear in Chapter 2.

Robust stability can be defined as the ability of a closed loop system to preserve its stability in case of the presence of modelling error. The robust control provides the stability of the control system as long as uncertain parameters and disturbances in the system and modelling errors are defined within some limit [6].

In this thesis, the work is concentrated on the robust stability of time delayed control systems. Various ways to handle model uncertainty and design a robustly stabilizing controller in literature will be introduced.

1.1.3 Modelling of Uncertainty

In frequency domain, model uncertainty can be quantized by analysing the frequency response of the transfer functions and perturbations in them. This perturbations can be mainly expressed as additive, multiplicative, feedback multiplicative or co-prime factor uncertainty [7]. If we use " Δ " as the notation for perturbation of the nominal model, then Δ is a transfer function with an upper bound on its H_∞ norm.

It is assumed that perturbation of the nominal model is below a level $\|\Delta\|_\infty < \frac{1}{\gamma}$ where $\gamma \geq 0$.

Clearly the controller would satisfy stability if Δ=0 , which points $\gamma \to \infty$. The robustness criteria here is that how large the perturbation Δ can become unless stability is destroyed. The H_∞ norm of the largest perturbation Δ before the system is unstable is called the *stability margin* of the system [8].

Robust stabilization problem can be described as finding a controller C such that closed loop system is stable for all plant P with $\|\Delta\|_\infty < \frac{1}{\gamma}$ for smallest $\gamma \geq 0$ so that stability margin of the closed loop system is maximized. This controller C is called a *robustly stabilizing* or *optimally robustly stabilizing* controller and it can be formulated for the uncertainty models [9]. Solution of this problem can be obtained by mixed sensitivity minimization, discussed later in Chapter 2.

1.1.4 Estimation Theory

Estimation theory deals with the estimation of unknown states or parameters given the measured data which may be empiric and uncertain [10]. The state of a system is a variable that carries the complete information about an internal condition of the system at a given time. These states can be position, velocity, altitude etc. State estimation can be used in many areas of engineering and science. The possible applications of state estimation theory includes [11].

- Tracking problems e.g. determination of position or velocity states of a moving target.
- Statistical inference e.g.
- Parameter identification, state estimation, stochastic control
- System identification e.g. determination of model parameters
- Communication theory e.g. determination of the transmitted signal from the noisy observation of the received signal
- Signal/image processing e.g. determination of parameters of an image or signal

There are 3 estimation problems that may be subjects of interests.

- *Filtering*- is the estimation of the current state given all the relevant data (measurements) so far.
- *Smoothing*- is the estimation of the some past state given all the relevant data(measurements) so far.
- *Prediction*- is the estimation of some future state given all the relevant data(measurements) so far.

In many control systems, the controlling action is derived by feedback, which involves process measurements derived from the system. Frequently, these measurements will contain random inaccuracies or be contaminated by unwanted signals, therefore filtering is necessary in order to make the controlling action close to the desired performance. Filtering is the estimation of the current state of a dynamic system through a process of eliminating the undesired signals such as noise in the measurements in order to get the best estimate. There are some filtering methods in literature described below.

1.1.5 Optimal Filtering

Minimization of the possible error in estimating the current state of a dynamic system is one of the typical optimization problems. Optimal estimation is an algorithm which processes the measurements to estimate the desired parameters under a certain criterion [12]. Optimal estimation methods assume that the observation data and the process noise that affects the measurements follow a known model and there is an error criteria to be minimized. Optimal state estimation can be used for both linear and non-linear systems, although is more straightforward for linear systems. Optimal filters are used in telecommunication systems, radar tracking systems, speech processing. Model based optimal filtering methods such as Kalman Filters, Hidden Markov Model filters, particle filters, find wide applications.

1.1.6 H_∞ Filtering

Linear estimation methods were mainly concentrated on H_2 minimization of the estimation error. However, H_2 filtering requires a known signal characteristic for exogenous inputs and model for the process dynamics. Where the exogenous input or the noise characteristics are unknown, H_2 filtering may not be appropriate. Therefore in cases where model uncertainty is present, being less sensitive to parameter variations than H_2 filtering methods is one of the main reasons why mini-max estimation and robust filtering is used. H_∞ norm minimization can be used for estimating the states of a system under noisy measurements. The aim is to obtain an estimator which processes the available measurements to estimate the system states such that resulting estimation error has H_∞ norm below a prescribed positive value.

In order to better formulate the problem, consider a system with state space description given in equation 1.1:

$$\dot{x} = Ax + Bw_d \ , \quad x(0) = x_0 \ \ , \ \ y(t) = Cx + Dw_n \ , \ \ z(t) = Lx \tag{1.1}$$

where y is the measured output, z is the signal to be estimated, w_d is disturbance and w_n is noise.

H_∞ optimal filtering problem is to find a H_∞ optimal estimation $\hat{z}(t)$ such that transfer function from exogenous inputs -such as disturbance and measurement noise- to the error defined with $e = z - \hat{z}$, has the smallest H_∞ norm. This transfer function is typically denoted with T_{we} , so the aim is to minimize $\|T_{we}\|_\infty$ and find a resulting filter, which is a linear time invariant operator from $y(t)$ to $\hat{z}(t)$. Note that y(t) contains all the measurements available to the estimator at time t. A general solution to this estimation problem is not available for infinite dimensional systems in general and a solution is defined only for specific cases. Therefore as a general solution, a suboptimal solution for estimation problem is searched for. In other words, given a scalar $\gamma > 0$, the problem is now to find an estimator that achieves this aim by approximating the solution as closely as desired to by iterating values of γ [13].

H_∞ filtering is less sensitive to slight changes in system dynamics and uncertainty in the exogenous signal statistics.

1.2 Literature Survey

1.2.1 Analysis of Time Delay Systems

Time delay phenomena was first recognized in biological process and then found in many engineering systems such as mechanical transmissions and networked controlled systems etc. Due to its recurrence in engineering applications, many researches were then carried on time delay phenomena.

Robust control of time delayed systems has been an active research area which is divided into many branches as stability analysis, stabilization design, H_∞ control, H_∞ filtering, Kalman filtering and stochastic control, all of which are based on stability.

Stability is a very basic issue in control theory. In [14] the problem of the robust stability of a linear system with a time-varying delay and uncertainties is considered. In order to construct a parameter-dependent Lyapunov functional for the system, a new method of dealing with a time-delay system without uncertainties in the derivative terms of the state but in the derivative of the Lyapunov functional is derived.

Research on the stability of time delayed systems first began with using frequency domain methods, then later included time domain methods as well. Frequency domain methods determine the stability of a system by analysing the locations of the roots of the systems characteristic equation as in [15].

The other way is by analysing the complex Lyapunov matrix function equation [16]. It is shown that stability conditions for time delayed systems are equivalent to robust stability analysis on an uncertain system free of delay by the use of scaled small gain lemma with constant delays, although these methods only work for systems with constant delay. The main time domain methods are the Lyapunov -Krasovskii functional and Razumikin function methods which are the most common methods for stability analysis of time delay systems [17, 18, 19, 20, 21].

Due to complexity in establishing Lyapunov-Krasovskii functionals and Lyapunov functions, the stability criteria were obtained in form of existence conditions and obtaining a general solution was not possible. After the derivation of Riccati equations, linear matrix inequalities(LMI) [22, 23, 24, 25], widespread usage of Matlab toolboxes provided solutions to construct Lyapunov-Krasovskii functionals and Lyapunov functions.

1.2.2 H_∞ Control Solutions

H_∞ problem was first introduced by Zames [26] who found small gain theorem and suggested that problem of sensitivity minimization can be described as an optimization problem and can be separated from stabilization problem by using a parametrization of all stabilizing controllers. The theory is developed for input-output systems in a general setting of Banach algebras, and then specialized to a class of multi-variable, time-invariant systems characterized by $n \times n$ matrices of H_∞ frequency response functions, either with or without zeros in the right half-plane. The approach is based on the use of a weighted semi norm on the algebra of operators to measure sensitivity, and on the concept of an approximate inverse.

In [27] an outline of stability theory for input-output problems using functional methods is given. In order to derive open loop conditions for the boundedness and continuity of feedback systems without placing restrictions on linearity or time invariance, basic researches about time delayed systems including topics such as existence and uniqueness of solutions to dynamic equations, stability analysis for trivial solutions etc. is established and laid the foundation of analysis and design of time-delayed systems.

In [28] it is shown that H_∞ problem requires the solution of two algebraic Riccati equations (AREs). Simple state-space formulas are derived for all controllers solving the following standard H_∞ problem that for a given number $\gamma > 0$, all controllers such that the H_∞ norm of the closed-loop transfer function is (strictly) less than γ is to be found. It is shown that a controller exists if and only if the unique stabilizing solution to two algebraic Riccati equations are positive definite and the spectral radius of their product is less than γ^2. Besides, as a guide, standard H_2 solution is also developed.

In [29] another solution to H_∞ problem is shown as well as how ARE are reached. In this article, the history of the relationship between modern optimal control and robust control is briefly reviewed. Robust control came into use after the observed inadequacies of the optimal control. After the acceptance of the controversial parts of the robust control, optimal and robust control theory have been used together.

Later in [30], H_∞ problem was reduced to Linear Matrix Inequalities(LMI). It is shown that continuous and discrete-time H_∞ control problems can be solved through elementary manipulations on linear matrix inequalities(LMI). Two new feature of this approach is that, a solution can be obtained for both regular and singular problems and an LMI-based parametrization can be made for all H_∞ suboptimal controllers including reduced order. Rather than usual indefinite Riccati equations, the conditions for the solution includes Riccati inequalities. Alternatively, these solvability conditions can be expressed as a system of three LMI's.

In [31], in accordance with two decoupled Riccati equations, four coupled Riccati equations are required. With this new method, instead of a pair of matrix Riccati equations as in full-order LQG case, the optimal steady-state fixed-order dynamic compensator is characterized by four matrix equations which includes two modified Riccati equations and two modified Lyapunov equations coupled by a projection whose rank is precisely equal to the order of the compensator and which determines the optimal compensator gains.

1.2.3 Filtering Solutions

As an alternative to existing techniques and algorithms, in [32] the merit of the H_∞ approach to the linear equalization of communication channels is investigated. Then in [33] the multiple input multiple output (MIMO) decision feedback equalization(DFE) problem in digital communications is approached from an estimation point of view. Using the standard and simplifying assumption that all previous decisions are correct, an explicit parametrization of all optimal DFEs is obtained. In particular, it is shown that, under the above assumption, minimum mean square error(MMSE) DFEs are optimal.

In [34] two most commonly used methodologies, the stochastic H_2 approach and the deterministic (worst-case) H_∞ approach are discussed. Despite the fundamental differences in the philosophies of these two approaches, it is shown that if indefinite metric spaces are considered, they can be treated in the same way and are essentially the same. The benefits and consequences of this unification are pursued in detail, with discussions of how to generalize well-known results from H_2 theory to H_∞setting, as well as new results and insight, the development of new algorithms, and applications to adaptive signal.

1.2.4 Mixed Sensitivity Minimization Solutions

Mixed sensitivity design of a linear multi-variable control system implies shaping the sensitivity functions to achieve the design targets of closed-loop system performance and robustness. Both H_∞ and H_2 optimization may be used for this purpose. The best-known method for H_2version of the mixed sensitivity problem is the LQG problem which is explained in [35] where low and high frequency shaping and partial pole assignment methods are used. In the mixed sensitivity problem, given a plant P, the aim is to find a controller C such that a desired H_∞performance is achieved.

The H_∞ and H_2 mixed sensitivity problems that we discuss are all special cases of the standard H_∞ and H_2 problems. The actual algorithm that is used to solve the problems is more or less identical. The famous two Riccati equation algorithm for the standard H_∞ problem [36] is widely implemented but suffers from some limitations as certain stabilizability and detectability conditions need to be satisfied. The algorithm cannot handle mixed sensitivity problems with non-proper weighting functions and only suboptimal as opposed to optimal solutions are obtained. Algorithms based on polynomial matrix fraction representations or descriptor representations of the generalized plant [37] can deal with more general problems but no fully reliable implementations are available for now.

Mixed sensitivity H_∞ stability robustness means the minimization of the mixed sensitivity criterion and results in optimal robustness. Mixed sensitivity problem is not only used for robustness optimization but also for design for performance [38].

1.2.5 Organization of the Thesis

The thesis is organized as follows. In Chapter 2, problem definition is given. In Chapter 3, the proposed method to solve the problem at hand is presented. In Chapter 4, H_∞ optimal controller solution is investigated under different input, noise and delay scenarios and compared with the performance of H_2 optimal controller solution. Concluding remarks are given in Chapter 5.

Chapter 2

Problem Definition

Consider a classical continuous time delay system with measurement and process noise. Let's start the problem definition by characterizing the H_∞ norm of the concerned transfer function using state space system equations. Let A, B, C and D be real matrices which define the system state space equations as in equations 2.1, 2.2, 2.3. All eigenvalues of A are supposed to be on the left half plane (LHP).

$$\dot{x}(t) = Ax\,(t) + Bw_p(t) \tag{2.1}$$

$$y\,(t) = Cx\,(t-h) - dw_0(t) \tag{2.2}$$

$$z\,(t) = Lx(t) \tag{2.3}$$

As for the process noise, $w_p(t)$, it is generated by passing an input $w_1(t)$ through a weighting filter $W_1(s)$ so that

$$w_p = W_1 w_1\ . \tag{2.4}$$

It will be assumed that $w_1(t)$ is a finite energy signal.

Here the aim is to design a filter $Q(s)$ such that estimated system control output $\hat{z}$ is given by

$$\hat{z} = Qy \tag{2.5}$$

The estimation error $e(t)$ is given by the difference of observation and estimation.

$$e(t) = z(t) - \hat{z}(t) \tag{2.6}$$

The exogenous inputs of the system are L_2 bounded signals and collected in w as denoted in equation 2.7.

$$w = \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} \tag{2.7}$$

By designing the filter Q(s) in such a way that, estimation error with respect to input signals mapped in L_2 norm is minimized as defined in equation 2.8

$$\underset{Q\ \in H_\infty}{min} \ \underset{w \in L_2}{sup} \ \frac{\|e\|_2}{\|w\|_2} = \underset{Q\ \in H_\infty}{min} \|T_{we}\|_\infty \tag{2.8}$$

Note that H_∞ is the induced norm of mapping L_2 signals to L_2 signals [4, 8, 10, 12, 39].

2.1 Block Diagram Representation

The system described with state space equations 2.1, 2.2, 2.3 can be represented in s-domain as a block diagram illustrated in Figure 2.1 .

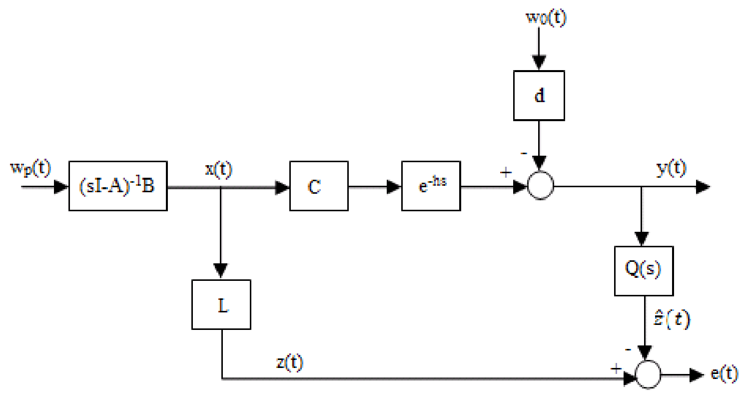

Figure 2.1: Block diagram of the estimation problem

Estimation error in s-domain can be rewritten as

$$E\left(s\right)= \ L\left(s\right)\left(sI-A\left(s\right)\right)^{-1}B\left(s\right)w_p\left(s\right)-Q(s)\left[-dw_0\left(s\right)+C(s)(sI-A\left(s\right))^{-1}B\left(s\right)e^{-hs}w_p\left(s\right)\right] \tag{2.9}$$

. Taking input vector w out of the matrix and rewriting equation 2.9 as a multiplication two vectors yields equation 2.10 .

$$E\left(s\right)=\left[L(sI-A)^{-1}B-C(sI-A)^{-1}Be^{-hs}Q\left(s\right) \qquad dQ(s)\right]\begin{bmatrix} w_p\left(s\right) \\ w_0\left(s\right) \end{bmatrix} \tag{2.10}$$

Here, transfer function from error to input T_{ew} can be represented as in equation 2.11.

$$T_{ew}(s)=\left[\frac{N_w(s)}{D_p(s)}-\frac{N_p(s)}{D_p(s)}e^{-hs}Q(s) \qquad dQ(s)\right] \tag{2.11}$$

Where N_w , D_p and N_p are obtained from the factorization of the terms $L(sI-A)^{-1}B$ and $C(sI-A)^{-1}B$. The aim of this filtering problem is to minimize this transfer function $\left\|T_{ew}\right\|_{\infty}$ over $Q\epsilon H_\infty$

2.2 Dual Problem of H_∞ Optimal Controller

Now, let's consider the closed loop control system whose block diagram is depicted in Figure 2.2

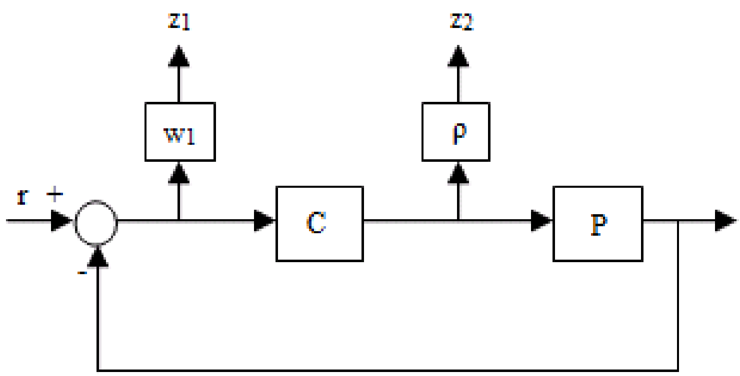

Figure 2.2: Control problem block diagram

The system output signals are collected in z as :

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{2.12}$$

Here plant P is assumed to be stable, controller C is constructed by filter Q, which is also stable

$$C = \frac{Q}{1 - PQ} \tag{2.13}$$

The aim is to minimize the transfer function $\frac{\|z\|_2}{\|r\|_2}$, which means the minimization of $\|T_{zr}\|_\infty$ in H_∞ norm.

Transfer function from z to r can be represented in vector form as equation 2.14 .

$$T_{zr} = \begin{bmatrix} W_1\ (1 - PQ) \\ \rho Q \end{bmatrix} \tag{2.14}$$

and can be rewritten as equation 2.15

$$T_{zr} = \begin{bmatrix} W_1 - W_1 PQ \\ \rho Q \end{bmatrix} \tag{2.15}$$

Here, we assume that $N_W(s)$ is a stable polynomial. Normally, this condition is satisfied by almost all meaningful choices of L[8] . Let's rewrite the transfer function in equation 2.11 as equation 2.16 .

$$T_{ew} = \left[\frac{N_w(s)}{D_p(s)} \left(1 - \frac{N_p(s)}{N_w(s)} e^{-hs} Q(s) \right) \quad dQ(s) \right] \tag{2.16}$$

Here if we let

$$W_1\,(s) = \frac{N_w(s)}{D_p(s)} = L(sI - A)^{-1}B \tag{2.17}$$

which is minimum phase, then plant transfer function is given as equation 2.18

$$P = \frac{N_p(s)}{N_w(s)} = \left(\frac{C(sI - A)^{-1}B}{L(sI - A)^{-1}B} \right) e^{-hs} \tag{2.18}$$

which is stable.

If the noise scaling factor ρ is equated to d, i.e. $\rho = d$, then the transfer function in equation 2.11 becomes transpose of the transfer function in equation 2.15

$$T_{ew} = {T_{zr}}^T \tag{2.19}$$

Then the aim is to minimize T_{zr} to find optimal filter $Q(s) \in H_\infty$.

2.3 Two-Block Mixed Sensitivity Problem

We discuss the design of multi-variable feedback systems as in the block diagram of Figure 2.3. The plant is represented by P, and the controller by C. The signal w_1 represents the disturbances and w_2 the measurement noise. Throughout, P is assumed to be square, rational and invertible.

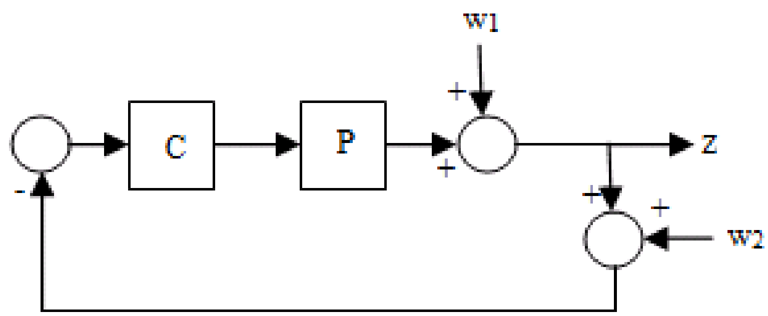

Figure 2.3: Two block problem: find C stabilizing P such that $\|T_{ww}\|_\infty$ is minimized.

Performance and robustness are characterized by various well-known closed-loop functions, in particular the sensitivity function S, the complementary sensitivity function T, and the input sensitivity function U, successively defined by equation 2.20, 2.21, 2.22 .

$$S = (I + PC)^{-1} \tag{2.20}$$

$$T = PC(I + PC)^{-1} \tag{2.21}$$

$$U = C(I + PC)^{-1} \tag{2.22}$$

S is the transfer matrix from the disturbance input w_1 to the control system output y, T is its complement $T = I - S$ and $-U$ is the transfer matrix from the disturbance w_1 to the plant input r.

If the weights are chosen correctly for a mixed sensitivity problem, the standard 2-block H_∞ control problem is obtained [40]. Note that generally plant P is stable or at least has a low dimensional instability.

If the effect of the disturbance w_1 on the output y is desired to be minimized, the controller C has to be chosen such that the sensitivity S is small in the frequency band where w_1 has most of its power.

If the tracking error e in the system is defined as $e = r - y$ then,

$$e = r - y = (I + PC)^{-1}(r - w_1) + PC(I + PC)^{-1}w_2 \tag{2.23}$$

From equation 2.23, it is seen that S has to be kept small in both the disturbance and the tracking band. On the other hand complementary sensitivity function T must be kept small to avoid sensor noise effect.

This relation has an important influence on the ultimate performance of the control system. If S is chosen very close to zero for reasons of disturbance rejection and good tracking, T becomes very close to 1 which results in full sensor noise in the output and vice versa. Therefore optimality will be some compromise between control constraints.

Note that

- The transfer function may be multi-variable and thus we encounter matrices.
- The crucial quantities S and T involve matrix inversions $(I + PC)^{-1}$
- The controller C may only be chosen from the set of stabilising controllers.

When we are dealing with a stable transfer function P, whose inner-outer factorization is $P = P_o P_i$, last two problems can be avoided. Mixed sensitivity minimization problem in general is given by equation 2.24 [41].

$$\min_{Q \epsilon H_\infty} \quad \left\| \begin{bmatrix} W_1 \ (1 - PQ) \\ W_2 P_o Q \end{bmatrix} \right\|_\infty \tag{2.24}$$

Note that equation 2.24 is same as the equation 2.16 if

$$W_2 = d \ {P_o}^{-1} \tag{2.25}$$

In particular, there are some essential design limitations related to the locations of the open-loop plant poles and zeros. If the plant has right-half plane open-loop zeros, then the magnitude of the smallest right-half plane zero is an upper bound for the closed-loop bandwidth. This is because good performance basically involves inversion of the plant. Since a right-half plane zero implies instability of the inverse plant, this inversion may only be accomplished for low frequencies that are smaller than the magnitude of the smallest right-half plane open-loop zero. This provides an upper bound to the width of the band over which S may be made small. If the plant has right-half plane poles then the gain may only be allowed to drop off for frequencies greater than the magnitude of the largest right-half plane pole. Otherwise the plant cannot be stabilized. This constitutes a lower bound for the frequency band outside which T drops off to small values.

In particular when $L = C$, $P = e^{-hs}$, $P_0 = 1$, then $W_1 = C(sI - A)^{-1}B$.

In this study, in order to present the proposed method for target tracking, one dimensional movement will be considered. Besides, target tracking problem constitutes an example for the state estimation of any linear system. Therefore, the method proposed here can be adapted to more general state estimation problems as described in [42]. State equations of a moving vehicle in 1-D, can be written as equation 2.26 for position $x_p(t)$, and velocity $x_p(t)$.

$$\dot{x}_p \ = x_v \ , \qquad \dot{x}_v = \frac{1}{M}\ F_x - \epsilon x_v \tag{2.26}$$

Here M represents the mass of the vehicle, F_x shows the force applied to the vehicle and ϵ is the friction constant. Note that, for 3-D movement, state equations for the position $x_p(t)$ and velocity $x_v(t)$ can be written in the same way as equation 2.27 and equation 2.28.

$$\dot{y}_p \ = y_v, \qquad \dot{y}_v = \frac{1}{M}\ F_y - \epsilon y_v \tag{2.27}$$

$$\dot{z}_p \ = z_v, \qquad \dot{z}_v = \frac{1}{M}\ F_z - \epsilon z_v \tag{2.28}$$

Assume that F_x, F_y and F_z can be independently applied. Then equation 2.26, equation 2.27 and equation 2.28 are identical.

In equation 2.26, the term $\frac{1}{M}F_x$ that provide the manoeuvre of the target can be thought as a process noise- that is to say an unknown disturbance effect for target tracking-, $w_p(t)$. Then the state equations of the position and velocity of the target can be rewritten with the definition of $x_1 = x_p$ and $x_2 = x_v$ as in equation 2.29, where $w_p(t)$ is process noise-unknown term $\frac{1}{M}F_x$.

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2 = -\epsilon x(t) + w_p(t) \tag{2.29}$$

The output is given by equation 2.30 where $y(t)$ represents the taken measurements, $w_0(t)$ represents measurement noise and h_0 represents the delay at the output.

$$y(t) = x_1(t - h_0) + w_0(t) \tag{2.30}$$

If it is wanted to predict the position h_1 seconds ahead of time, the variable to be tracked $z(t)$ can be defined as in equation 2.31.

$$z(t) = x_1(t + h_1) \tag{2.31}$$

Now the problem is converted to the design of a causal filter Q(s) as in 2.4 such that $sup\frac{\|e\|_2}{\|w\|_2}$ is minimized.

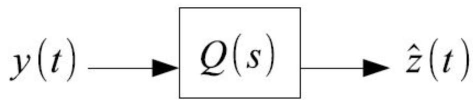

Figure 2.4: Optimal Filter

In order to estimate the prospective position of the system, the measurement y(t) can be imposed to a causal filter Q(s). In this situation, the estimation error which is

$$e(t) = z(t) - \hat{z}(t) \tag{2.32}$$

Since the process noise that provides the movement of the target and measurement noise will not be at the same levels, a scaling factor ρ can be defined. Then the signals that are exogenous inputs to the system can be compiled in matrix form as in equation 2.33. Here, $F_1(s)$ can be regarded as a filter and $\rho > 0$ as a constant scaling factor.

$$\begin{bmatrix} w_p \\ w_0 \end{bmatrix} = \begin{bmatrix} F_1 & 0 \\ 0 & \rho \end{bmatrix} w = \begin{bmatrix} F_1 w_1 \\ \rho w_2 \end{bmatrix} \tag{2.33}$$

$$w(t) = \begin{bmatrix} w_p(t) \\ w_0(t) \end{bmatrix} := \begin{bmatrix} w_1(t) \\ \rho w_2(t) \end{bmatrix} \tag{2.34}$$

In this situation when error is defined as in equation 2.35, H_∞ optimal filter design makes the cost function $\sup_{w \neq 0} \frac{\|e\|_2}{\|w\|_2}$ minimum.

$$e(t) = -\hat{z}(t) + z(t) \tag{2.35}$$

The error under equations 2.29 - 2.35 can be shown as in equation 2.36 in frequency domain.

$$E(s) = e^{h_1 s} X_1(s) - Q(s)(e^{-h_0 s} X_1(s) + w_0(s)) \tag{2.36}$$

If sum of the measurement delay h_0 and estimation time h_1 is defined as $h := h_0 + h_1$, equation 2.36 can be rewritten as equation 2.37.

$$e^{-h_1 s} E(s) = (I - Q(s)e^{-hs}) X_1(s) - Q(s) w_0(s) \tag{2.37}$$

The position of the system in frequency domain regarding equation 2.29 is found as in equation 2.38.

$$X_1(s) = \frac{1}{s+\epsilon} W_p(s) \tag{2.38}$$

In this manner, target tracking problem is transformed to H_∞ model adaptation problem with equation 2.39

$$\sup_{w \neq 0} \frac{\|e\|_2}{\|w\|_2} = \left\| \left[\ \tfrac{F_1(s)}{s+\epsilon} \left(1 - e^{-hs} Q(s)\right) - \rho Q(s) \ \right] \right\|_\infty \tag{2.39}$$

Typically, process noise, $w_p(t)$, can be modelled as a signal which consists of short duration pulses that makes the target short duration manoeuvred as in Figure 2.5.

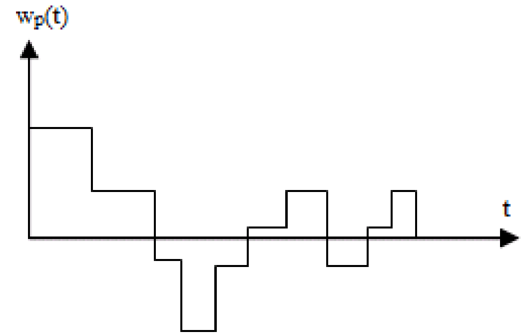

Figure 2.5: Finite energy signal, $w_p(t)$

Here we assume that $w_p(t)$ is generated by $w_1(t)$ which is a finite energy signal: i.e. ,

$$W_p(s) = F_1(s)W_1(s) \tag{2.40}$$

If it is taken as $F_1(s) = 1$ and $W_1(s) = \frac{1}{s(s+\epsilon)} \cong \frac{1}{s^2}$, then the inequality equation 2.39 becomes as equation 2.41

$$\sup_{w \neq 0} \frac{\|e\|_2}{\|w\|_2} = \left\| \left[\; H_Q(s) - \rho Q(s) \; \right] \right\|_\infty \tag{2.41}$$

where $H_Q(s)$ is given by equation 2.42.

$$H_Q(s) = \frac{1}{s^2}\left(1 - e^{-hs}Q(s)\right) \tag{2.42}$$

Here, transfer function from the input $w_p(t)$ to output $e(t-h_1)$, $H_Q(s)$, can be named as *error system*.

Chapter 3

H_∞ Optimal Filter Design

3.1 Calculation of Filter Parameters

Finding $Q(s)$ that makes the norm at the right hand side of equation 2.42 minimum is equivalent to H_∞ mixed sensitivity minimization problem defined in 2.41.

$$\begin{aligned}\gamma_0 = \inf_{Q\epsilon H_\infty} & \left\| \begin{bmatrix} W_1(s) \\ 0 \end{bmatrix} - \begin{bmatrix} e^{-hs} \\ \rho \end{bmatrix} Q(s) \right\|_\infty \\ = \inf_{Q\epsilon H_\infty} & \left\| \begin{bmatrix} W_1(s) \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ \rho \end{bmatrix} e^{-hs} Q(s) \right\|_\infty \end{aligned} \tag{3.1}$$

$$\gamma_0 = \inf_{Q\epsilon H_\infty} \left\| \begin{bmatrix} W_1(s)\left(1 - e^{-hs}Q(s)\right) \\ -\rho Q(s) \end{bmatrix} \right\|_\infty \tag{3.2}$$

Given two weighting functions $W_1(s)$ and $W_2(s)$, $P(s)$ is an irrational transfer function, mixed sensitivity minimization problem is given by equation 3.3.

$$\gamma_0 = \inf_{C \in \mathcal{C}(P)} \left\| \begin{bmatrix} W_1(1+PC)^{-1} \\ W_2PC(1+PC)^{-1} \end{bmatrix} \right\|_\infty \tag{3.3}$$

where $\mathcal{C}(P)$ is the set of all controllers $C(s)$ for which the feedback system formed by C and P is stable. Stability of the closed loop system means that closed loop system transfer functions S, CS and PC is also stable in H_∞. The optimal controller solving equation 3.3 is denoted by C_{opt}.

Typically $W_1(s)$ represent the class of reference signals to be tracked and is a low order low-pass filter. $W_2(s)$ represent a boundary on the multiplicative plant uncertainty and is an improper low order high-pass filter [8, 39].

In this case $W_1(s)$, $W_2(s)$, $P(s)$ is given respectively as in equations 3.4 - 3.6.

$$W_1(s) = \frac{1}{s(s+\epsilon)} \cong \frac{1}{s^2} \tag{3.4}$$

$$W_2(s) = -\rho \tag{3.5}$$

$$P(s) = e^{-hs} \tag{3.6}$$

In what follows we provide a general solution of problem (3.3) from [41] - [43].

The plant is assumed to have finitely many poles in $C+$ and no poles on the imaginary-axis. In this case, $P(s)$ can be factored as in equation 3.7.

$$P(s) = \frac{M_n(s)N_o(s)}{M_d(s)} \tag{3.7}$$

where M_n is an inner (all-pass) function, $N_o(s)$ is an outer (minimum phase) function and $M_d(s)$ is a rational inner function.

If RHP zeros of $M_d(s)$ as unstable poles of the plant, are denoted as $a_1, ..., a_l \in C+$, it is assumed that $a_1, ..., a_l$ are distinct for simplicity of the mathematical calculations.

Since W_1 is rational, it can be factorized as $W_1(s) = nW_1(s)/dW_1(s)$, with two coprime polynomials nW_1 and dW_1 where it is assumed that $deg(nW_1) \leq deg(dW_1) =: n_1 \geq 1$. Defining a function $E_\gamma(s)$ as

$$E_\gamma(s) := \frac{W_1(-s)W_1(s)}{\gamma^2} - 1 \tag{3.8}$$

and replacing $W_1(s)$ in equation 3.8 gives equation 3.9

$$E_\gamma = \frac{1}{s^4\gamma^2} - 1 \tag{3.9}$$

for our system $n_1 = 2$.

Let $\beta_1, ..., \beta_{2n_1}$ be the zeros of $E_\gamma(s)$, enumerated in such a way that $-\beta_{n_1} + k = \beta_k \in C+$, for $k = 1, ..., n_1$.

Note that each β_k is dependent on $\gamma > 0$, which is a candidate for γ_{opt}. We assume that for $\gamma = \gamma_{opt}$, the zeros of E_γ are distinct. Note that this condition is satisfied generically (if not, a small perturbation in the problem data changes γ_{opt} which moves the locations of $\beta_1, ..., \beta_{n1}$).

Now define a rational function which depends on $\gamma > 0$ and the weights W_1 and W_2,

$$F_\gamma(s) := \gamma \frac{dW_1(-s)}{nW_1(-s)} G_\gamma(s) \tag{3.10}$$

where $G_\gamma \in H_\infty$ is an outer function determined from the spectral factorization given in equation 3.11.

$$G_\gamma(-s)G_\gamma(s) = \left(1 + \frac{W_2(-s)W_2(s)}{W_1(-s)W_1(s)} - \frac{W_2(-s)W_2(s)}{\gamma^2}\right)^{-1} \tag{3.11}$$

Placing W_1 and W_2 in equation 3.11, yields equation 3.12.

$$G_\gamma * G_\gamma = \left(1 + s^4\rho^2 - \frac{\rho^2}{\gamma^2}\right)^{-1} \tag{3.12}$$

Right hand side of equation 3.12 can be factorized as in equation 3.13.

$$\left(\frac{1}{s^4\rho^2 + 1 - \frac{\rho^2}{\gamma^2}}\right) = \left(\frac{1}{\rho s^2 + a_\gamma s + \sqrt{1 - \frac{\rho^2}{\gamma^2}}}\right)\left(\frac{1}{\rho s^2 - a_\gamma s + \sqrt{1 - \frac{\rho^2}{\gamma^2}}}\right) \tag{3.13}$$

If definition for a_γ^2 in equation 3.14 is made,

$$2\rho\sqrt{1 - \frac{\rho^2}{\gamma^2}} = a_\gamma^2, \tag{3.14}$$

then a_γ is found as

$$a_\gamma = \sqrt{2\rho\sqrt{1 - \frac{\rho^2}{\gamma^2}}}. \tag{3.15}$$

Hence, $G_\gamma(s)$ can be rewritten as equation 3.16

$$G_\gamma(s) = \frac{1}{\rho s^2 + a_\gamma s + \sqrt{1 - \frac{\rho^2}{\gamma^2}}}. \tag{3.16}$$

Replacing $W_1(s)$ and $G_\gamma(s)$ in equation 3.10, the rational function $F_\gamma(s)$ can be rewritten as in equation 3.17.

$$F_\gamma(s) = \gamma \frac{s^2}{\rho s^2 + \sqrt{2\rho\sqrt{1 - \frac{\rho^2}{\gamma^2}}} + \sqrt{1 - \frac{\rho^2}{\gamma^2}}}. \tag{3.17}$$

Note that $E_\gamma(s)$ can be also written as equation 3.18 and its zeros are enumerated as in [42] as equation 3.19.

$$E_\gamma(s) = \frac{1 - s^4\gamma^2}{s^4\gamma^2} \tag{3.18}$$

$$\beta_i = \frac{1}{\sqrt{\gamma}}, \frac{j}{\sqrt{\gamma}}, \frac{-1}{\sqrt{\gamma}}, \frac{-j}{\sqrt{\gamma}} \qquad i = 1, 2, 3, 4 \tag{3.19}$$

With the above definitions, the optimal controller can be expressed as equation 3.20

$$C_{opt}(s) := E_\gamma(s) M_d(s) \gamma \frac{F_\gamma(s) L(s)}{1 + M_n(s) F_\gamma(s) L(s)} N_0(s)^{-1} \tag{3.20}$$

where $\gamma = \gamma_{opt}$ and $L(s)$ is a transfer function of the form

$$L(s) = \frac{[1s \ldots s^{n-1}]\psi_2}{[1s \ldots s^{n-1}]\psi_1} \quad n = n_1 + l \tag{3.21}$$

where the coefficient vectors

$$\psi_1 = [\psi_{10} ... \psi_{1(n-1)}]^T \qquad and \qquad \psi_2 = [\psi_{20} ... \psi_{2(n-1)}]^T \tag{3.22}$$

are to be determined from the interpolation conditions given in [43]. Preserving $L(s)$ for now, $C_{opt}(s)$ can be written as equation 3.23.

$$C_{opt}(s) = \frac{Q}{1 - PQ} = \frac{1 - s^4\gamma^2}{s^4\gamma^2} \frac{\gamma \frac{s^2}{\rho s^2 + a_\gamma s + \sqrt{1 - \frac{\rho^2}{\gamma^2}}} L(s)}{1 + P\gamma \frac{s^2}{\rho s^2 + a_\gamma s + \sqrt{1 - \frac{\rho^2}{\gamma^2}}} L(s)} \tag{3.23}$$

Note that

$$\left(\frac{1}{s^4\gamma^2} - 1\right) \frac{F_\gamma L}{1 + e^{-hs} F_\gamma L} = \frac{Q}{1 - e^{-hs} Q} = C_{opt}. \tag{3.24}$$

Since the filter is given by

$$Q = C(1 + PC)^{-1} = \frac{E_\gamma \frac{F_\gamma L}{1 + e^{-hs} F_\gamma L}}{1 + e^{-hs} \frac{E_\gamma F_\gamma L}{1 + e^{-hs} F_\gamma L}} = \frac{E_\gamma F_\gamma L}{1 + e^{-hs} F_\gamma L \ \frac{1}{s^4\gamma^2}} \tag{3.25}$$

placing E_γ and F_γ in equation 3.25 yield Q(s) as:

$$\begin{aligned} Q\left(s\right) &= \frac{\left(1 - s^4\gamma^2\right) \frac{\gamma s^2}{\rho s^2 + a_\gamma s + \sqrt{1 - \frac{\rho^2}{\gamma^2}}} L\left(s\right)}{s^4\gamma^2 + e^{-hs} \frac{\gamma s^2}{\rho s^2 + a_\gamma s + \sqrt{1 - \frac{\rho^2}{\gamma^2}}} L\left(s\right)} \\ &= \frac{\left(1 - s^4\gamma^2\right) \ L(s)}{\gamma s^2 \ \left(\rho s^2 + a_\gamma s + \sqrt{1 - \frac{\rho^2}{\gamma^2}}\right) + e^{-hs} L(s)} \end{aligned} \tag{3.26}$$

Note that

$$Q(0) = \frac{L(0)}{L(0)} = 1. \tag{3.27}$$

Remember that from equation 3.25, as we already assumed $Q(s)$ is a stable filter, now we should check the pole locations of the optimal controller. Considering the denominator of equation 3.25 as

$$1 - e^{-hs}Q\left(s\right) = 1 - \frac{\left(1-\gamma^2 s^4\right) L(s)e^{-hs}}{\gamma s^2 \left(\rho s^2 + a_\gamma s + \sqrt{1-\frac{\rho^2}{\gamma^2}}\right) + e^{-hs}L(s)}, \tag{3.28}$$

rewriting equation 3.28 yields equation 3.29.

$$\frac{\gamma s^2 \left(\rho s^2 + a_\gamma s + \sqrt{1-\frac{\rho^2}{\gamma^2}}\right) + \gamma^2 s^4 L(s)e^{-hs}}{\gamma s^2 \left(\rho s^2 + a_\gamma s + \sqrt{1-\frac{\rho^2}{\gamma^2}}\right) + e^{-hs}L(s)} = \gamma s^2 \frac{\left(\rho s^2 + a_\gamma s + \sqrt{1-\frac{\rho^2}{\gamma^2}}\right) + \gamma s^2 L(s)e^{-hs}}{\gamma s^2 \left(\rho s^2 + a_\gamma s + \sqrt{1-\frac{\rho^2}{\gamma^2}}\right) + e^{-hs}L(s)} \tag{3.29}$$

From equation 3.29, we see that denominator has two zeros at $s = 0$ as desired, so that we obtain $\frac{1}{s^2}\left(1 - e^{-hs}Q\left(s\right)\right)$ is stable.

Interpolation conditions of this system is given by equation 3.30

$$1 + e^{-hs}\frac{\gamma s^2}{\rho s^2 + a_\gamma s + \sqrt{1-\frac{\rho^2}{\gamma^2}}}L(s)\Bigg|_{s=\beta_i} = 0 \tag{3.30}$$

where $\beta_1 = \frac{1}{\sqrt{\gamma}}$ and $\beta_2 = \frac{j}{\sqrt{\gamma}}$, a_γ is given by $a_\gamma = \sqrt{2\rho\sqrt{1-\frac{\rho^2}{\gamma^2}}}$.

As interpolation conditions are 2^{nd} order, we obtain

$$L\left(s\right) = \pm\frac{1-\varphi s}{1+\varphi s}. \tag{3.31}$$

Solving two set of interpolation conditions by replacing $L(s)$ yields equation 3.32.

$$\begin{cases} 1+\varphi\frac{1}{\sqrt{\gamma}} \pm \quad e^{\frac{-h}{\sqrt{\gamma}}} \quad \frac{\left(1-\varphi\frac{1}{\sqrt{\gamma}}\right)}{\frac{\rho}{\gamma}+\frac{a\gamma}{\sqrt{\gamma}}+\sqrt{1-\frac{\rho^2}{\gamma^2}}} = 0 \\ 1+\varphi j\frac{1}{\sqrt{\gamma}} \pm \quad e^{\frac{-hj}{\sqrt{\gamma}}} \quad \frac{(-1)\left(1-\varphi\frac{1}{\sqrt{\gamma}}\right)}{-\frac{\rho}{\gamma}+j\frac{a\gamma}{\sqrt{\gamma}}+\sqrt{1-\frac{\rho^2}{\gamma^2}}} = 0 \end{cases} \tag{3.32}$$

Combining set of equations in equation 3.32 gives equation 3.33.

$$\left(1 \pm \frac{e^{\frac{-h}{\sqrt{\gamma}}}}{\frac{\rho}{\gamma}+\frac{a\gamma}{\sqrt{\gamma}}+\sqrt{1-\frac{\rho^2}{\gamma^2}}}\right) + \left(1 \pm \frac{-e^{\frac{-h}{\sqrt{\gamma}}}}{\frac{\rho}{\gamma}+\frac{a\gamma}{\sqrt{\gamma}}+\sqrt{1-\frac{\rho^2}{\gamma^2}}}\right)\frac{\varphi}{\sqrt{\gamma}} = 0 \tag{3.33}$$

Taking ϕ out of equation 3.33 gives equation 3.34.

$$\varphi = \sqrt{\gamma}\left(\frac{1 \pm \frac{e^{\frac{-h}{\sqrt{\gamma}}}}{\frac{\rho}{\gamma}+\frac{a\gamma}{\sqrt{\gamma}}+\sqrt{1-\frac{\rho^2}{\gamma^2}}}}{\pm\frac{e^{\frac{-h}{\sqrt{\gamma}}}}{\frac{\rho}{\gamma}+\frac{a\gamma}{\sqrt{\gamma}}+\sqrt{1-\frac{\rho^2}{\gamma^2}}} - 1}\right) = \frac{\sqrt{\gamma}}{j}\left(\frac{1 \pm \frac{-e^{\frac{-jh}{\sqrt{\gamma}}}}{\frac{-\rho}{\gamma}+j\frac{a\gamma}{\sqrt{\gamma}}+\sqrt{1-\frac{\rho^2}{\gamma^2}}}}{\pm\frac{-e^{\frac{-jh}{\sqrt{\gamma}}}}{\frac{-\rho}{\gamma}+j\frac{a\gamma}{\sqrt{\gamma}}+\sqrt{1-\frac{\rho^2}{\gamma^2}}} - 1}\right) \tag{3.34}$$

At this point let's define two auxiliary functions r_γ and q_γ for the ease of calculations.

$$r_\gamma = \frac{e^{\frac{-h}{\sqrt{\gamma}}}}{\frac{\rho}{\gamma}+\sqrt{\frac{2\rho}{\gamma}\sqrt{1-\frac{\rho^2}{\gamma^2}}}+\sqrt{1-\frac{\rho^2}{\gamma^2}}} = \frac{e^{-j\frac{h}{\sqrt{\rho}}\sqrt{\frac{\rho}{\gamma}}}}{\frac{\rho}{\gamma}+\sqrt{\frac{2\rho}{\gamma}\sqrt{1-\frac{\rho^2}{\gamma^2}}}+\sqrt{1-\frac{\rho^2}{\gamma^2}}} \tag{3.35}$$

$$q_\gamma = \frac{e^{-j\frac{h}{\sqrt{\gamma}}}}{\frac{\rho}{\gamma}-\sqrt{1-\frac{\rho^2}{\gamma^2}}-j\sqrt{\frac{2\rho}{\gamma}\sqrt{1-\frac{\rho^2}{\gamma^2}}}} = \frac{e^{-j\frac{h}{\sqrt{\rho}}\sqrt{\frac{\rho}{\gamma}}}}{\frac{\rho}{\gamma}-\sqrt{1-\frac{\rho^2}{\gamma^2}}-j\sqrt{\frac{2\rho}{\gamma}\sqrt{1-\frac{\rho^2}{\gamma^2}}}} \tag{3.36}$$

Then equation 3.34 can be rewritten as

$$\varphi = \sqrt{\gamma}\left(\frac{1 \pm r_\gamma}{\pm r_\gamma - 1}\right) = \frac{\sqrt{\gamma}}{j}\left(\frac{1 \pm q_\gamma}{\pm q_\gamma - 1}\right). \tag{3.37}$$

Calculating the equations for (+) and (-) sign gives equation 3.38 and equation 3.39 respectively.

$$1_+ \;: j\left(\frac{1+r_\gamma}{r_\gamma-1}\right) = \left(\frac{1+q_\gamma}{q_\gamma-1}\right) \leftrightarrow \frac{q_\gamma-1}{1+q_\gamma} = -j\left(\frac{r_\gamma-1}{1+r_\gamma}\right) = j\left(\frac{1-r_\gamma}{1+r_\gamma}\right) \tag{3.38}$$

$$2_- \quad : j\left(\frac{1-r_\gamma}{-r_\gamma-1}\right) = \left(\frac{1-q_\gamma}{-q_\gamma-1}\right) \leftrightarrow \left(\frac{1-q_\gamma}{1+q_\gamma}\right) = j\left(\frac{1-r_\gamma}{1+r_\gamma}\right) \tag{3.39}$$

Reorganizing them gives equation 3.40 and equation 3.41 respectively.

$$1_+ \quad : j\left(\frac{1-r_\gamma}{1+r_\gamma}\right) = \left(\frac{q_\gamma-1}{1+q_\gamma}\right) \tag{3.40}$$

$$2_- \quad : j\left(\frac{1-r_\gamma}{1+r_\gamma}\right) = \left(\frac{1-q_\gamma}{1+q_\gamma}\right) \tag{3.41}$$

Combining the two equations above gives the equation 3.42.

$$j\left(\frac{1-r_\gamma}{1+r_\gamma}\right) \pm \left(\frac{1-q_\gamma}{1+q_\gamma}\right) = 0 \tag{3.42}$$

Where r_γ and q_γ are given by equation 3.43 and equation 3.44 respectively with variables x_γ and h_ρ defined as $x_\gamma = \rho/\gamma$ and $h_\rho = h/\sqrt{\rho}$.

$$r_\gamma = \frac{e^{-h_\rho\sqrt{x_\gamma}}}{x_\gamma + \sqrt{1-{x_\gamma}^2} + \sqrt{2x_\gamma\sqrt{1-{x_\gamma}^2}}} \tag{3.43}$$

$$q_\gamma = \frac{e^{-jh_\rho\sqrt{x_\gamma}}}{x_\gamma - \sqrt{1-{x_\gamma}^2} - j\sqrt{2x_\gamma\sqrt{1-{x_\gamma}^2}}} \tag{3.44}$$

$$j\left(\frac{1-r_\gamma}{1+r_\gamma}\right) \mp \left(\frac{1-q_\gamma}{1+q_\gamma}\right) = 0 \tag{3.45}$$

$$\left(\frac{1-r_\gamma}{1+r_\gamma}\right) = \pm j\left(\frac{1-q_\gamma}{1+q_\gamma}\right) \tag{3.46}$$

If the absolute difference in the two sides of equality in equation 3.46 is defined as

$$f_\pm = \left|\frac{1-r_\gamma}{1+r_\gamma} \pm -j\frac{1-q_\gamma}{1+q_\gamma}\right| \tag{3.47}$$

the minimum cost γ_{opt} is found from the minimum value of $x \in (0,1)$ which makes $f_{\pm}$ equal to zero with either (+) or (-) sign.

In other words, in the interval $x \in (0,1)$, the solutions of $f_{+} = 0$ and $f_{-} = 0$ are investigated, the minimum solution x gives us the optimum$\gamma_{opt} = \rho/x$ value. The sign that gives this minimum x value is taken as the same sign in the representation of $L(s)$ which defines the optimal filter.

γ_{opt} is the largest γ which solves equation 3.46 with the $(+)$ or $(-)$ sign. The corresponding sign determines $L(s)$ and hence C_{opt} is obtained via equation 3.23.

Matlab implementation of the solution to equation 3.46 showed that smallest x_{γ} which means the largest γ is achieved with the $(+)$ sign.

When $h = 1$ and $\rho = 0.25$ numerical values are taken as an example, the smallest x value that makes $f_{\pm}$ function minimum is shown in Figure 3.1. Note that the smallest x that makes $f_{\pm}$ zero is obtained for $(+)$ sign and at $x = 0.2882$. Therefore $\gamma_{opt} = \ 0.2882\sqrt{0.25} = 0.1441$ is obtained.

In the figures below, solution for $f_{\pm}$ is shown for different values of h and ρ. It is seen that in all the cases the x value that makes $f_{\pm}$ smallest always obtained with $(+)$ sign.

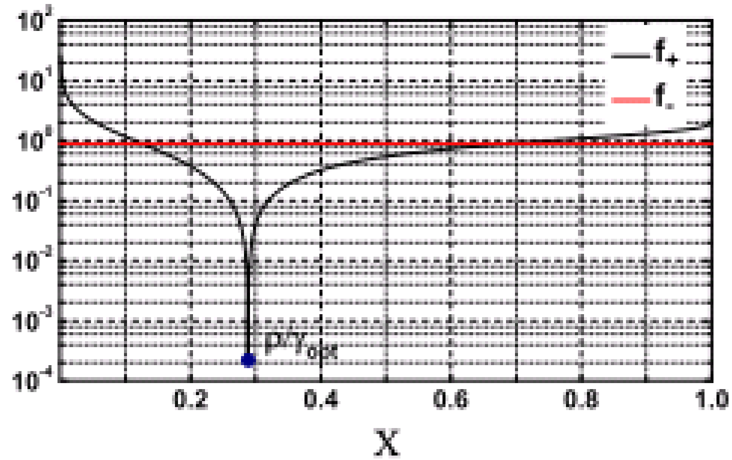

Figure 3.1: $f_{\pm}$ vs. x values for $h = 1$ and $\rho = 0.25$

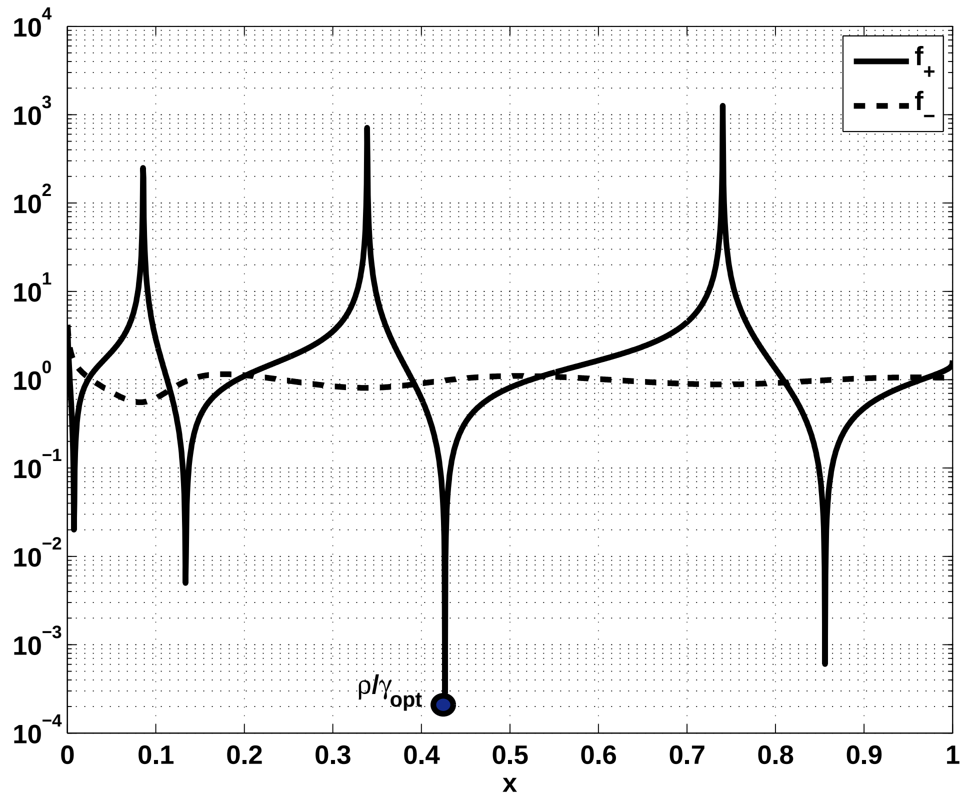

Figure 3.2: $f_{\pm}$ vs. x values for $h = 10$ and $\rho = 0.25$

Note that in Figure 3.2, instead of the smallest value of x leading to a zero in f_+ or f_- , the indicated value is taken, which will result in instability of the closed loop system with related H_∞ controller. This will be later shown in Chapter 4.

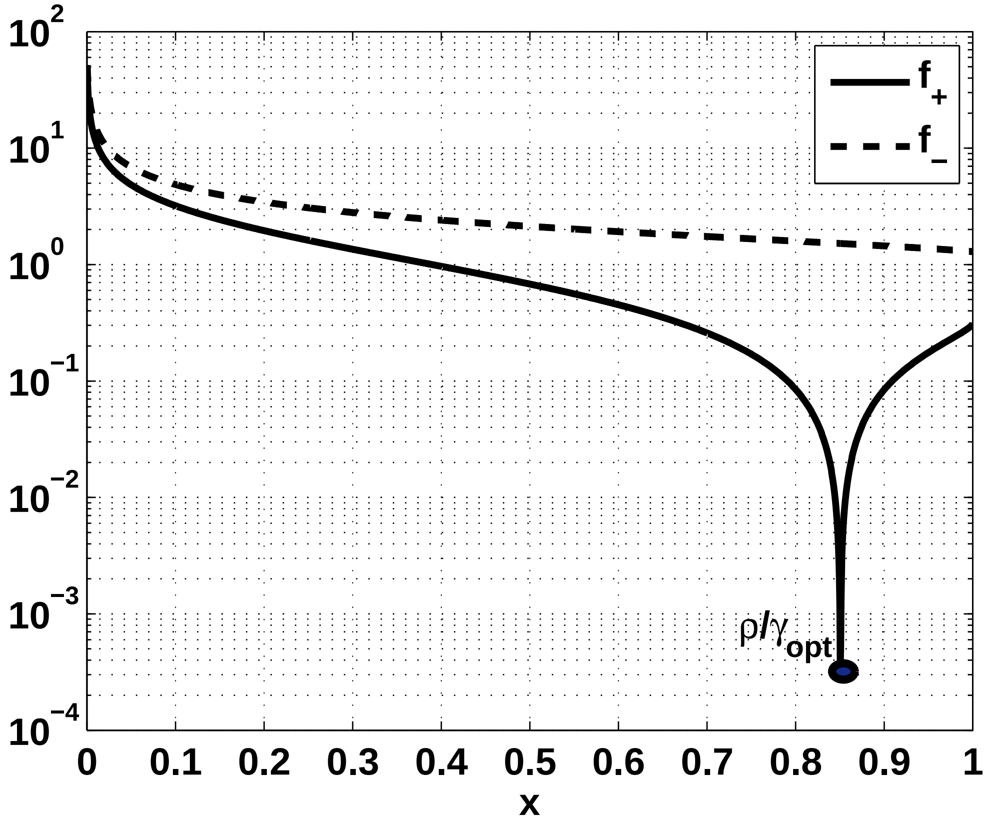

Figure 3.3: $f_{\pm}$ vs. x values for $h = 1$ and $\rho = 10$

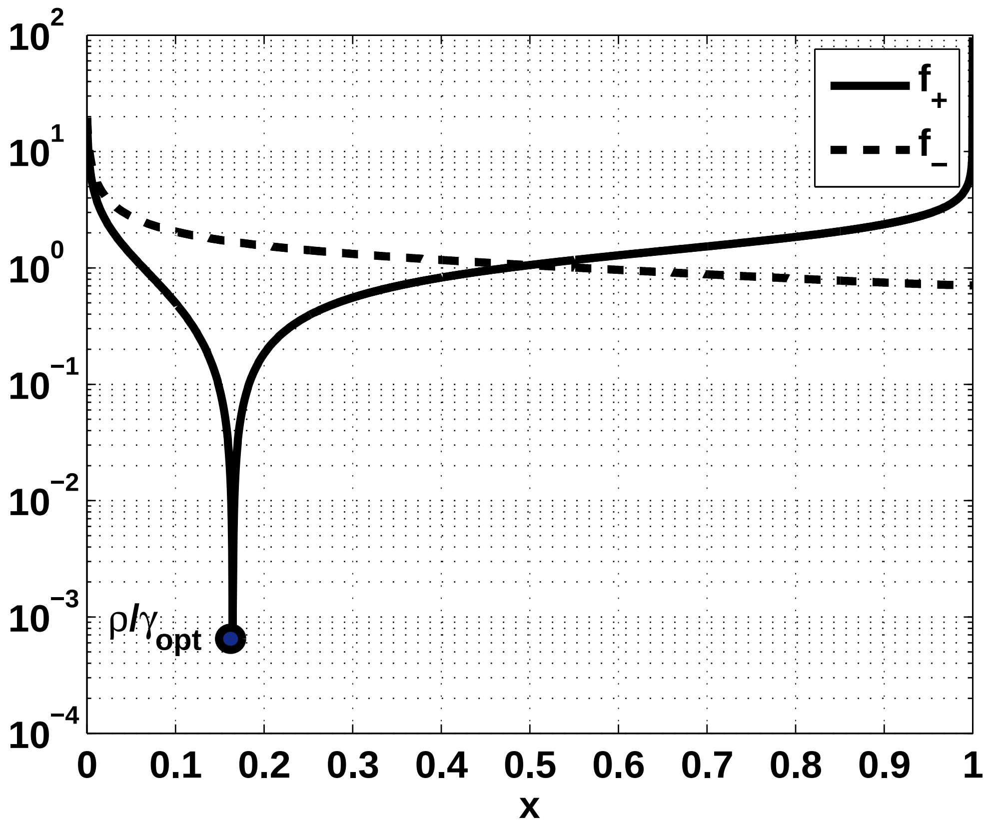

Figure 3.4: $f_{\pm}$ vs. x values for $h = 10$ and $\rho = 10$

Taking the (+) sign in equation 3.31 gives L(s) as equation 3.48 .

$$L\left(s\right) = \frac{1 - \varphi s}{1 + \varphi s} = \frac{1 - \frac{\varphi}{\sqrt{\rho}} s \sqrt{\rho}}{1 + \frac{\varphi}{\sqrt{\rho}} s \sqrt{\rho}} \tag{3.48}$$

It turns out that in our case $\varphi < 0$, and hence

$$L^{-1}(s) = \frac{1 + \frac{\varphi}{\sqrt{\rho}} s \sqrt{\rho}}{1 - \frac{\varphi}{\sqrt{\rho}} s \sqrt{\rho}} \tag{3.49}$$

is stable. Therefore resulting optimal filter is also stable as proven below in equations 3.50 - 3.53.

The optimal filter is given by

$$Q\left(s\right)=\frac{1}{\frac{\gamma s^{2}\ \left(\rho s^{2}+a_{\gamma}s+\sqrt{1-\frac{\rho^{2}}{\gamma^{2}}}\right)L^{-1}(s)+e^{-hs}}{(1-\gamma^{2}s^{4})}} \tag{3.50}$$

which can be rewritten as following equations.

$$Q(s)=\left(\frac{-\gamma s^{2}\ \left(\rho s^{2}+a_{\gamma}s+\sqrt{1-\frac{\rho^{2}}{\gamma^{2}}}\right)}{(1-\gamma^{2}s^{4})}+\frac{\frac{2}{1-\varphi s}\gamma s^{2}\left(\rho s^{2}+a_{\gamma}s+\sqrt{1-\frac{\rho^{2}}{\gamma^{2}}}\right)+e^{-hs}}{(1-\gamma^{2}s^{4})}\right)^{-1} \tag{3.51}$$

$$Q(s)=\left(\frac{\rho}{\gamma}+\frac{-\gamma s^{2}\left(a_{\gamma}s+\sqrt{1-\frac{\rho^{2}}{\gamma^{2}}}\right)-\frac{\rho}{\gamma}}{(1-\gamma^{2}s^{4})}+\frac{\frac{2}{1-\varphi s}\gamma s^{2}\left(\rho s^{2}+a_{\gamma}s+\sqrt{1-\frac{\rho^{2}}{\gamma^{2}}}\right)+e^{-hs}}{(1-\gamma^{2}s^{4})}\right)^{-1} \tag{3.52}$$

$$Q\left(s\right)=\frac{\gamma}{\rho}(1-R_{Q}(s))^{-1} \tag{3.53}$$

where $R_Q(s)$ is defined as

$$R_{Q}\left(s\right)=\frac{1+\frac{\gamma^{2}}{\rho}s^{2}\left(a_{\gamma}s+\sqrt{1-\frac{\rho^{2}}{\gamma^{2}}}\right)-\frac{2}{1-\varphi s}\left(\frac{\gamma^{2}}{\rho}s^{2}\left(\rho s^{2}+a_{\gamma}s+\sqrt{1-\frac{\rho^{2}}{\gamma^{2}}}\right)\right)-\frac{\gamma}{\rho}e^{-hs}}{1-\gamma^{2}s^{4}}. \tag{3.54}$$

From the Nyquist graph of $R_Q(s)$, it can be shown that Q is stable.

3.2 H_2 Optimal Filter Design

The H_2 norm computation in the mixed sensitivity minimization problem described in previous section can be performed as in equation 3.55.

$$\gamma_{Q}:=\left\|\begin{bmatrix} W_{1}(s)\left(1-e^{-hs}Q(s)\right)\\ -\rho Q(s)\end{bmatrix}\right\|_{2}=\left\|\begin{bmatrix} W_{1}\\ 0\end{bmatrix}-\begin{bmatrix} W_{1}e^{-hs}\\ \rho\end{bmatrix}Q\right\|_{2}=\left\|\begin{bmatrix} W_{1}\\ 0\end{bmatrix}-\begin{bmatrix} W_{1}\\ \rho\end{bmatrix}e^{-hs}Q\right\|_{2} \tag{3.55}$$

If a function G is defined as equation 3.56

$$W_1{}^* \ W_1 + \ \rho^2 = G^*G \tag{3.56}$$

replacing W_1 in equation 3.56 yields equation 3.57 below.

$$\frac{1}{s^4} + \ \rho^2 = \ G^*G \ \rightarrow \ G^*G = \frac{1+ \ \rho^2 s^4}{s^4} \tag{3.57}$$

From there, following a set of equation manipulations as in equations 3.58 and 3.59.

$$1+ \ \rho^2 s^4 = \left(\rho s^2 + bs + 1\right)\left(\rho s^2 - bs + 1\right) = \ \rho^2 s^4 + \ \left(2\rho - b^2\right) s^2 + 1 \tag{3.58}$$

$$b = \sqrt{2\rho} \tag{3.59}$$

Function G is obtained as below.

$$G = \frac{1+\sqrt{2\rho}s + \rho s^2}{s^2} \tag{3.60}$$

Equation 3.55 can be rewritten as below:

$$\gamma_Q = \left\| \begin{bmatrix} w_1 \\ 0 \end{bmatrix} = \begin{bmatrix} w_1 G^{-1} \\ \rho G^{-1} \end{bmatrix} G e^{-hs} Q(s) \right\|_2 \tag{3.61}$$

And after using some matrix manipulations as in equation 3.62,

$$\begin{bmatrix} W^* G^{-*} & \rho G^{-*} \\ -\rho G^{-1} & W G^{-1} \end{bmatrix} \begin{bmatrix} W G^{-1} & -\rho G^{-*} \\ \rho G^{-1} & W^* G^{-*} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.62}$$

we obtain the mixed sensitivity cost function as in equation 3.63 below for H_2 case.

$$\left\| \begin{bmatrix} W^* W G^{-*} - G e^{-hs} Q \\ -\rho G^{-1} W \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} G(W^* W G^{-*} G^{-1} - e^{-hs} Q) \\ -\rho G^{-1} W \end{bmatrix} \right\|_2 \tag{3.63}$$

Note that

$$-\rho G^{-1} W = \ - \ \rho \ \frac{1}{\rho s^2 + \sqrt{2\rho}s + 1} \ \in \ H_2 \tag{3.64}$$

and

$$W^{*}G^{-*} = \frac{1}{\rho s^{2} - \sqrt{2\rho} s + 1} \tag{3.65}$$

In order to find a solution for optimal filter Q, we need to solve equation 3.66

$$\inf_{Q\epsilon H_{\infty}\cap H_{2}} \qquad \left\| \frac{1}{s^{2}} \left(\frac{1}{\rho s^{2}-\sqrt{2\rho}s+1} - \left(\rho s^{2} + \sqrt{2\rho} s + 1\right) e^{-hs} Q(s) \right) \right\|_{2} \tag{3.66}$$

Which is same as Equation 3.67

$$\inf_{Q\epsilon H_{\infty}\ \cap H_{2}} \qquad \left\| \left(\frac{\left(\rho s^{2} + \sqrt{2\rho} s + 1\right)}{s^{2}} \right) \left(\frac{e^{hs}}{\rho^{2} s^{4} + 1} - Q(s) \right) \right\|_{2} \tag{3.67}$$

Optimal filter Q(s) can be formulated as:

$$Q\left(s\right) = \frac{\left(s+\epsilon\right)^{2}}{\rho s^{2} + \sqrt{2\rho} s + 1} Q_{1}(s) \tag{3.68}$$

Then the cost function becomes as equation 3.69.

$$\left\| \frac{1}{\left(s+\epsilon\right)^{2}} \left(\frac{1}{\rho s^{2} - \sqrt{2\rho} s + 1} \right) - e^{-hs} Q_{1}(s) \right\|_{2} \tag{3.69}$$

Note that first term in equation 3.69 is the projection on to H_2 space.

If cost function is decomposed into its partial fractions , it follows equations 3.70, 3.71, 3.72, 3.73, 3.74, 3.75

$$\frac{A\left(s+\epsilon\right)}{\left(s+\epsilon\right)^{2}} + \frac{B}{\left(s+\epsilon\right)^{2}} + \frac{C}{s-r_{1}} + \frac{\bar{C}}{s+\bar{r_{1}}} = \frac{1}{\left(s+\epsilon\right)^{2}\left(\rho s^{2} - \sqrt{2\rho} s + 1\right)} \tag{3.70}$$

$$B \cong 1 \tag{3.71}$$

$$A\left(s+\epsilon\right) + B + \frac{C(s+\epsilon)^{2}}{s-r_{1}} + \frac{\bar{C}(s+\epsilon)^{2}}{s-\bar{r_{1}}} = \frac{1}{\rho s^{2} - \sqrt{2\rho} s + 1} \tag{3.72}$$

$$A = \frac{d}{ds}\left(\frac{1}{\rho s^2 - \sqrt{2\rho}s + 1}\right)\Bigg|_{s=\epsilon} = \left.\frac{-\left(2\rho s - \sqrt{2\rho}\right)}{\rho s^2 - \sqrt{2\rho}s + 1}\right|_{s=\epsilon} \cong \sqrt{2\rho} \tag{3.73}$$

$$C(sI - A)^{-1}B - e^{-hs}Ce^{Ah}(sI - A)^{-1}B = F(s) \tag{3.74}$$

$$C(sI - A)^{-1}B - e^{-hs}C(sI - A)^{-1}B - e^{-hs}C\left(e^{Ah} - I\right)(sI - A)^{-1}B = F(s) \tag{3.75}$$

As an example, lets consider the system given in equations below:

$$C = \begin{bmatrix} 1 & \sqrt{2\rho} \end{bmatrix} \tag{3.76}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.77}$$

$$B = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.78}$$

Then

$$C(sI - A)^{-1}B = \frac{1 + \sqrt{2\rho}s}{s^2} \tag{3.79}$$

is obtained and e^{Ah} can be written as Taylor series expansion in equation 3.80.

$$e^{Ah} = I + Ah + \frac{A^2h^2}{2!} + \ \ldots + H.O.T. \tag{3.80}$$

Calculating the unknown terms as in equations 3.81, 3.82, 3.83, 3.84, 3.85

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow e^{Ah} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \tag{3.81}$$

$$\left(e^{Ah}-I\right)\cong\begin{bmatrix}0 & h\\ 0 & 0\end{bmatrix} \tag{3.82}$$

$$(sI-A)^{-1}B\cong\begin{bmatrix}1\\ s\end{bmatrix}\frac{1}{s^2} \tag{3.83}$$

$$C\left(e^{Ah}-I\right)\cong\begin{bmatrix}0 & h\end{bmatrix} \tag{3.84}$$

$$C\left(e^{Ah}-I\right)(sI-A)^{-1}B\cong\frac{hs}{s^2} \tag{3.85}$$

and replacing them in place, yields F(s) as equation 3.86.

$$F\left(s\right)=\left(\frac{\sqrt{2\rho}s+1}{s^2}\right)-\frac{e^{-hs}\left(\left(\sqrt{2\rho}s+1\right)+hs\right)}{s^2} \tag{3.86}$$

Equation 3.86 can be rewritten as equation 3.87.

$$F\left(s\right)=\left(\frac{\sqrt{2\rho}s+1}{s^2}\right)-e^{-hs}\frac{\left(\sqrt{2\rho}s+h\right)s+1}{s^2} \tag{3.87}$$

Then cost function becomes

$$\gamma_Q=\quad\left\|\frac{\sqrt{2\rho}}{s+\epsilon}+\frac{1}{(s+\epsilon)^2}+\frac{C}{s-r_1}+\frac{\bar{C}}{s-\bar{r_1}}-e^{-hs}Q_1(s)\right\|_2 \tag{3.88}$$

with

$$Q_1\left(s\right)=\mathcal{L}\left\{q_1(t)\right\}. \tag{3.89}$$

Note that

$$e^{-hs}Q_1\left(s\right)=\frac{\sqrt{2\rho}}{s}+\frac{1}{s^2}-\left[FIR\right]_{[0,h]} \tag{3.90}$$

where $[FIR]_{[0,h]}$ is a stable function whose impulse response is non-zero only on $[0,h]$.

Optimal filter becomes :

$$Q\left(s\right)=\left(\frac{s^2}{\rho s^2+\sqrt{2\rho}s+1}\right)\left(\frac{\sqrt{2\rho}s+1}{s^2}-\ [FIR]\right)e^{hs} \tag{3.91}$$

$$Q(s)=\left(\left(\frac{\sqrt{2\rho}s+1}{\rho s^2+\sqrt{2\rho}s+1}\right)-\frac{s^2}{\rho s^2+\sqrt{2\rho}s+1}\,[FIR]\right)e^{hs} \tag{3.92}$$

H_2 solution to this problem is obtained by the following set of equations.

$$Q\left(s\right)=\left(\frac{1+(h+\sqrt{2\rho})s}{\rho s^2+\sqrt{2\rho}s+1}\right)=1+\frac{s(h-\rho s)}{\rho s^2+\sqrt{2\rho}s+1} \tag{3.93}$$

Note that $Q(0)=0$ as expected and

$$\frac{1-e^{-hs}Q(s)}{s^2}\ =\ \frac{1-\left(1-hs+\frac{(hs)^2}{2}+\ \ldots\right)Q(s)}{s^2} \tag{3.94}$$

.

Expanding right hand side of equation 3.94 yields

$$\frac{1-e^{-hs}Q(s)}{s^2}=\frac{(1-Q\left(s\right))+hsQ\left(s\right)-\frac{(hs)^2}{2}Q\left(s\right)+H.O.T}{s^2} \tag{3.95}$$

which can be rewritten as

$$\frac{1-e^{-hs}Q(s)}{s^2}=\frac{1}{s^2}\left(\frac{\rho s^2+\sqrt{2\rho}s+1-1-\left(h+\sqrt{2\rho}\right)s}{\rho s^2+\sqrt{2\rho}s+1}\right)+\frac{hQ(s)}{s}-\ \frac{h^2}{2!}Q\left(s\right)+H.O.T \tag{3.96}$$

From 3.96 , it follows that

$$\frac{1}{s^2}\left(1-e^{-hs}Q\left(s\right)\right)\Big|_{s=0}\ =\ \rho+h\left(h+\sqrt{2\rho}\right)-\frac{h^2}{2}\ =\ \rho+h\sqrt{2\rho}+\frac{h^2}{2} \tag{3.97}$$

Thus H_2 optimal filter $Q(s)$ can be implemented as in equation 3.98

$$Q\left(s\right)=\frac{1}{\rho}\left(\frac{1+\left(h+\sqrt{2\rho}\right)s}{s^{2}+\sqrt{\frac{2}{\rho}}s+\frac{1}{\rho}}\right)=C(sI-A)^{-1}B \tag{3.98}$$

where

$$C=\begin{bmatrix} \frac{1}{\rho} \\ \left(\frac{h}{\rho}+\sqrt{\frac{2}{\rho}}\right) \end{bmatrix} \tag{3.99}$$

$$A=\begin{bmatrix} 0 & 1 \\ -\frac{1}{\rho} & -\sqrt{\frac{2}{\rho}} \end{bmatrix} \tag{3.100}$$

$$B=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.101}$$

The system equations using filter Q(s) can be represented as in as in equations 3.102 - 3.106 .

$$z\left(t\right)=x_{p}(t+h_{1}) \tag{3.102}$$

$$y\left(t\right)=\ x_{p}\left(t-h_{0}\right)+\ \rho w_{0} \tag{3.103}$$

$$\dot{x}_{p}(t)\ =x_{v}(t) \tag{3.104}$$

$$\dot{x}_{v}=w_{p}(t) \tag{3.105}$$

$$\frac{1}{\rho} y(t) = \frac{1}{\rho} x_p (t - h_0) + w_0 \tag{3.106}$$

State estimation equations can be written as in equations 3.107 - 3.109.

$$\dot{\hat{x}}_1 (t) = \hat{x}_2 (t) \tag{3.107}$$

$$\dot{\hat{x}}_2 (t) = -\frac{1}{\rho} \hat{x}_1 (t) - \sqrt{\frac{2}{\rho}} \hat{x}_2 (t) + y(t) \tag{3.108}$$

$$\hat{z} (t) = \frac{1}{\rho} \hat{x}_1 (t) + \left(\frac{h}{\rho} + \sqrt{\frac{2}{\rho}} \right) \hat{x}_2 (t) \tag{3.109}$$

An alternative realization can be made through state equations 3.110 - 3.112 .

$$\dot{\hat{x}}_1 (t) = \hat{x}_2 (t) \tag{3.110}$$

$$\dot{\hat{x}}_2 (t) = -\frac{1}{\rho} \hat{x}_1 (t) - \sqrt{\frac{2}{\rho}} \hat{x}_2 (t) + \frac{1}{\rho} y(t) \tag{3.111}$$

$$\hat{z} (t) = \hat{x}_1 (t) + \left(h + \sqrt{2\rho} \right) \hat{x}_2 (t) \tag{3.112}$$

Defining

$$\frac{1}{\rho} \hat{y} (t) = \frac{1}{\rho} \hat{x}_1 (t) + \sqrt{\frac{2}{\rho}} \hat{x}_2 (t) \tag{3.113}$$

yields equations 3.114 and 3.115 .

$$\hat{y} (t) = \hat{x}_1 (t) + \sqrt{2\rho}\, \hat{x}_2 (t) \tag{3.114}$$

$$\hat{z}(t) = \hat{y}(t) + h\hat{x}_2(t) \tag{3.115}$$

Then, state equations for this case can be written as in equations 3.116 - 3.117

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\rho} \end{bmatrix} (y(t) - \hat{y}(t)) \tag{3.116}$$

$$z(t) = x_p(t + h_1) = \ y(t+h) - \rho w_0(t+h) \tag{3.117}$$

Then equations 3.118 - 3.121 are obtained for state estimation.

$$z(t-h) = y(t) - \rho w_0(t) \tag{3.118}$$

$$\hat{z}(t) = \ \hat{y}(t) + h\hat{x}_1(t) \tag{3.119}$$

$$z(t) = y(t+h) - \rho w_0(t+h) \tag{3.120}$$

$$\hat{z}(t) = \hat{y}(t) + h\hat{x}_2(t) \tag{3.121}$$

Chapter 4

Simulation Results

For the parameter values $h = 1$ and $\rho = 0.25$, H_∞ and H_2 optimal filters are calculated as described in previous section. A process noise that provides a short duration manoeuvre is applied to the system. On the other hand as a measurement noise, low power white noise is applied. Figure 4.1 - Figure 4.4 shows the estimation results for these inputs. Consequently it is observed that H_∞ optimal filter results in less tracking error compared to H_2optimal filter.

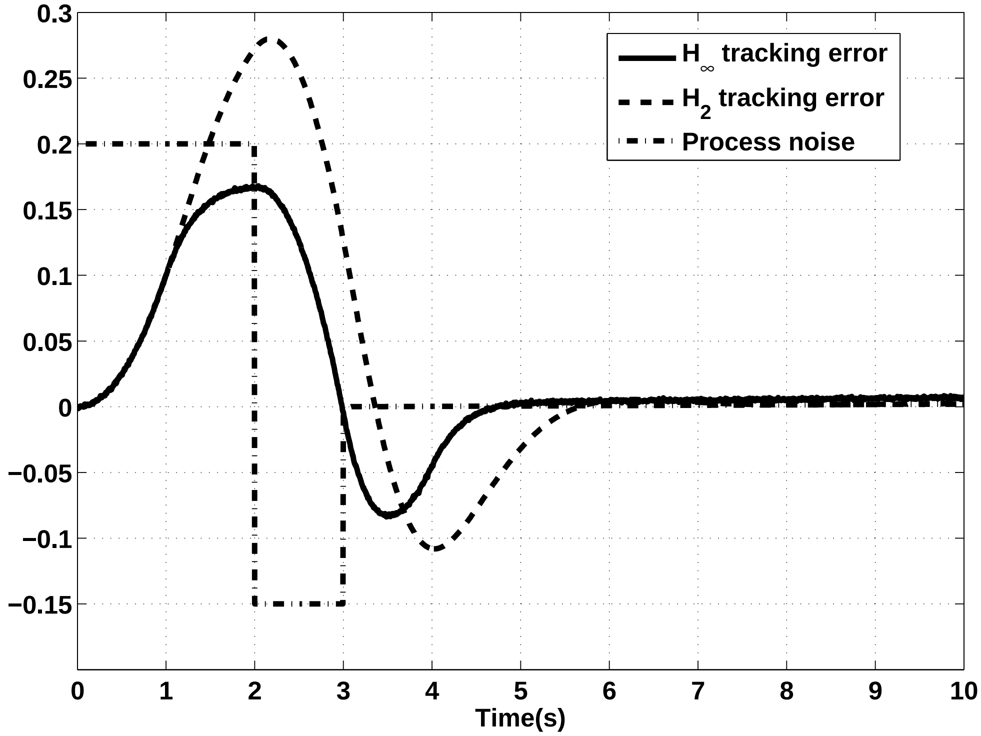

Figure 4.1: Tracking error for $h = 1$ and $\rho = 0.25$

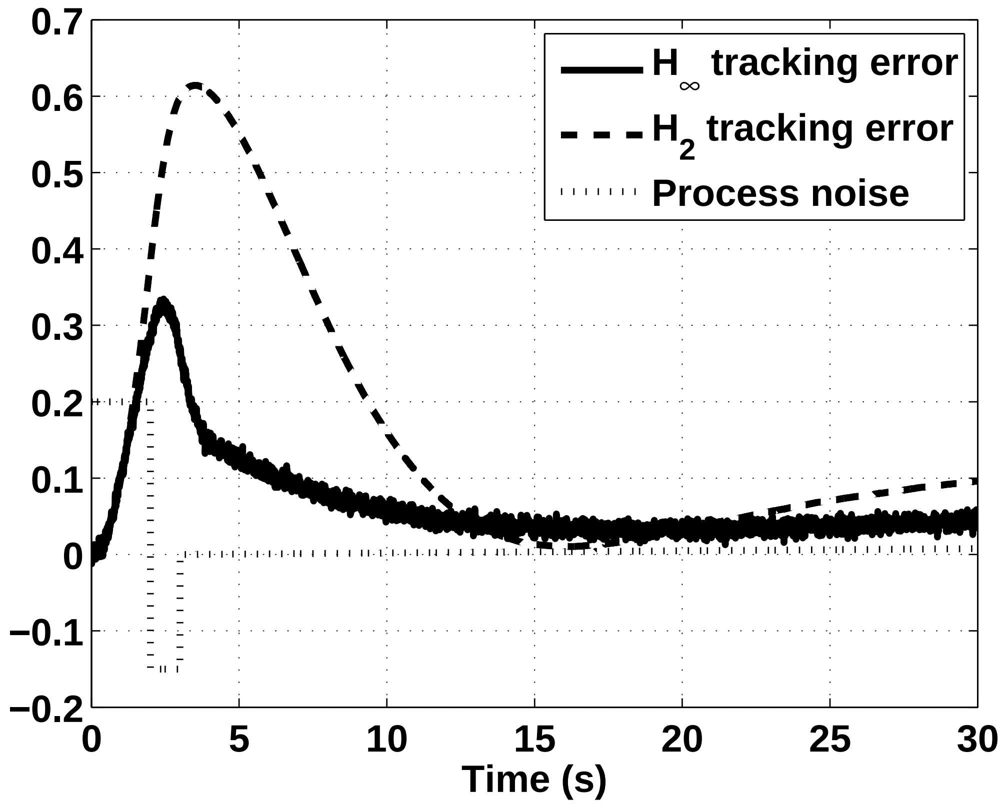

Figure 4.2: Tracking error for $h = 1$ and $\rho = 10$

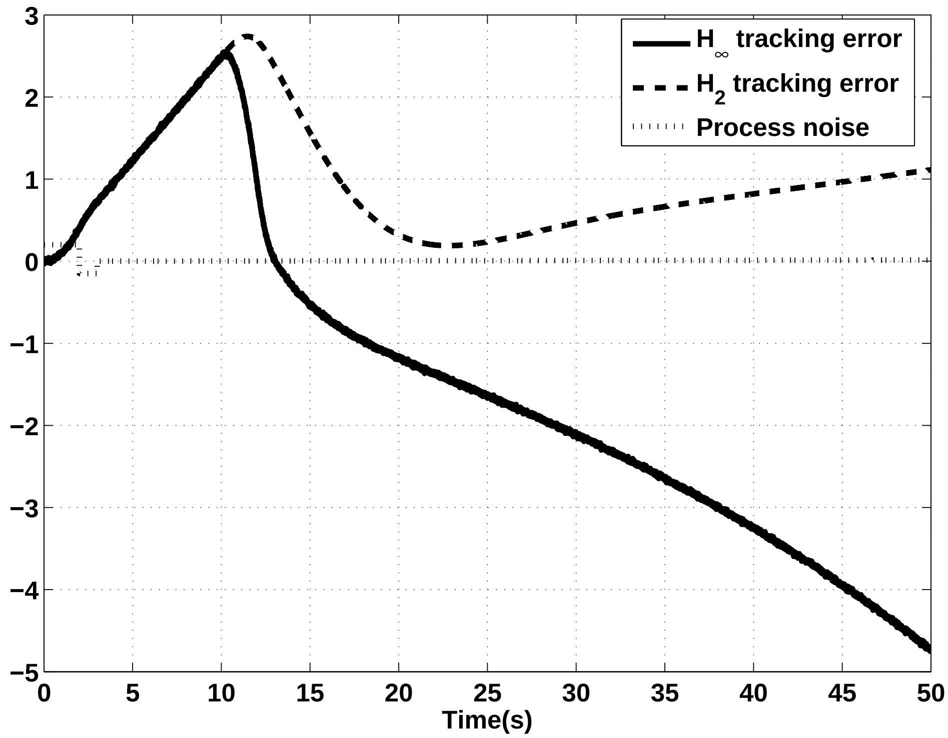

Figure 4.3: Tracking error for $h = 10$ and $\rho = 10$

Note that, for this simulation case instead of the smallest x solution in Figure 3.4, another x value was taken as a solution. Since it was not the optimal solution, simulations show that this non-optimal solution makes the closed loop system unstable.

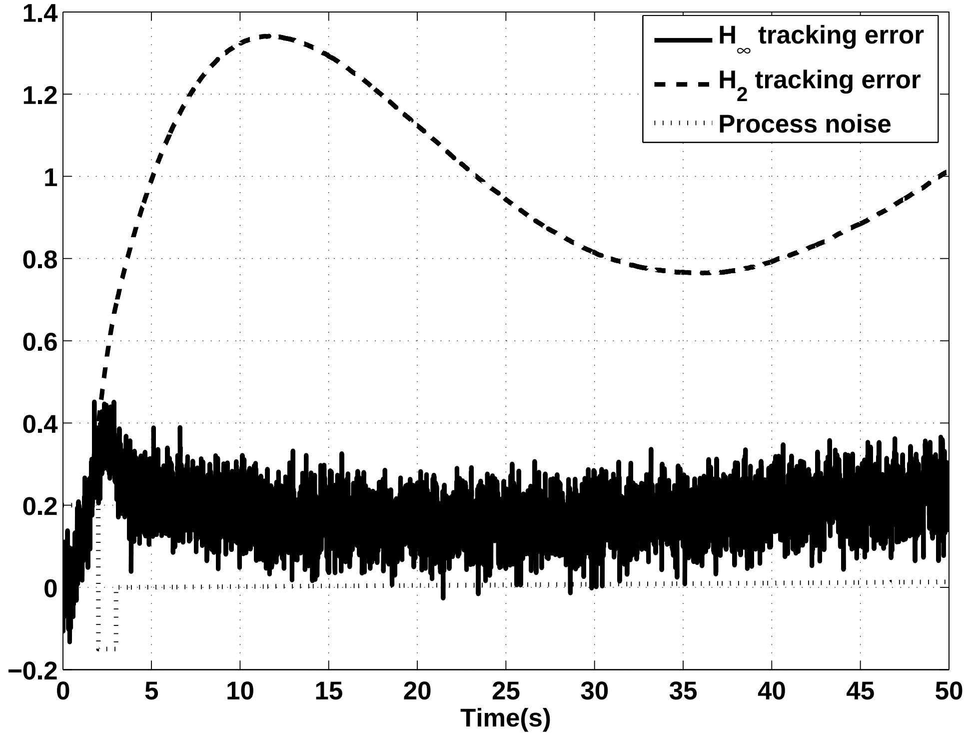

Figure 4.4: Tracking error for $h = 1$ and $\rho = 100$

Bode plot of error system obtained from the H_∞ optimal and H_2 optimal filter designed for $h = 1$ and $\rho = 0.25$ values is given in Figure 4.5.

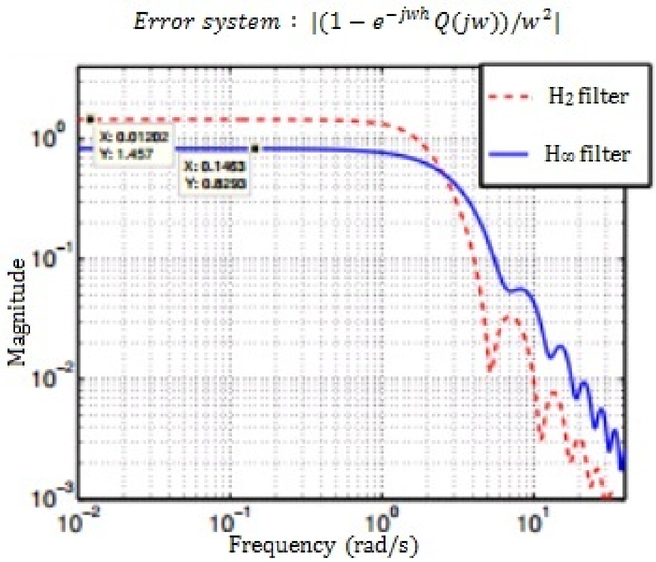

Figure 4.5: Bode plot of error system for $h = 1$ and $\rho = 0.25$

As it can be seen from this figure, for low frequencies the gain in the H_2 optimal filter is 1.75 times worse than the H_∞ optimal filter. However for frequencies higher than 2.5 rad/s, the error corresponding to the disturbance effect is more when H_∞ optimal filter is used. As a result, for disturbance $w_p(t)$ with low frequency ingredient, one should choose to use H_∞ optimal filter. If disturbance effect is a high frequency ingredient signal, one should choose to use H_2 optimal filter.

Chapter 5

Conclusion

In this study, targets which have manoeuvre dynamics modelled in continuous time are discussed. Target tracking under time delayed observations was previously discussed in [42] and in this study it is shown that simplifications can be made on the H_∞ optimal filter design proposed in [42]. The filter obtained here requires the solution of a scalar equation derived from the analysis of problem data. Stable realization of the obtained filter is based on the feedback connection of a first order sub-system and a FIR- filter in a certain way. On the other side, optimal filter in [42] consists of second order sub-blocks whose parameters can only be calculated with numeric methods. Therefore the FIR filter structure of it is more complex than the filter structure obtained in this study. Results obtained from the transformation to target tracking problem show that proposed H_∞ filter effectively suppresses the process noise with low frequency signal content. On the other hand, H_2 optimal filtering is more effective for suppressing measurement noise with high frequency content.

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