

ON MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ CONTROLLER DESIGN FOR SYSTEMS WITH TIME DELAY

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On Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Controller Design for Systems With Time Delay

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.



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ABSTRACT

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This study discusses mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller design problem for uncertain Linear Time-Invariant (LTI) systems with a time delay. More precisely, the goal is to find an internally stabilizing controller that minimizes the $\mathcal{H}_2$ performance measure subject to robust stability condition which bounds the $\mathcal{H}_\infty$ norm of the weighted closed loop transfer function. Two different methods are used to find the optimal $\mathcal{H}_2$ controller. The first method is inspired from the $\mathcal{H}_\infty$ control design to reduce the two block problem into one block. Second method, however is the implementation of the Mirkin's formula. These methods are compared and their pros and cons are discussed. The key point in the optimal controller is that, it is in Smith predictor form that includes an internal feedback in the form of an Finite Impulse Response (FIR) filter. These types of controllers are easy to implement and programmed in physical systems. A case study is considered to show the exact way to controller design. The simulation results and the effects of delay term on the performance measure are also provided.

Keywords: Time-delay, Robust Control, Smith Predictor Controller, Infinite Dimensional System.

ÖZET

ZAMAN GECİKMELİ SİSTEMLER İÇİN KARIŞIK $\mathcal{H}_2/\mathcal{H}_\infty$ KONTROLÖR TASARIMI

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Bu çalışmada, belirsiz Doğrusal Zamanla Değişmeyen zaman gecikmeli sistemler için, karmaşık $\mathcal{H}_2/\mathcal{H}_\infty$ kontrolcü tasarımı problemi konusu işlenmiştir. Burada hedef, ağırlıklı kapalı döngü transfer fonksiyonunun $\mathcal{H}_\infty$ normunu sınırlayan gürbüz kararlılık şartına bağlı olarak $\mathcal{H}_2$ performans ölçüsünü en aza indiren, sistemin iç kararlılığını sağlayan kontrolcüyü bulmaktır. En uygun $\mathcal{H}_2$ kontrolcünün bulunması için iki farklı yöntem kullanılmıştır. Bu yöntemler karşılaştırılmış ve yöntemlerin avantajları ve dezavantajları tartışılmıştır. En uygun kontrolcüdeki önemli nokta, bu kontrolcünün sonlu dürtü yanıtı süzgeç şeklinde bir dahili geri bildirim içeren Smith öngörücü formunda olmasıdır. Bu türdeki kontrolcüler, fiziksel sistemlere rahatça programlanabilir ve uygulanabilir. Kontrolcü tasarımının tam olarak gösterimi amacıyla örnek bir durum ele alınmıştır. Buna ek olarak, simülasyon sonuçları ve zaman gecikmesinin performans ölçüsü üzerindeki etkilerine yer verilmiştir.

Anahtar sözcükler: Zaman-gecikmeli Sistemler, Gürbüz Kontrol, Smith Kestirim Tabanlı Denetleyici, Sonsuz Boyutlu Sistemler.

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Chapter 1

Introduction

Time delay is present in many physical processes, such as mechanical, electrical, communication as well as biological systems. For instance, in post-surgical therapy, when a patient takes drugs to reduce the effects of bleed after surgery, it is crucial for immune system to react fast enough in order not to cause problem. In this system, the amount and types of drug which act as control action should be designed to make the blood pressure in a certain range. It is clear that, delay is inherent in such systems [1, 2]. The presence of delay in a system, may result in complexity in controlling, oscillations, instability and poor performance [3]. Delay can also be replaced by part of a plant in modeling process [4].

Due to the importance of this problem, this thesis mainly discusses mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller design for systems with time delay. There are various results in the literature for this problem [5, 6, 7], some of them also discuss two block problem and optimal controller design. However, most of them only consider systems without delay. We showed that the time delay problem can be reduced into a delay free setup for which available solutions can be used. We also study the two-block $\mathcal{H}_2$ problem for a class of first order stable system with time delay. For this purpose, classical results such as Nehari's theorem [8] and spectral factorization are used to construct optimal controller. The proposed optimal controller is in Smith predictor form [9], which makes it easy to implement it in real systems. In addition, we show that our proposed controller has the same form as the controller achieved

using the idea of [10].

The primary goal of this work is the design of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller for a class of LTI time delay systems. In order to gain a better understanding of the problem discussed in this work and why it is important, we review some fundamental results.

1.1 $\mathcal{H}_\infty$ Robust Control

Normally, in the conventional control system design, for stability, phase and gain margins are two key factors to show how well the proposed controller works to ensure stability in the presence of model uncertainties. However, neither the uncertainties and the perturbations are quantified nor performance is taken into account in terms of disturbance and noise effects. Also, most of SISO controller design methods cannot be extended to MIMO systems. These are some of the basic limitations of classical control design. One of the successful and widely used methods to tackle these issues is $\mathcal{H}_\infty$ robust control. This type of controllers have the following features:

- Stability is guaranteed.
- Certain level of performance of the controlled system is guaranteed.
- System will remain in stable zone irrespective of uncertainties in the plant or disturbances.

It is important to note that there is always a compromise between performance and robustness. Therefore in control theory applications, $\mathcal{H}_\infty$ methods are used when one is seeking good performance level with robust stability at the same time. This type of control is widely studied in literature; see for instance [11]. This problem is defined as a mathematical optimization problem and various contributions are dedicated to solve this problem in the literature. Some important

methods that studied this problem are small gain theorem, convex optimization, loop shaping, and Linear Matrix Inequality (LMI). Robustness in control problems is equivalent to stability of a system regardless of disturbances and uncertainties in the system, which is the basic usage of controller. Furthermore, once we have a stabilizing controller, we look for some performance objectives. Common performance objectives in control are tracking, noise attenuation, and disturbance rejection. Tracking means that the output should track the reference input signal. And a system has noise attenuation and disturbance rejection when the sensor noise and disturbance does not affect the output. The robust control provides the stability of the control system as long as uncertain parameters and disturbances in the system and modeling errors are defined within some limit [12]. These are the common types of problems considered in literature, and researchers provide optimal and suboptimal solution using different types of controllers.

1.2 $\mathcal{H}_2$ Control

An optimization-based approach to linear feedback control system design using the $\mathcal{H}_2$ norm, or energy of the impulse response, to quantify closed-loop performance, is called the $\mathcal{H}_2$ control problem. This type of controller design is one of the oldest and yet important control design. The $\mathcal{H}_2$ optimization, due to its convex nature, usually leads to a unique optimal solution. State-space methods such as Riccati equations and matrix inequalities are standard solutions for the solution of $\mathcal{H}_2$ optimal control problem. In [13], both regular and singular $\mathcal{H}_2$ problems using Riccati equation and matrix inequalities are presented in a continuous-time setting. Connections to so-called LQR and LQG control problems are also described.

1.3 Time Delay

Consider the signal u acting on a time delay element $\mathcal{D}_h$ in Fig. 1.1:

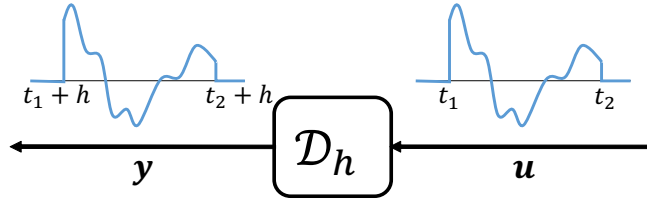


Figure 1.1: Delay element in continuous time signal

$$y = \mathcal{D}_h u : \Rightarrow y(t) = u(t - h) \quad (1.1)$$

In time domain representation, delay is written in the form in (1.1). So, if there is a time delay in a linear system then its transfer function includes the term e^{-sh} where $h > 0$ is the delay time.

Time-delay is present in many practical applications such as process control, biological systems, and economics models [4]. In addition, a higher order system can be replaced by a lower order system with a time-delay, [14]. Therefore, it is important to consider time delays while designing the controllers otherwise time delays may cause instability in the system.

Whether it is present in state variables or in the input, existence of time delay in the overall system may result in poor performance, difficulty in controlling and complexity in system behavior such as oscillations, instability and poor performance. Stability of a closed-loop system can be strongly affected by small delays, although even large time delays can stabilize the other systems. Therefore controllability, observability, robustness, optimization, stability and robust stabilization of time delay systems has been an important issue. The control and stability issues with delay systems are more complex than of the finite dimensional systems. When the delays are not small, the exact delay system model should be used and controllers should be designed on this infinite dimensional system model. However, there are some ways to approximate the delay, in reality they do not work as expected. Two commonly used types of approximations are

Pade approximation and Taylor series expansion as briefly discussed below.

We can write

$$e^{-s} = \frac{e^{-s/2}}{e^{s/2}}.$$

Truncated Taylor expansion of numerator and denominator yields

$$e^{-s} \approx \frac{\sum_{i=0}^n \frac{1}{i!2^i} (-s)^i}{\sum_{i=0}^n \frac{1}{i!2^i} s^i}.$$

For $n = 1$, $n = 2$ and $n = 3$

$$\begin{aligned} n = 1 : \quad e^{-sh} &\approx \frac{1 - \frac{sh}{2}}{1 + \frac{sh}{2}} \\ n = 2 : \quad e^{-sh} &\approx \frac{1 - \frac{sh}{2} + \frac{s^2 h^2}{8}}{1 + \frac{sh}{2} + \frac{s^2 h^2}{8}} \\ n = 3 : \quad e^{-sh} &\approx \frac{1 - \frac{sh}{2} + \frac{s^2 h^2}{8} - \frac{s^3 h^3}{48}}{1 + \frac{sh}{2} + \frac{s^2 h^2}{8} + \frac{s^3 h^3}{48}} \\ &\vdots \end{aligned}$$

Note that, for $n > 4$ it becomes unstable.

General formula for Pade approximation is

$$e^{-s} \approx \frac{\sum_{i=0}^n \frac{(2n-i)!n!}{(2n)!(n-i)!i!} (-s)^i}{\sum_{i=0}^n \frac{(2n-i)!n!}{(2n)!(n-i)!i!} s^i}.$$

This yields

$$\begin{aligned} n = 1 : \quad e^{-sh} &\approx \frac{1 - \frac{sh}{2}}{1 + \frac{sh}{2}} \\ n = 2 : \quad e^{-sh} &\approx \frac{1 - \frac{sh}{2} + \frac{s^2 h^2}{12}}{1 + \frac{sh}{2} + \frac{s^2 h^2}{12}} \\ n = 3 : \quad e^{-sh} &\approx \frac{1 - \frac{sh}{2} + \frac{s^2 h^2}{10} - \frac{s^3 h^3}{120}}{1 + \frac{sh}{2} + \frac{s^2 h^2}{10} + \frac{s^3 h^3}{120}} \\ &\vdots \end{aligned}$$

As opposed to Taylor series approximation, Pade approximation is stable for all n .

The Bode plot of the exact system and its second and forth order Pade approximation of $G(s) = \frac{e^{-sh}}{s+1}$ is compared in Fig. 1.2, and its corresponding approximation error in Fig. 1.3.

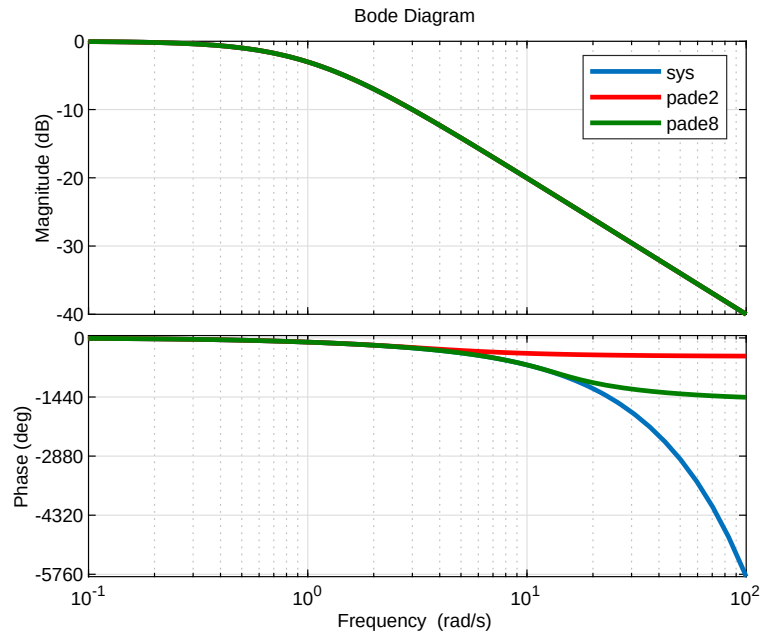


Figure 1.2: Bode plot of $\frac{e^{-sh}}{s+1}$ and it's second and eighth order Pade approximation

1.4 Modeling Uncertainty

Mathematical modeling process of a system is always with assumptions and simplifications, in which a final nominal model is chosen for controller design procedure. However, in robust control theory these uncertainties in modeling are considered in design process. In which, the controller is designed so that the real system will be stable, that is called robust stability. To define it more precisely,

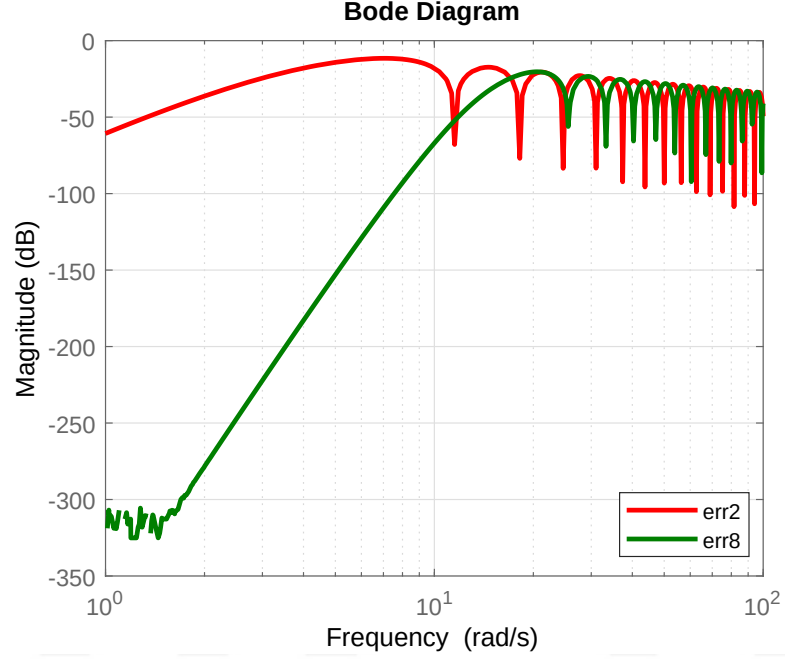


Figure 1.3: Approximation error of second and eighth order

we need some background on the modeling uncertainty.

In modeling procedure it is possible to keep track of the assumptions and simplifications, so it is possible to derive an uncertainty bound on the nominal plant P_0 , [15]. The uncertainty description is denoted by Δ_P , associated with P_0 . Two common types of uncertainties is described below.

1.4.1 Dynamic Uncertainty Description

Consider the nominal plant transfer function $P_0(s)$, and assume that the true plant $P(s)$ is LTI. Then the modeling uncertainty is

$$\Delta_P(s) = P(s) - P_0(s). \quad (1.2)$$

This uncertainty has the following description:

- (i) the number of poles of $P_0(s) + \Delta_P(s)$ in the right half plane is assumed to be the same as the number of right half plane poles of $P_0(s)$
- (ii) there is a known function $W(s)$ whose magnitude bounds the magnitude of $\Delta_P(s)$ on the imaginary axis :

$$|\Delta_P(j\omega)| < |W(j\omega)| \quad \text{for all } \omega.$$

This type of uncertainty is called *dynamic uncertainty*.

The $\mathcal{H}_\infty$ norm of the largest perturbation before the system is unstable is called the *stability margin* of the system, [16].

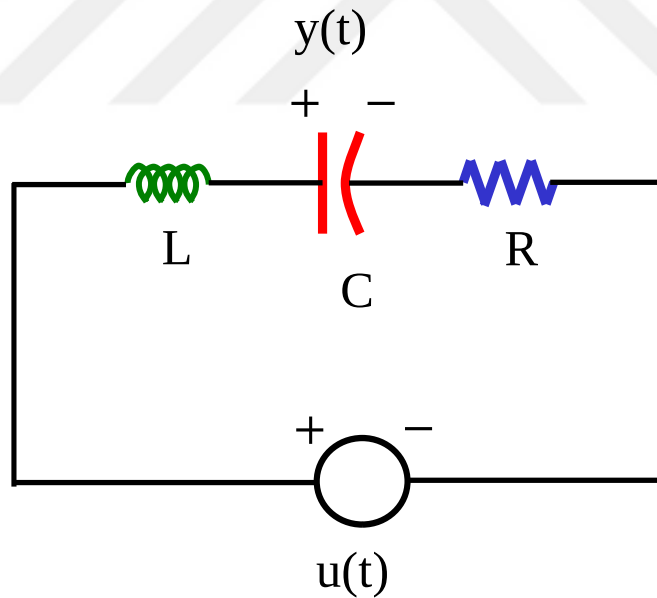


Figure 1.4: RLC Circuit

1.4.2 Parametric Uncertainty Description

Parametric uncertainty occurs in special types of plants which have fixed structure transfer function in which the coefficients of the numerator and denominator are

in a certain range of change. For instance, consider RLC circuit or spring damper system. In this case the true plant $P(s)$ and nominal plant $P_0(s)$ are

$$P(s) = \frac{1}{LCs^2 + RCs + 1}, \quad P_0(s) = \frac{1}{L_0C_0s^2 + R_0C_0s + 1}$$

where, R_0, L_0, C_0 are the nominal values of R, L, C , respectively. Thus, uncertainties in these physical parameter of the real system, appears as uncertainty in coefficients in the transfer function. These types of uncertainty are called parametric uncertainty. They can be transformed into dynamic uncertainty with some conservatism [15], using $\Delta_P(s) = P(s) - P_0(s)$ for all values of R, L and C .

1.5 Literature Review

For systems with time delay, it is crucial to take into account the exact delay in control design, actually if it is ignored in most cases it causes poor performance or even it leads to instability. However, it is difficult to take the exact delay term in design process due to the complexity of analysis of such systems. Therefore, specific methods and tools are under consideration in the literature to satisfactorily take into account the presence of delay at the design stage of the control system.

Time delay in control systems has two major drawbacks. Firstly, it renders control system analysis and design difficult. Secondly, time-delay makes it difficult to achieve the performance specifications [17]. In the literature, time-delay compensators are usually employed with classical PI or PID controllers to improve the performance of the feedback system [18]. The Smith Predictor is one of the first and commonly used time-delay compensator to improve the performance of closed-loop system.

In [19] there are some results on this problem using PID types of controllers, which tunes the parameters to achieve the optimal parameter.

Since the controller structure is fixed, by taking the $\mathcal{H}_\infty$ norm of the closed loop transfer function. The optimization over the free parameter can be done to obtain

a non-conservative result [20].

Especially, there are many numerical methods for designing low order controllers for systems with time delay, see [21, 22, 23, 18]. In [24] and [25], there are communication networks application. In [26] and [27], they extend the method in [20] to cover fractional order systems with time delay.

Stability is a very basic issue in control theory. In [28], an uncertain time-varying delay system is studied to find the robust stability condition. They construct a parameter-dependent Lyapunov functional with the condition on the derivative of the Lyapunov functional rather than on the derivative of the state. Research on the stability of time delayed systems first began with frequency domain methods, then later included time domain methods as well. Frequency domain methods determine the stability of a system by analyzing the locations of the roots of the systems characteristic equation [29].

The other way is by analyzing the complex Lyapunov matrix function equation [30]. It is shown that stability conditions for time delayed systems are equivalent to robust stability analysis on an uncertain delay free system by the use of scaled small gain lemma with constant delays, although these methods only work for systems with constant delay. The main time domain methods are the Lyapunov-Krasovskii functional and Razumikin function methods which are the most common methods for stability analysis of time delay systems [31, 32, 4, 33, 34].

1.5.1 $\mathcal{H}_\infty$ Control Approach for Time Delay Systems

The $\mathcal{H}_\infty$ control theory was first developed in [35] using small gain theorem, formulating problem of sensitivity minimization as an optimization problem and considering the minimization of the ∞ -norm of the sensitivity function of a single-input-single-output linear feedback system.

In the literature, there are several methods to solve the $\mathcal{H}_\infty$ problem for time delay systems. Especially, when the plant has the form $e^{-hs}P_0(s)$ where P_0 is

a rational SISO plant, optimal control problem is solved in [36]. Particularly, in [37] the issues for the weighted $\mathcal{H}_\infty$ minimization problem is concerned and possible solution to these issues are discussed. It has also been shown in [37] that the weighted $\mathcal{H}_\infty$ minimization problem reveals the difficulty of the distributed nature of the problem, nevertheless certain version of this problem can be solved by simple Nevanlinna-Pick interpolation.

Operator theoretic methods also presented in [38], later in [39] simpler formula for the $\mathcal{H}_\infty$ optimal controller is derived. As another approach to this problem, state-space solution is given in [40, 41]. Notably, [41] solve the weighted $\mathcal{H}_\infty$ minimization problem for MIMO systems using J-spectral factorization method. In addition, the Finite Impulse Response (FIR) structure in their designed controller makes it implementable in physical system. Later, the extended version to multi-delay system is considered in [42]. In [43], controller is developed for the sensitivity minimization problem for pseudorational plants. In [11], they used inner-outer factorization of the plant to obtain controller for general infinite dimensional plants. A simplification and a compact form of this method is presented in [44]. In [45], the optimal $\mathcal{H}_\infty$ performance has been computed and the optimal $\mathcal{H}_\infty$ controller for general SISO time-delay systems using the approach in [11, 44] has been designed. Moreover, SISO plants admitting a coprime-inner/outer factorization has been classified. Finally, conditions to check admissible plants and explicit factorization terms have been given in [45].

In [46], the authors solve for the optimal and suboptimal two block $\mathcal{H}_\infty$ problems for distributed systems with finitely many unstable modes using a system of linear equation. This system of linear equations is generated using the state space realization of the delay free part of the system. This method is more reliable numerically comparing to earlier results.

In [47], an unstable system with a time delay is studied. In their proposed controller design, the authors try to both sensitivity reduction and robustness to certain unmodeled dynamics. They formulate the control objectives as an $\mathcal{H}_\infty$

optimal control problem.

In [48] an outline of stability theory for input-output problems using functional methods is given. In order to derive open loop conditions for the boundedness and continuity of feedback systems without placing restrictions on linearity or time invariance, basic explorations about time delayed systems including topics such as existence and uniqueness of solutions to dynamic equations, stability analysis for trivial solutions etc. is carried out and it laid the foundation of analysis and design of time-delayed systems.

In [49], it is shown that $\mathcal{H}_\infty$ problem requires the solution of two algebraic Riccati. It is also proved that the existence of a controller is guaranteed if and only if the positive definite unique stabilizing solution to two algebraic Riccati equations exists and a spectral radius condition is satisfied.

In [16], another solution to $\mathcal{H}_\infty$ problem is shown as well as how Algebraic Riccati Equations (ARE) are reached. In this article, the history of the relationship between modern optimal control and robust control is briefly reviewed. Robust control came into use after the observed inadequacies of the optimal control. After the acceptance of the controversial parts of the robust control, optimal and robust control theory have been used together.

Later in [50], $\mathcal{H}_\infty$ problem was reduced to Linear Matrix Inequalities(LMI). It is shown that continuous and discrete-time $\mathcal{H}_\infty$ control problems can be solved through elementary manipulations on linear matrix inequalities(LMI). Two new features of this approach is that, a solution can be obtained for both regular and singular problems and an LMI-based parametrization can be made for all $\mathcal{H}_\infty$ suboptimal controllers including reduced order. Rather than usual indefinite Riccati equations, the conditions for the solution includes Riccati inequalities. Alternatively, these solvability conditions can be expressed as a system of three

LMI's.

1.5.2 Mixed Sensitivity Minimization

Mixed sensitivity design of a linear time invariant control system is the problem of shaping the sensitivity function $S(s)$, and complementary sensitivity function $T(s)$ at the same time to achieve the design targets of closed-loop system performance and robustness. Both $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norm optimization may be used for this purpose. Each of them has their own advantages and disadvantages. The $\mathcal{H}_2$ norm optimization of this problem is usually referred as Linear-Quadratic-Gaussian control (LQG) [52] or Linear-Quadratic-Regulation control (LQR) for regulation purposes [53]. For instance, LQG problem in [54], studies low and high frequency shaping and partial pole assignment methods. Basically, in the mixed sensitivity problem, given a plant P and appropriate weights, the aim is to find a controller C such that a desired $\mathcal{H}_\infty$ or $\mathcal{H}_2$ performance is achieved, for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ problem, respectively.

The mixed sensitivity problems that we discuss here are much more difficult than the standard problems available in the literature. The difficulty comes from the time delay element in the system. In fact, in this case we are dealing with infinite dimensional system rather than finite dimensional system. However, the algorithm used to solve the problems is more or less similar to the classical solutions. One algorithm used here include the famous two Riccati equation for the standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem. More detail explanation can be found in [16, 55]. These solutions are widely used and implemented in physical systems. Same as any theorem, there are some assumption on the system, such as certain stabilizability and detectability conditions that need to be satisfied. In addition, the algorithm cannot be used for mixed sensitivity problems with non-proper weighting functions. Thus, only suboptimal solution can be achieved. On the

other hand, there are some other algorithms which are based on polynomial matrix fraction representations or descriptor representations of the generalized plant [56], in which it can deal with more general problems, the problem of this method is that it is not fully reliable for implementations.

In [10], the $\mathcal{H}_2$ solution is expressed by solving two algebraic Riccati equations and an additional one-block problem. Later, [57] addresses they addressed a numerical problem with the modified Smith predictor when the plant has fast stable poles and solves this issue by introducing the unified Smith predictor. See [57] for the derivation of augmented plant comprising of a dead time plant and unified Smith predictor.

1.6 One Block Problem

In these types of problems, the aim is to focus mainly on one objective at a time. This objective is either robustness or performance, which makes it much easier problem to solve compared to mixed sensitivity problem, which is actually two block problem. It is crucial to know the methods for solving one block problem. The importance of this fact, comes from the fact that there is a method called spectral factorization which transforms the two block problem into one block problem. The details of this methods and how it works is presented in chapter 2.

1.7 Organization of the Thesis

The thesis is organized as follows. In Chapter 2, problem definition and preliminary results are given. In Chapter 3, two alternative solution for the $\mathcal{H}_2$ optimal control problem is presented. In Chapter 4, the proposed mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller design method is investigated. To illustrate these features two design

examples are included in Chapter 5, which also include a comparison between methods. Concluding remarks and future works are given in Chapter 6.

1.8 Notation

Throughout, I and 0 denote an identity matrix and a zero matrix of appropriate dimensions respectively. By $\|M\|$ we denote the spectral norm of a matrix M . For a transfer function matrix $G(s)$, the notation $G^*(s)$ stands for its complex conjugate transpose. Also the notations $\|G(s)\|_2$ and $\|G(s)\|_\infty$ are used for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, respectively.

Chapter 2

Problem Definition and Preliminary Results

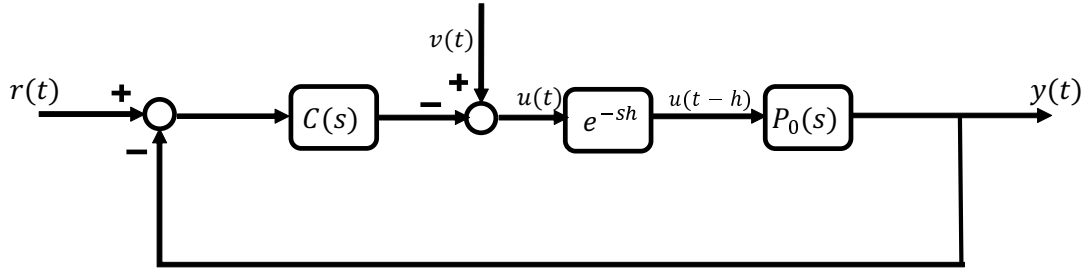


Figure 2.1: Feedback System with Time Delay

This chapter is dedicated to formulation of the two block mixed sensitivity problem. Two different methods are used, in which they end up with same controller. Additionally, suboptimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller problem is also addressed. In both problems, the designed controllers are in Smith predictor form.

Throughout the thesis, our aim is to design a controller for stable dead-time plant P_h ,

$$P_h(s) = P_0(s)e^{-sh}, \quad \text{where } h > 0 \quad (2.1)$$

and $P_0(s)$ is the rational stable plant.

2.1 $\mathcal{H}_2$ Two Block Mixed Sensitivity Optimal Controller Design

The $\mathcal{H}_2$ two block mixed sensitivity minimization problem is defined as finding $C = C_{opt}$ stabilizing (C, P_h) and minimizing

$$\gamma_{opt} = \inf_{(C, P_h)_{stable}} \left\| \begin{bmatrix} W_1(1 + P_h C)^{-1} \\ W_2 P_h C(1 + P_h C)^{-1} \end{bmatrix} \right\|_2 \quad (2.2)$$

where W_1 and W_2 are the sensitivity and uncertainty weights respectively, which are assumed to be known. A good example for these weights would be

$$W_1 = \frac{1}{s + \epsilon_1}, \quad 0 < \epsilon_1 \ll 1$$

and

$$W_2 = ks + \epsilon_2, \quad k > 0, \epsilon_2 > 0.$$

The γ_{opt} is called optimal performance level.

Let's define $S = (1 + P_h C)^{-1}$ as the sensitivity transfer function, and the $T = P_h C(1 + P_h C)^{-1}$ as the complementary transfer function. Then the original problem becomes:

$$\gamma_{opt} = \inf_{(C, P_h)_{stable}} \left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_2. \quad (2.3)$$

This chapter mathematically defines the problem with preliminaries results. These results are used in the construction of optimal control solution in the following chapters.

2.2 Control Problem with Feedback Control

The standard feedback control system considered in this thesis is illustrated in Figure 2.1, where the controller C and plant P are in the form

$$C(s) = \frac{N_c}{D_c}$$

and

$$P(s) = e^{-sh} P_0(s), \quad \text{where} \quad P_0(s) = \frac{N_p}{D_p}$$

in which (N_c, D_c) and (N_p, D_p) are coprime pairs. In our formulation these coprime factors are functions in $\mathcal{H}_\infty$, coprime factors in $\mathcal{H}_\infty$ do not have common zeros in the right half plane (including $+\infty$). The open loop transfer function is

$$G(s) = e^{-sh} G_0(s),$$

where $G_0(s) = P_0(s)C(s)$ corresponds to the delay free case (i.e. $h = 0$).

The response of a delay element e^{-sh} to an input $u(t)$ is simply $u(t - h)$ which is the h units of time delayed version of $u(t)$. Since the $\mathcal{L}_p[0, \infty)$ norm of $u(t - h)$ is equal to the $\mathcal{L}_p[0, \infty)$ norm of $u(t)$. So, by definition the induced system norm of the delay term is unity. Thus, the delay element is a stable system.

The function e^{-sh} is analytic in the entire complex plane and has no poles or zeros. The frequency response of the delay element is determined by the following equation:

$$\begin{aligned} |e^{-j\omega h}| &= 1 \quad \text{for all } \omega \\ \angle e^{-j\omega h} &= -\omega h. \end{aligned}$$

The magnitude of delay element is equal to one which confirms the norm preserving property of it. In addition, it also implies that the transfer function e^{-sh} is all-pass. For $\omega > 0$ the phase is negative and it is linearly decreasing.

The following identities are used to draw the Bode and Nyquist plots of the transfer function $G(s)$:

$$|G(j\omega)| = |G_0(j\omega)| \quad \text{for all } \omega$$

$$\angle G(j\omega) = -\omega h + \angle G_0(j\omega).$$

These facts are useful when dealing with time delay systems.

2.3 Smith predictor

In 1957, Smith, [9] introduced a novel controller structure for systems with delay in which it takes out the delay term from the characteristic equation of closed loop transfer function. This structure has a feedback loop inside the controller. The Smith predictor controller structure in a feedback system is depicted in Fig. 2.2. In a case of a long time delay, since it is not feasible to get the needed information for prediction from output, thus, control input information is used for prediction. This controller feedback loop uses the information of plant $P(s)$, which is assumed to be stable, and time delay. In a case where these data are not available, the estimated data can be used. For example, in such case the approximated plant $\hat{P}(s)$ and time delay $\hat{T}_d$ are used in controller.

In Fig. 2.2, $C(s)$ is the Smith predictor controller of the form

$$C(s) = \frac{C_0(s)}{1 + C_0(s)\hat{P}(s)(1 - e^{-\hat{T}_d s})} \quad (2.4)$$

If we assume that the plant $P(s)$ and time delay T_d are known

$$\hat{P}(s) = P(s) \quad (2.5)$$

$$\hat{T}_d = T_d \quad (2.6)$$

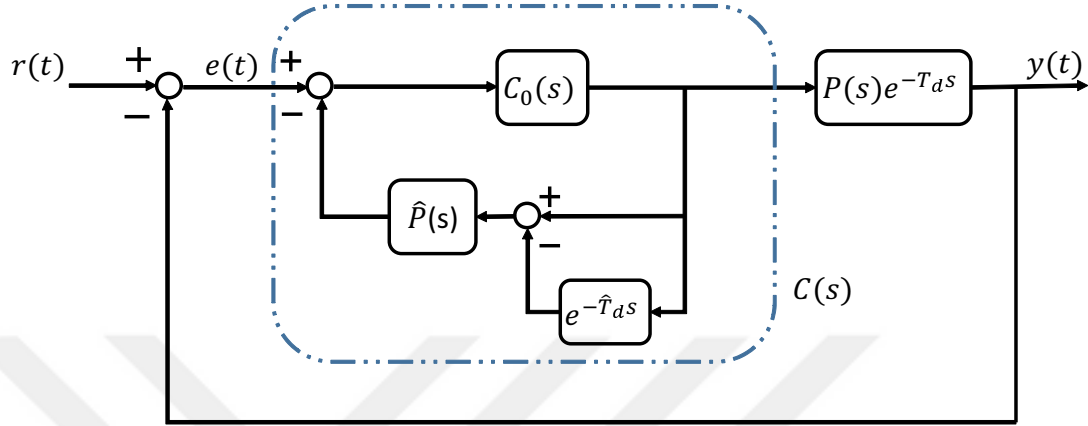


Figure 2.2: Block diagram of Smith predictor controller structure in a feedback system

the closed loop transfer function from input $r(t)$ to $y(t)$ becomes

$$T(s) = \frac{C(s)P(s)e^{-T_d s}}{1 + C(s)P(s)e^{-T_d s}} \quad (2.7)$$

$$= \frac{\frac{C_0(s)}{1 + C_0(s)P(s)(1 - e^{-T_d s})} P(s)e^{-T_d s}}{1 + \frac{C_0(s)}{1 + C_0(s)P(s)(1 - e^{-T_d s})} P(s)e^{-T_d s}} \quad (2.8)$$

$$= \frac{C_0(s)P(s)}{1 + C_0(s)P(s)} e^{-T_d s}. \quad (2.9)$$

Therefore, the delay term is taken out of the closed loop transfer function characteristic equation. This helps to design a controller as if the plant has no delay, in other words, we can assume we are dealing with a system shown in Fig. 2.3, where C_0 is designed to stabilize the non-delayed plant $P(s)$.

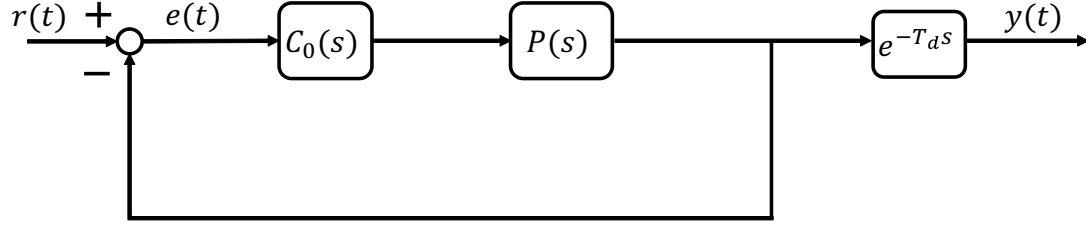


Figure 2.3: Equivalent Block diagram of Smith predictor controller structure in Input-Output view point

2.4 Linear Fractional Transformation

In the literature, there are various types of system representation useful in control systems. Here we present linear fractional transformation. This will be used later, when loop shifting is discussed.

2.4.1 State Space Representation of System

State space representation is used to represent a linear dynamical system. This representation is useful especially when a system has multiple input multiple output. This representation replace n^{th} order differential equation with a single first order matrix differential equation. The state space representation of a system with n states, p inputs, and q outputs is given by two equations :

$$\dot{X}(t) = AX(t) + BU(t) \quad (2.10)$$

$$Y(t) = CX(t) + DU(t) \quad (2.11)$$

where:

- $X(t) \in \mathbb{R}^n$: States Vector
- $U(t) \in \mathbb{R}^{n \times p}$: Input Vector
- $Y(t) \in \mathbb{R}^q$: Output Vector.

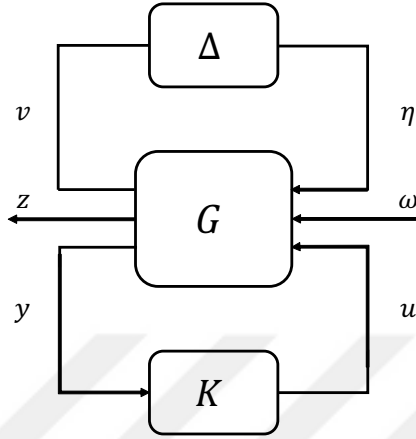


Figure 2.4: Linear Fractional Transformation (LFT) General Framework

The closed loop transfer matrix of the system in (2.10) is

$$G(s) = C(sI - A)^{-1}B + D. \quad (2.12)$$

To expedite calculations involving transfer matrices, we will use the following notation:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := C(sI - A)^{-1}B + D. \quad (2.13)$$

Consider a more complex system

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \triangleq \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix},$$

where $G_{ij}(s) = C_i(sI - A)^{-1}B_j + D_{ij}$ for $i, j \in \{1, 2\}$.

2.4.2 Linear Fractional Transformation

Linear Fractional Transformation (LFT) is a useful tool in writing transfer function of a compact block diagram in a short form. Consider the block diagram in Fig 2.4, consider transfer matrices G , K , and Δ of appropriate dimensions,

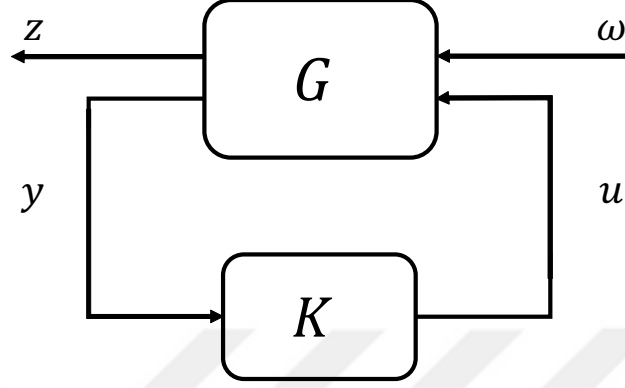


Figure 2.5: Lower Fractional Transformation

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \quad (2.14)$$

then lower and upper fractional transformations are defined as:

$$\begin{aligned} \mathcal{F}_l(P, K) &:= P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \\ \mathcal{F}_u(P, \Delta) &:= P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12} \end{aligned} \quad (2.15)$$

In the block diagram in Fig 2.5, the $\Delta(s)$ is removed, now $T_{z\omega}$, the transfer function from ω to z is given by the $\mathcal{F}_l(G, K)$

$$T_{z\omega} = \mathcal{F}_l(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \quad (2.16)$$

2.5 Technical Lemma

The following lemma is a well-known orthogonal projection result.

Lemma 2.1 *Suppose $T_1(s), K(s) \in \mathcal{H}_2$, $T_2(s) \in \mathcal{H}_\infty$, $T_2^*(s)T_2(s) = I$, and $T_2^*(s)T_1(s)$ is anti-stable, then the following holds:*

$$\|T_1(s) + T_2(s)K(s)\|_2^2 = \|T_1(s)\|_2^2 + \|K(s)\|_2^2. \quad (2.17)$$

It follows that $\|T_1(s) + T_2(s)K(s)\|_2$ is minimized over $K(s) \in \mathcal{H}_2$ for $K(s) = 0$.

Proof.

$$\begin{aligned} & \|T_1(s) + T_2(s)K(s)\|_2^2 \\ &= \langle T_1(s), T_1(s) \rangle + 2\operatorname{Re} \langle T_1(s), T_2(s)K(s) \rangle + \langle T_2(s)K(s), T_2(s)K(s) \rangle \\ &= \|T_1(s)\|_2^2 + 2\operatorname{Re} \langle T_2^*(s)T_1(s), K(s) \rangle + \|K(s)\|_2^2 \end{aligned}$$

Since $T_2^*(s)T_1(s) \in \mathcal{H}_2^\perp$, we have that $\langle T_2^*(s)T_1(s), K(s) \rangle = 0 \ \forall \ K(s) \in \mathcal{H}_2$. $\square$

Chapter 3

Two Alternative Solutions of the $\mathcal{H}_2$ Optimal Control Problem

In this chapter, optimal solution to the two block $\mathcal{H}_2$ problem is solved. Two different methods are used, then the differences between the methods and their advantages and drawbacks are explained. It is important to note that, in the whole process of solving for optimal controller, no approximation is used for the time delay element in the system.

The original problem is:

$$\gamma_{opt} = \inf_{(C, P_h)_{stable}} \left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_2. \quad (3.1)$$

3.1 First Method

We borrowed the procedure used in $\mathcal{H}_\infty$ mixed sensitivity problem to reduce two block problem into one block problem. The first step is to parametrize all stabilizing controller using Youla-Kučera method. In this case since the plant $P_h(s)$

is stable, we have:

$$\begin{aligned} C &= \frac{Q}{1 - P_0 Q e^{-sh}} \\ S &= 1 - P_0 Q e^{-sh} \\ T &= P_0 Q e^{-sh} \end{aligned}$$

Then we have

$$\begin{aligned} W_1 S &= W_1 (1 - P_0 Q e^{-sh}) \\ &= W_1 - W_1 P_0 Q e^{-sh} \end{aligned} \tag{3.2}$$

similarly

$$W_2 T = W_2 (P_0 Q e^{-sh}) \tag{3.3}$$

Putting them into (3.1)

$$\left\| \begin{bmatrix} W_1 - W_1 P_0 Q e^{-sh} \\ W_2 P_0 Q e^{-sh} \end{bmatrix} \right\|_2 \tag{3.4}$$

is equal to

$$\left\| \begin{bmatrix} W_1 \\ 0 \end{bmatrix} - \begin{bmatrix} W_1 \\ -W_2 \end{bmatrix} P_0 Q e^{-sh} \right\|_2. \tag{3.5}$$

Now, by spectral factorization i.e.

$$W_1^* W_1 + W_2^* W_2 = G^* G \tag{3.6}$$

we define unitary matrix L (i.e. $LL^* = I$) as

$$L = \begin{bmatrix} W_1^* G^{-*} & -W_2^* G^{-*} \\ W_2^* G^{-1} & W_1^* G^{-1} \end{bmatrix}. \tag{3.7}$$

Let us rewrite the minimization with respect to the free parameter $Q(s)$, then the problem becomes

$$\gamma_{opt} = \inf_{Q \in \mathcal{H}^\infty} \gamma_2(Q). \tag{3.8}$$

Since the unitary matrix does not change the norm, we multiply (3.5) by L , we obtain

$$\gamma_2(Q) = \left\| \begin{bmatrix} W_1 W_1^* G^{-*} \\ W_1 W_2 G^{-1} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} G P_0 Q e^{-sh} \right\|_2 \quad (3.9)$$

$$= \left\| \begin{bmatrix} W_1 W_1^* G^{-*} - G P_0 Q e^{-sh} \\ W_1 W_2 G^{-1} \end{bmatrix} \right\|_2 \quad (3.10)$$

take the square of both side

$$\gamma_2^2(Q) = \|W_1 W_2 G^{-1}\|_2^2 + \|W_1 W_1^* G^{-*} - G P_0 Q e^{-sh}\|_2^2 \quad (3.11)$$

it is clear that the first term in (3.11) is independent of Q , furthermore, the two-block problem defined in (3.1) is converted into one block problem which is much easier to solve. Thus, we can follow the one block problem procedure to solve for optimal $Q(s) \in \mathcal{H}_\infty$. For a case study plant in chapter 5 this problem is solved parametrically.

3.2 Second Method

Here is another method which eventually leads to the same controller found in first method. This method is the implementation of the work done in [10]. The controller structure found here has Smith predictor form as in first method. Note that, we will work directly with the infinite dimensional model and no approximation is used for the time delay element.

In this method, loop shifting is used to reduce the problem to an equivalent delay-free problem which can be treated by well developed methods for delay free systems. Also, it can be shown that, the set of all stabilizing controllers is in the observer-predictor form, where the open loop prediction is used to generate an estimate of the state vector of the plant [10].

3.2.1 Loop Shifting

Consider linear time invariant (LTI) setup in Fig. 3.1, where $\tilde{P}$ and $\tilde{K}$ are generalized plant and controller, respectively. By definition, this system is said to be internally stable if all the nine transfer matrices mapping from $\tilde{\omega}$, $\tilde{v}_1$, and $\tilde{v}_2$ to $\tilde{z}$, $\tilde{y}$, and $\tilde{u}$ are stable. In this case $\tilde{P}$ is rational (i.e. finite dimensional) so the stability problem is well understood, see, e.g., [16, 15].

Consider the general dead-time LFT system with a single delay in Fig. 3.2, where P is the rational part of the generalized plant given by its state space realization

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right]. \quad (3.12)$$

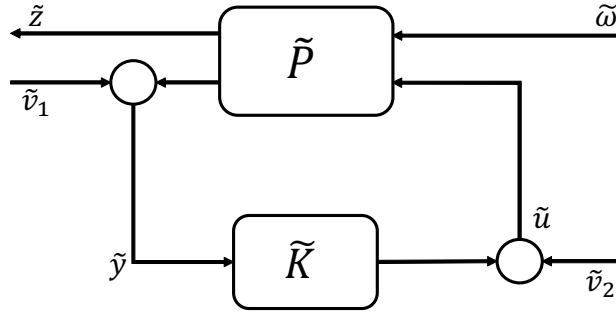


Figure 3.1: Controller implementation

In the Fig. 3.2, the controller K is a proper transfer function, and e^{-sh} is the delay element.

The stability problem of the dead time system in Fig. 3.2 can be reduced to an equivalent problem as in the system in Fig. 3.1.

To this end, define the following auxiliary system:

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & P_{12} \\ P_{21} & \tilde{P}_{22} \end{bmatrix}, \quad (3.13)$$

where $\tilde{P}_{11}$ and $\tilde{P}_{22}$ are rational and the off-diagonal blocks are same as for P .

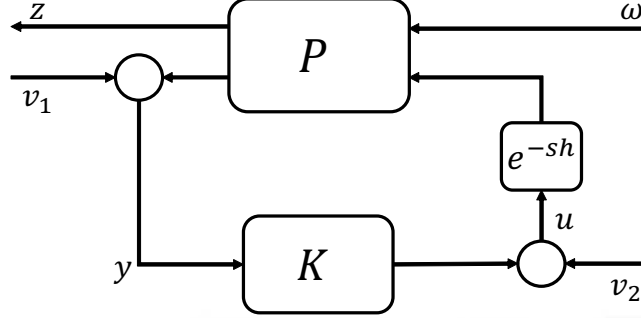


Figure 3.2: Controller implementation

Lemma 3.1 ([10]) *Let $\tilde{P}_{11}(s)$ be proper and $\tilde{P}_{22}$ be strictly proper rational transfer matrices such that $\Pi_1 \doteq P_{11} - e^{-sh}\tilde{P}_{11}$ and $\Pi_2 \doteq \tilde{P}_{22} - e^{-sh}P_{22}$ are stable. Define*

$$\tilde{K} \doteq (I + K\Pi_2)^{-1}K. \quad (3.14)$$

Then,

- (i) $\tilde{K}$ is proper if and only if so is K ;
- (ii) a proper K internally stabilizes the system in Fig. 3.1 if and only if a proper $\tilde{K}$ internally stabilizes the system in Fig. 3.2;
- (iii) for any K the following equality holds:

$$\mathcal{F}_l(P, e^{-sh}K) = \Pi_1 + e^{-sh}\mathcal{F}_l(\tilde{P}, \tilde{K}). \quad (3.15)$$

According to lemma 3.1, in order to find the equivalent setup, appropriate $\tilde{P}_{11}$ and $\tilde{P}_{22}$ are needed to have stable Π_1 and Π_2 . In general, there is no unique solution for that, however, there are some useful candidates for spacial cases. Here we analyze different scenarios. When P_{22} is stable, two straightforward solutions would be either $\tilde{P}_{22} = 0$ or $\tilde{P}_{22} = P_{22}$. While the latter yields in the Smith predictor form, the former is the Internal Model Controller (IMC) structure [58]. On the other hand, in the case where P_{22} is unstable, the selection of $\tilde{P}_{22}$ will be tricky. In [14] modified Smith predictor is introduced as taking $\tilde{P}_{22} = P_{22}F$, where F is a suitable transfer function in which the $P_{22}(F - e^{-sh})$ is stable.

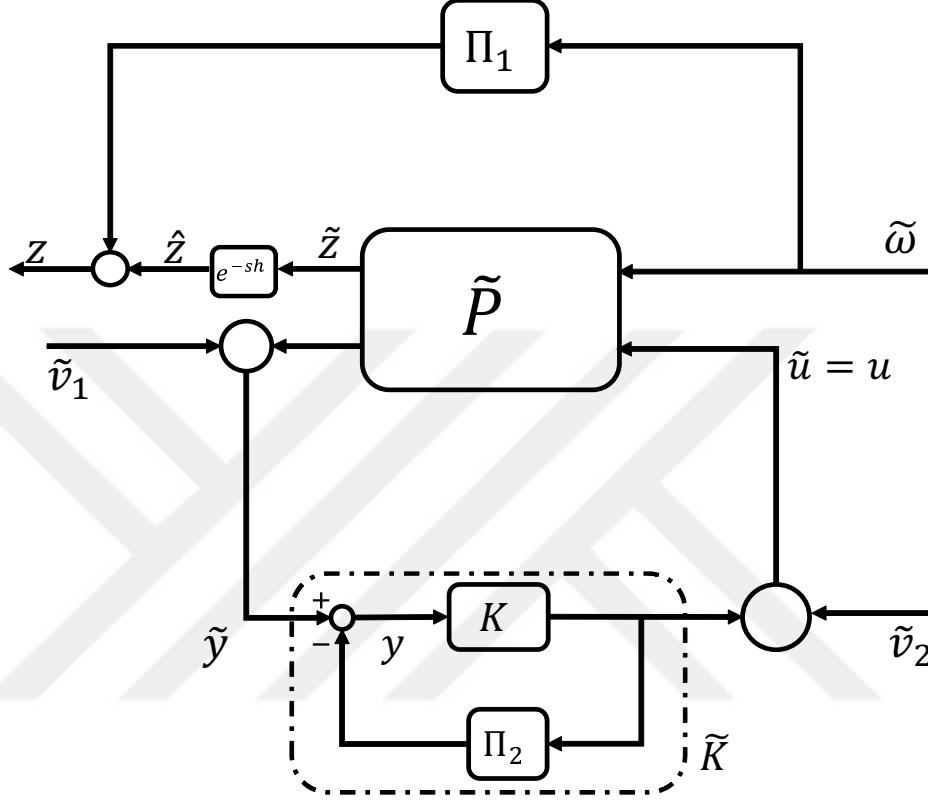


Figure 3.3: Equivalent System

In [10], to satisfy the conditions on Π_1 and Π_2 , the concept of FIR filter is used. FIR filters are systems with finite impulse response which are always stable. Therefore, we only need to find suitable $\tilde{P}_{11}$ and $\tilde{P}_{22}$ to make Π_1 and Π_2 FIR systems. The following are the appropriate choices:

$$\tilde{P}_{11} = \left[\begin{array}{c|c} A & e^{Ah}B_1 \\ \hline C_1 & 0 \end{array} \right] = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 e^{Ah} & 0 \end{array} \right], \quad (3.16)$$

$$\tilde{P}_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 e^{-Ah} & 0 \end{array} \right] = \left[\begin{array}{c|c} A & e^{-Ah}B_2 \\ \hline C_2 & 0 \end{array} \right], \quad (3.17)$$

Thus Π_1 will be

$$\Pi_1 = D_{11} + C_1(sI - A)^{-1}(I - e^{-(sI-A)h})B_1$$

and

$$\Pi_2 = C_2 e^{-Ah}(I - e^{-(sI-A)h})(sI - A)^{-1}B_2 - D_{22}e^{-sh}.$$

Regardless of the stability of P_{22} , with these choices for $\tilde{P}_{11}$ and $\tilde{P}_{22}$, the conditions for Π_1 and Π_2 in the lemma 3.1 are satisfied.

The auxiliary system defined in 3.13 will become

$$\tilde{P} = \left[\begin{array}{c|cc} A & e^{Ah}B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2e^{-Ah} & D_{21} & 0 \end{array} \right]. \quad (3.18)$$

Note that the generalized plant of the equivalent setup has the same order as the original plant P .

The system in 3.18 can be stabilized by an output feedback controller if and only if the pair (A, B_2) is stabilizable and the pair (C_2e^{-Ah}, A) is detectable. In [16], it is shown that the latter condition is equivalent to the detectability of the pair (C_2, A) .

Now, the following theorem deals with the parametrization of all stabilizing controllers.

Theorem 3.1 ([16]) *There exists a proper K achieving internal stability of the system in Fig.3.2 if and only if (A, B_2) is stabilizable and (C_2, A) is detectable. Furthermore, let F and L be such that $A + B_2F$ and $A + LC_2$ are Hurwitz. Then all the stabilizing controllers can be parameterized as the transfer matrix from y to u in Fig.3.3, where*

$$J = \left[\begin{array}{c|cc} A + B_2F + e^{Ah}LC_2e^{-Ah} & -e^{Ah}L & B_2 \\ \hline F & 0 & I \\ -C_2e^{-Ah} & I & 0 \end{array} \right] \quad (3.19)$$

and

$$\Pi_2 = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2e^{-Ah} & 0 \end{array} \right] - e^{-sh} \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array} \right] \quad (3.20)$$

and with any $Q \in \mathcal{H}_\infty$. Furthermore, the set of all closed-loop transfer matrices from ω to z achievable by an internally stabilizing controller is

$$\mathcal{F}_l(P, e^{-sh}K) = \Pi_1 + e^{-sh}(T_{11} + T_{12}QT_{21}), \quad (3.21)$$

where

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A_F & -B_2F & e^{Ah}B_1 & B_2 \\ 0 & e^{Ah}A_L e^{-Ah} & e^{Ah}B_L & 0 \\ \hline C_F & -D_{12}F & 0 & D_{12} \\ 0 & C_2 e^{-Ah} & D_{21} & 0 \end{array} \right] \quad (3.22)$$

and

$$\Pi_1 = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} - e^{-sh} \begin{bmatrix} A & B_1 \\ C_1 e^{Ah} & 0 \end{bmatrix}, \quad (3.23)$$

where the notations $A_F \doteq A + B_2F$, $A_L \doteq A + LC_2$, $B_L \doteq B_1 + LD_{21}$, and $C_F \doteq C_1 + D_{21}F$ are used.

Proof. The proof is in [16].

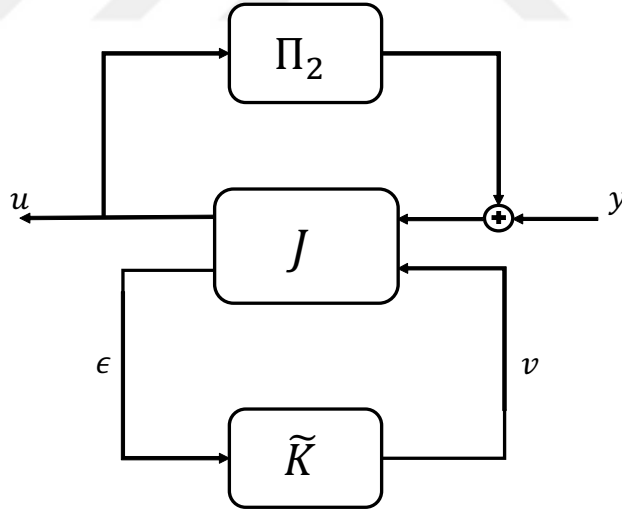


Figure 3.4: Youla-Kučera controller parametrization for time delay system

In theorem 3.1, it is shown that every time delay system has the form depicted in Fig. 3.3.

Now, the system is converted to the desired delay free form and available controller design can be used. A case study example is considered in chapter 5.

Chapter 4

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Controller Design

In this chapter, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ suboptimal controller is designed for stable plant $P_h(s)$ of the form

$$P_h(s) = P_0(s)e^{-sh}. \quad (4.1)$$

4.1 Problem Formulation

In (4.2) the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem is formulated:

$$\begin{aligned} \gamma_{subopt} &= \inf_{(C, P_h)_{stable}} \|W_1 S\|_2 \\ \text{subject to } & \|W_2 T\|_\infty < 1 \end{aligned} \quad (4.2)$$

The value of γ_{subopt} determines the performance level. The condition $\|W_2 T\|_\infty < 1$ guarantees the robust stability in the presence of plant and model uncertainties. To find the set of all stabilizing controllers, Youla-Kučera controller parametrization is used. Since the plant is stable, set of all stabilizing controllers parametrized as:

$$C(s) = \frac{X(s) + D(s)Q(s)}{Y(s) - N(s)Q(s)} \quad (4.3)$$

where $Q \in \mathcal{H}_\infty$ and $Q \neq YN^{-1}$. Also, $X(s), Y(s) \in \mathcal{H}_\infty$ are transfer functions satisfying the Bezout identity

$$N(s)X(s) + D(s)Y(s) = 1. \quad (4.4)$$

Note that $N(s)$ and $D(s)$ are the coprime factorization of the plant

$$P_h(s) = N(s)D(s)^{-1} \quad (4.5)$$

in this case the plant is stable, simply $N(s) = P_h(s)$ and $D(s) = 1$ are the coprime factorization of the plant. To satisfy the Bezout identity $X(s) = 0$ and $Y(s) = 1$ are chosen.

Thus,

$$C(s) = \frac{X(s) + D(s)Q(s)}{Y(s) - N(s)Q(s)} = \frac{Q(s)}{1 - P_0(s)Q(s)e^{-sh}}, \quad Q \in \mathcal{H}_\infty. \quad (4.6)$$

First, let us consider the nominal performance condition, $\|W_1 S\|_2$.

4.1.1 Performance Condition

Recall that when the nominal plant P_0 is stable, the sensitivity function is

$$\begin{aligned} S &= \frac{1}{1 + P_h(s)C(s)} = 1 - P_0(s)Q(s)e^{-sh} \\ &\Rightarrow W_1 S = W_1 - W_1 P_0 Q e^{-sh} \end{aligned}$$

where $Q \in \mathcal{H}_\infty$ is the free design parameter. therefore, the problem reduces to

$$\|W_1 S\|_2 = \|W_1 - W_1 P_0 Q e^{-sh}\|_2. \quad (4.7)$$

4.1.2 Robust Stability Condition

Recall that the feedback system is robustly stable if and only if the nominal feedback system is stable and

$$\begin{aligned} \|W_2 T\|_\infty &< 1 \\ \text{with} & \\ T(s) &= \frac{P_h(s)C(s)}{1 + P_h(s)C(s)} \end{aligned} \quad (4.8)$$

By using (4.6)

$$T(s) = 1 - S(s) = P_h(s)Q(s) \quad (4.9)$$

From (4.8) and (4.9), we have

$$\begin{aligned} \|W_2 T\|_\infty &= \|W_2 P_0 e^{-sh} Q\|_\infty \\ \Rightarrow \|W_2 P_0 Q\|_\infty &< 1 \end{aligned} \quad (4.10)$$

which is equivalent to

$$|Q(j\omega)| < \frac{1}{|W_2(j\omega)P_0(j\omega)|} \quad \forall \omega \quad (4.11)$$

Now, the optimization reduces to minimizing (4.7) by designing $Q(s) \in \mathcal{H}_\infty$ subject to (4.11). This method is illustrated in chapter 5 on our case study example.

Remark 4.1 The discussion above holds for the case where P_0 is stable. For unstable plants with only one pole in the right half plane the approach can be extended as follows.

$$P(s) = \frac{e^{-sh}}{s-a} = P_i \left(\frac{N_o}{D_i} \right)$$

where $P_i = e^{-sh}$, $N_o = \frac{1}{s+a}$, and $D_i = \frac{s-a}{s+a}$.

Then solve Bézout identity for stable X and Y :

$$P_i N_o X + D_i Y = 1$$

For this case $X(s) = 2ae^{ah}$ and $Y(s) = \frac{s+a-2ae^{-h(s-a)}}{s-a}$ are the possible choices for X and Y in $\mathcal{H}_\infty$.

Then, the same procedure discussed above follows. The difference with the stable plant is that in unstable case, it is tricky to split the terms into two orthogonal terms. However, the idea of making FIR filter can help to find orthogonal terms.

Chapter 5

Design Examples

In this chapter we implement the solutions outlined in previous chapters. First we solve $\mathcal{H}_2$ two block mixed sensitivity problem with two different methods. Throughout this work, we consider a stable plant with a time delay h , of the form:

$$P_h = \frac{e^{-hs}}{\tau s + 1} \quad \tau > 0, \quad h > 0. \quad (5.1)$$

5.1 $\mathcal{H}_2$ Optimal Control Example

Our objective is to find a controller $C = C_{opt}$ to solve the following optimization problem:

$$J_2 = \inf_{(C,P)_{stable}} \left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_2 \quad (5.2)$$

where

$$W_1 = \frac{1}{s}$$

and

$$W_2 = ks + \epsilon, \quad k > 0, \epsilon > 0$$

are two known weighting transfer functions, $S = (1 + PC)^{-1}$ is the sensitivity transfer function, $T = 1 - S$ is the complementary transfer function.

5.1.1 First Method

In this part, we use the first method in chapter 3 for the system defined in (5.1). Youla-Kučera controller parametrization for this system yields

$$C = \frac{Q}{1 - \left(\frac{1}{\tau s + 1}\right) Q e^{-sh}}. \quad (5.3)$$

Then we have

$$\begin{aligned} W_1 S &= W_1 (1 - P_0 Q e^{-sh}) \\ &= \frac{1}{s} - \frac{1}{s} \left(\frac{1}{\tau s + 1}\right) Q e^{-sh}, \end{aligned} \quad (5.4)$$

similarly

$$W_2 T = W_2 (P_0 Q e^{-sh}) = (ks + \epsilon) \left(\frac{1}{\tau s + 1}\right) Q e^{-sh}. \quad (5.5)$$

Putting (5.4) and (5.5) into (5.2)

$$\left\| \begin{bmatrix} \frac{1}{s} - \frac{1}{s} \left(\frac{1}{\tau s + 1}\right) Q e^{-sh} \\ (ks + \epsilon) \left(\frac{1}{\tau s + 1}\right) Q e^{-sh} \end{bmatrix} \right\|_2 \quad (5.6)$$

can be written as

$$\left\| \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{s} \\ -ks + \epsilon \end{bmatrix} \left(\frac{1}{\tau s + 1}\right) Q e^{-sh} \right\|_2. \quad (5.7)$$

Now, by spectral factorization i.e.

$$\begin{aligned} W_1^* W_1 + W_2^* W_2 &= G^* G \\ \Rightarrow \left(\frac{1}{-s}\right) \left(\frac{1}{s}\right) + (k(-s) + \epsilon) (ks + \epsilon) &= \frac{-1}{s^2} + \epsilon^2 - k^2 s^2 = G^* G \\ \Rightarrow \frac{(k^2 s^4 - \epsilon^2 s^2 + 1)}{-s^2} &= \left(\frac{(ks^2 + as + 1)}{s}\right) \left(\frac{(ks^2 - as + 1)}{-s}\right) \\ \Rightarrow G(s) &= \frac{(ks^2 + as + 1)}{s}, \quad \text{with } a = \sqrt{\epsilon^2 + 2k}. \end{aligned}$$

We define unitary matrix L (i.e. $LL^* = I$) as

$$L = \begin{bmatrix} W_1^* G^{-*} & -W_2^* G^{-*} \\ W_2^* G^{-1} & W_1^* G^{-1} \end{bmatrix}. \quad (5.8)$$

Let us rewrite the minimization with respect to the free parameter $Q(s)$, then the problem becomes

$$\gamma_{opt} = \inf_{Q \in \mathcal{H}^\infty} \gamma_2(Q). \quad (5.9)$$

Since the unitary matrix does not change the norm, we multiply it to (5.7), then we obtain

$$\gamma_2(Q) = \left\| \begin{bmatrix} W_1 W_1^* G^{-*} \\ W_1 W_2 G^{-1} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} G P_0 Q e^{-sh} \right\|_2 \quad (5.10)$$

$$= \left\| \begin{bmatrix} W_1 W_1^* G^{-*} - G P_0 Q e^{-sh} \\ W_1 W_2 G^{-1} \end{bmatrix} \right\|_2 \quad (5.11)$$

$$= \left\| \begin{bmatrix} \frac{-1}{s^2} \frac{-s}{ks^2 - as + 1} - \left(\frac{ks^2 + as + 1}{s} \right) \left(\frac{1}{\tau s + 1} \right) Q e^{-sh} \\ \left(\frac{1}{s} \right) (ks + \epsilon) \frac{s}{ks^2 + as + 1} \end{bmatrix} \right\|_2 \quad (5.12)$$

take the square of both side

$$\begin{aligned} \gamma_2^2(Q) &= \left\| \left(\frac{1}{s} \right) (ks + \epsilon) \frac{s}{ks^2 + as + 1} \right\|_2^2 \\ &+ \left\| \frac{-1}{s^2} \frac{-s}{ks^2 - as + 1} - \left(\frac{ks^2 + as + 1}{s} \right) \left(\frac{1}{\tau s + 1} \right) Q e^{-sh} \right\|_2^2 \\ &= \left\| \frac{ks + \epsilon}{ks^2 + as + 1} \right\|_2^2 \\ &+ \left\| \left(\frac{1}{s} \right) \left(\frac{1}{ks^2 - as + 1} - \left(\frac{ks^2 + as + 1}{\tau s + 1} \right) Q e^{-sh} \right) \right\|_2^2. \end{aligned}$$

Since the first term is independent of the free parameter Q we define

$$\tilde{\gamma} = \left\| \left(\frac{1}{s} \right) \left(\frac{1}{ks^2 - as + 1} - \left(\frac{ks^2 + as + 1}{\tau s + 1} \right) Q e^{-sh} \right) \right\|_2.$$

Now we need to minimize $\tilde{\gamma}$. For this purpose, define

$$\tilde{Q}(s) = \frac{\tau s + 1}{ks^2 + as + 1} Q(s)$$

and

$$\left\| \frac{1}{s(ks^2 - as + 1)} - \left(\frac{1}{s} \right) \tilde{Q}(s)e^{-sh} \right\|_2 \quad (5.13)$$

$$= \left\| \frac{a - ks}{ks^2 - as + 1} + \frac{1}{s} - \left(\frac{1}{s} \right) \tilde{Q}(s)e^{-sh} \right\|_2 \quad (5.14)$$

$$= \left\| F(s) + \left(\frac{1}{s} \right) (1 - \tilde{Q}(s)e^{-sh}) \right\|_2 \quad (5.15)$$

$$\text{with } F(s) = \frac{a - ks}{ks^2 - as + 1} \in \mathcal{H}_2^\perp. \quad (5.16)$$

Therefore, we have the following interpolation condition

$$\tilde{Q}(0) = 1, \quad (5.17)$$

the (5.17) comes from the pole-zero cancellation at $s = 0$.

Thus, we can use $\tilde{Q}(s) = 1$ to satisfy the interpolation condition, so

$$Q_{opt}(s) = \frac{\tau s + 1}{ks^2 + as + 1}. \quad (5.18)$$

By putting $Q_{opt}(s)$ in (5.3), the optimal controller $C_{opt}(s)$ becomes

$$C_{opt}(s) = \frac{\left(\frac{\tau s + 1}{ks^2 + as + 1} \right)}{1 - \left(\frac{1}{\tau s + 1} \right) \left(\frac{\tau s + 1}{ks^2 + as + 1} \right) e^{-sh}}. \quad (5.19)$$

We can write the controller in the following form:

$$C = C_1 \frac{C_2}{1 + C_2 H_0} \quad (5.20)$$

where $C_1 = \frac{1 + \tau s}{s}$, C_2 is to be investigated and $H_0 = \frac{1 - e^{-hs}}{s}$ is the transfer function of FIR part. It can be easily seen that the controller can be implemented using a Smith-predictor structure as shown in Fig. 5.1.

by putting it into the (5.20) format, we get

$$C_{opt}(s) = \left(\frac{1 + \tau s}{s} \right) \frac{\left(\frac{1}{ks + a} \right)}{1 + \left(\frac{1}{ks + a} \right) \left(\frac{1 - e^{-sh}}{s} \right)} \quad (5.21)$$

so we will have

$$C_1(s) = \frac{1 + \tau s}{s}, \quad (5.22)$$

$$C_2(s) = \frac{1}{ks + a}, \quad (5.23)$$

$$H_0 = \frac{1 - e^{-sh}}{s}. \quad (5.24)$$

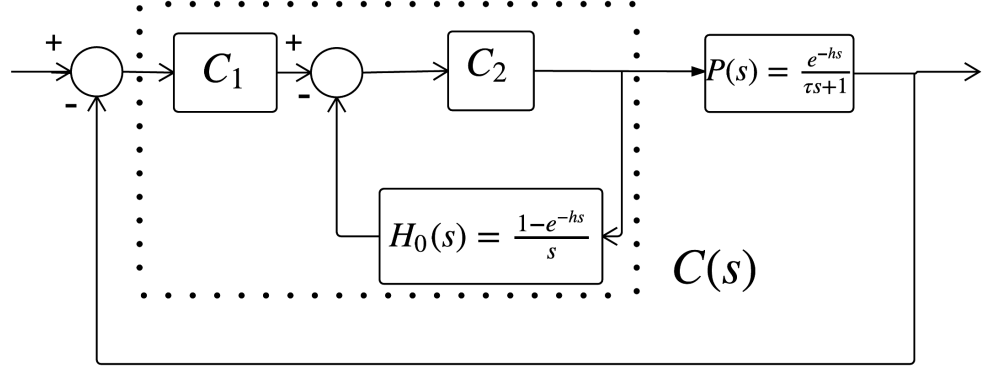


Figure 5.1: Controller implementation

Note that, this method gives us the optimal controller parametrically. Thus, we have designed optimal controller for a class of first order stable plants.

To illustrate the effect of delay in optimal performance level γ_{opt} , the optimal performance level γ_{opt} is depicted In Fig.5.2 with variation of the time delay h .

5.1.2 Second Method

To make a comparison with the first method, we use the block diagram in the Fig 5.3, so that we have equivalent optimization problem used in first method section. The aim of this section is to find the optimal $\mathcal{H}_2$ controller for the problem described in (5.25).

$$\gamma_{opt} = \inf_{\forall \text{ stabilizing controllers}} \|T_{z\omega}\|_2 \quad (5.25)$$

In Fig. 5.3 the block diagram of the system is illustrated, in which there are two weighting transfer functions

$$W_1(s) = \frac{1}{s}$$

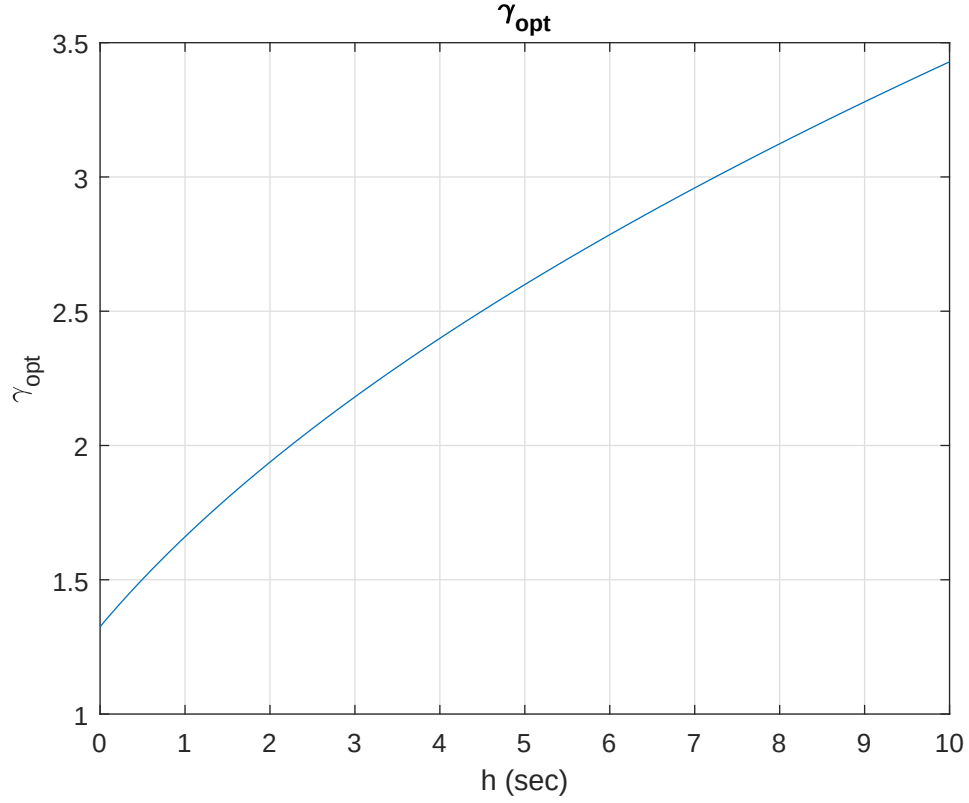


Figure 5.2: The optimal performance level γ_{opt} with respect to variation of time delay h

and

$$W_2(s) = ks + \epsilon, \quad k > 0, \epsilon > 0$$

The transfer matrix from $[\omega \ u]^T$ to $[z_1 \ z_2 \ y_p]^T$ is:

$$\begin{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ y_p \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} W_1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} -e^{-sh}P_0W_1 \\ e^{-sh}P_0W_2 \\ -P_0e^{-sh} + \Pi_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \omega \\ u \end{bmatrix} \quad (5.26)$$

The above transfer matrix has the form

$$\begin{bmatrix} \mathbf{z} \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \omega \\ u \end{bmatrix} \quad (5.27)$$

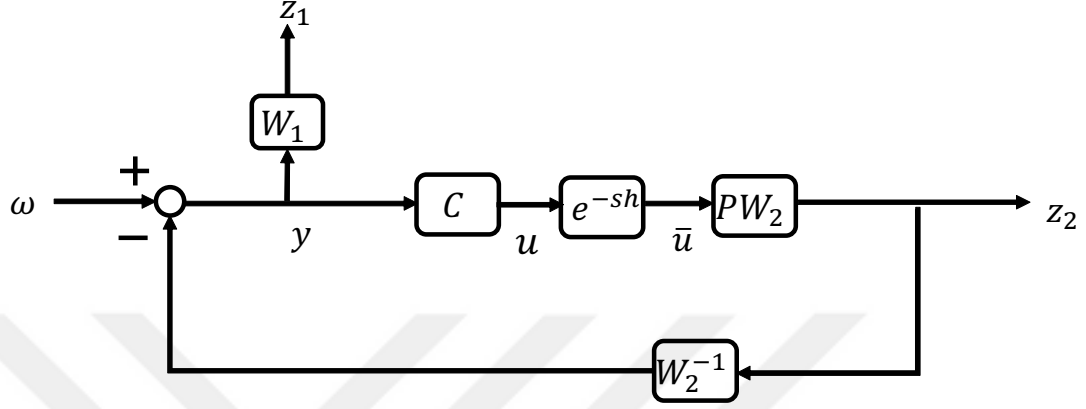


Figure 5.3: Stable system with time delay and weights

with $\mathbf{z} = [z_1 \ z_2]^T$ as the output vector.

Thus,

$$\begin{aligned}
 P_{11}(s) &= \begin{bmatrix} W_1(s) \\ 0 \end{bmatrix}, \\
 P_{12}(s) &= \begin{bmatrix} -e^{-sh}P_0W_1 \\ e^{-sh}P_0W_2 \end{bmatrix}, \\
 P_{21}(s) &= 1, \\
 P_{22}(s) &= \Pi_2(s) - e^{-sh}P_0(s)
 \end{aligned} \tag{5.28}$$

The next step is to find the equivalent setup for our system in the form shown in Fig. 5.5.

As discussed in chapter 3, the $\tilde{P}(s)$ is equal to:

$$\begin{bmatrix} \tilde{\mathbf{z}} \\ y_p \end{bmatrix} = \begin{bmatrix} e^{Ah}P_{11} & P_{12} \\ P_{21} & P_{22}^{aug} \end{bmatrix} \begin{bmatrix} \omega \\ u \end{bmatrix}. \tag{5.29}$$

In which

$$P_{22}^{aug} = -P_0e^{-sh} + \Pi_2 \quad \text{and} \quad \Pi_2 = -P_0 + P_0e^{-sh}$$

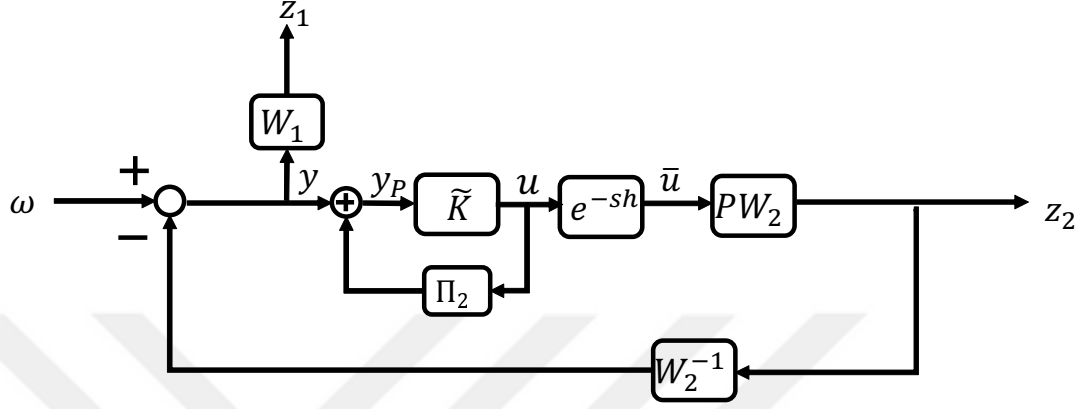


Figure 5.4: Stable system with time delay and weights

so, it yields to

$$P_{22}^{aug} = -P_0.$$

The $\tilde{P}(s)$ is the equivalent delay free generalized plant. Now, we can use the classical $\mathcal{H}_2$ solution for this system. Since this method does not give parametric solution, parameters in Table 5.1.2 are used for illustration.

Table 5.1: Parameter of the system for simulation

Parameter:	Value
τ	1
k	1
ϵ	0.1
h	0.1

The optimal controller for the equivalent system is:

$$\tilde{K}_{opt} = \frac{s+1}{s^2 + 1.418s}. \quad (5.30)$$

Note that $C(s) = \frac{\tilde{K}}{1-\tilde{K}\Pi_2}$. Therefore the optimal controller for the time delay

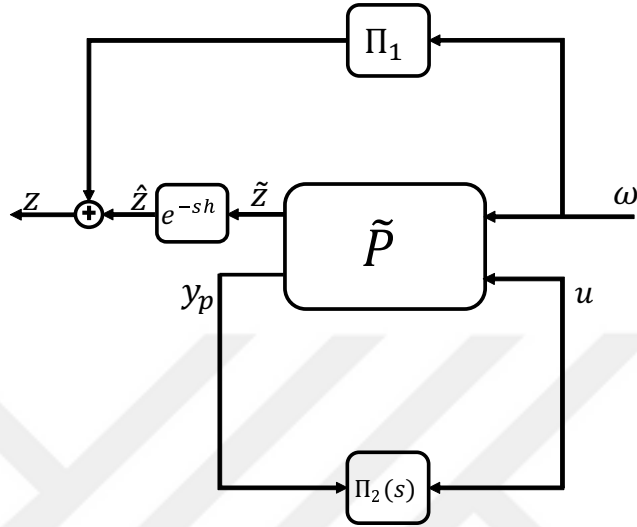


Figure 5.5: Equivalent Setup of the original system

system is:

$$C_{opt}(s) = \frac{\frac{s+1}{s^2+1.418s}}{1 - \frac{s+1}{s^2+1.418s} \frac{e^{-0.1s}-1}{s+1}}. \quad (5.31)$$

It can be seen that the optimal controller found in (5.31) is same as the controller found in (5.21). Hence the example illustrates the equivalency of these two methods. Unlike the first method, the second method does not give parametric optimal control.

5.2 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design example

In order to make this section self contained we first recall the problem definition and its transformed form from Chapter 4.

In (5.32) the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem is formulated.

$$\begin{aligned} \gamma &= \inf_{(C, P_h)_{stable}} \|W_1 S\|_2 \\ \text{subject to } & \|W_2 T\|_\infty < 1 \end{aligned} \quad (5.32)$$

First, let consider the $W_1 S$

$$W_1 S = W_1 - W_1 P_0 Q e^{-sh} \quad (5.33)$$

$$= \frac{1}{s} - \frac{1}{s} \frac{1}{\tau s + 1} Q e^{-sh} \quad (5.34)$$

$$= \frac{1 - e^{-sh}}{s} + \frac{e^{-sh}}{s} - \frac{1}{s} \frac{1}{\tau s + 1} Q e^{-sh} \quad (5.35)$$

$$= \frac{1 - e^{-sh}}{s} + \left(\frac{e^{-sh}}{s} \right) \left(1 - \frac{Q}{\tau s + 1} \right) \quad (5.36)$$

therefore, the problem reduces to

$$\|W_1 S\|_2^2 = \left\| \frac{1 - e^{-sh}}{s} + \left(\frac{e^{-sh}}{s} \right) \left(1 - \frac{Q}{\tau s + 1} \right) \right\|_2^2 \quad (5.37)$$

$$= \left\| \frac{1 - e^{-sh}}{s} \right\|_2^2 + \left\| \left(\frac{e^{-sh}}{s} \right) \left(1 - \frac{Q}{\tau s + 1} \right) \right\|_2^2 \quad (5.38)$$

the first term is a FIR system, also it has no Q in it, so in order to minimize $\|W_1 S\|_2^2$ simply we can choose $Q(s) = \frac{\tau s + 1}{\epsilon_1 s + 1} e^{-sh}$ for $0 < \epsilon_1 \ll 1$. This ϵ_1 is our free parameter to be chosen such a way that the robust stability condition is satisfied.

5.2.1 Robust Stability Condition

In order to analyze stability/robustness in the presence of dynamic uncertainty, $W_2(s) = \frac{ks + \epsilon}{\epsilon_2 s + 1}$ is taken as the upper bound on norm of multiplicative uncertainty, which is assumed to be stable. The feedback system formed by the nominal plant $P_h(s)$ is robustly stable if and only if

$$\begin{aligned} & \|W_2 T\|_\infty < 1 \\ & \text{with} \\ & T(s) = \frac{P_h(s)C(s)}{P_h(s)C(s) + 1} \end{aligned} \quad (5.39)$$

By using (5.3)

$$T(s) = 1 - S(s) = P_h(s)Q(s) \quad (5.40)$$

From (5.39) and (5.40), we have

$$\begin{aligned} \|W_2 T\|_\infty &= \|W_2 P_0 e^{-sh} Q\|_\infty \\ \Rightarrow \|W_2 P_0 Q\|_\infty &< 1 \end{aligned} \tag{5.41}$$

which is equivalent to

$$|Q(j\omega)| < \frac{1}{|W_2(j\omega)P_0(j\omega)|} \quad \forall \omega \tag{5.42}$$

Now, the optimization reduces to designing $Q(s)$ such a way that the inequality in (5.42) is guaranteed.

The optimal value of ϵ_1 is found to be $\epsilon_1 = 0.4$. The magnitude plot of $|Q(j\omega)|$ and $\frac{1}{|W_2(j\omega)P_0(j\omega)|}$ for the different values of ϵ_1 is illustrated in Fig 5.6. The figure clearly shows that $\epsilon_1 = 0.2$ violate the robust stability inequality condition and $\epsilon_1 = 0.6$ is not optimal value.

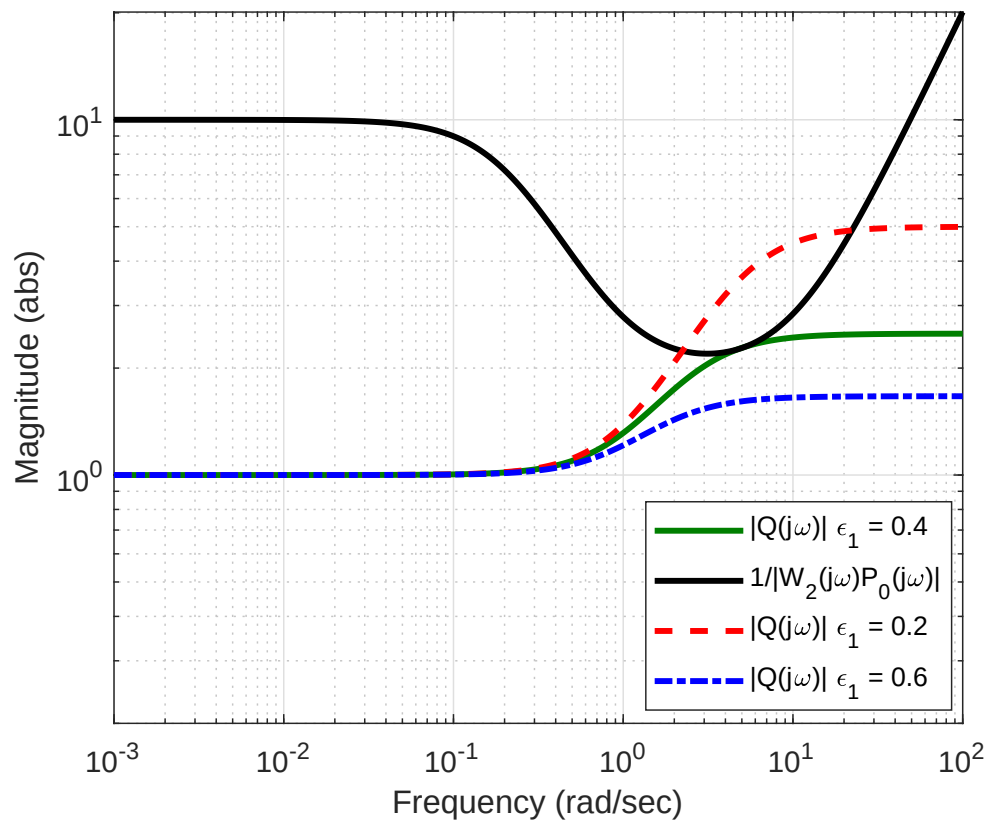


Figure 5.6: Magnitude plot of Q and robust stability check

Chapter 6

Conclusion

In this thesis, we deal with a uncertain time delay plant with known weight functions. This problem, still is one of the important topics in the field of control theory. In Chapter 3, two different methods are presented for $\mathcal{H}_2$ two block problem. These methods solved for the optimal controller. In the process of developing optimal $\mathcal{H}_2$ problem, a simple relation between the closed-loop transfer function of the original setup and the reduced delay free setup is established. In addition, the stability of dead-time systems in a general framework of linear fractional transformation (LFT) is studied. Chapter 4, proposes an approach for the solution of the mixed $\mathcal{H}_\infty/\mathcal{H}_2$ problem for time delay system. We have seen that all the above controllers are in Smith predictor form.

For the future study, these methods can be extended to more complex systems. For instance, one direction would be designing controller for unstable plants. Another more general extension can be considering MIMO systems.

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