ESSAYS ON UNIT ROOT TESTS IN TIME SERIES

A Ph.D. Dissertation

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Ankara
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to Sıla and my Family

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The Graduate School of Economics and Social Sciences
of
İhsan Doğramacı Bilkent University

by

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In Partial Fulfillment of the Requirements for the Degree of
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December 2018

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Economics.

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ABSTRACT

ESSAYS ON UNIT ROOT TESTS IN TIME SERIES

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December 2018

This dissertation consists of three essays which develop new unit root testing methods in time series. First one is about the effect of the persistent volatility breaks, i.e. non-stationary volatility, on the unit root inference in regulated time series. In this essay, we show that conventional bounded unit root tests become potentially unreliable in the presence of the non-stationary volatility. Then, as a remedy, we propose a new class of unit root tests that are robust to both the range constraints and the permanent volatility shifts present in the time series. While developing our new tests, we also extend the asymptotic theory for integrated time series. The second essay is about testing for seasonal unit roots. In this essay, we first construct a family of nonparametric seasonal unit root tests by utilizing fractional integration operator. Different from the well-known parametric seasonal unit root tests, the proposed tests are free from tuning parameters. Another contribution of this essay is on the fractional integration literature. We introduce a new fractionally transformed seasonal series. The third essay deals with the effect of the heteroscedastic innovations on the nonparametric seasonal unit root tests. We demonstrate that these tests spuriously reject the true seasonal unit root null hypothesis under the heteroscedasticity. To remove the aforementioned size distortions, we develop nonparametric wild bootstrap seasonal unit root tests. These tests are successful in correcting size problems under a broad class of heteroscedasticity observed in the seasonal time series. Moreover, we show that the proposed tests are asymptotically pivotal.

Keywords: Bootstrap Method, Nonparametric Test, Nonstationary Volatility, Regulated Time Series, Seasonal Unit Root.

ÖZET

ZAMAN SERİLERİNDE BİRİM KÖK TESTLERİ ÜZERİNE MAKALELER

GÖĞEBAKAN, Kemal Çağlar
Doktora, İktisat Bölümü
Tez Yöneticisi: Doç. Dr. Mehmet Taner Yiğit
Aralık 2018

Bu tez zaman serilerinde birim kök testi geliştiren üç makaleden oluşmaktadır. İlki durağan olmayan varyansın düzenlenmiş zaman serilerinde uygulanan birim kök testlerindeki etkisi üzerinedir. Bu makalede gösterilmiştir ki, düzenlenmiş standart birim kök testleri durağan olmayan varyans altında potansiyel olarak güvenilmezdir. Çözüm olarak zaman serilerinde gözlemlenen aralık kısıtlaması ve durağan olmayan varyansa karşı dayanıklı olan birim kök testi sınıfı önerilmiştir. Bu test sınıfı önerilirken, ayn zamanda bütünleşmiş zaman serileri için yeni bir asimptotik kuram geliştirilmiştir. İkinci makale mevsimsel birim kök testleri üzerinedir. Bu makalede ilk olarak, kesirsel bütünleşme operatörü kullanımıyla parametrik olmayan bir mevsimsel birim kök testi familyası oluşturulmuştur. Bilinen parametrik mevsimsel birim kök testlerinden farklı olarak önerilen testler ayar parametrelerinden bağımsızdır. Bu makalenin diğer katkısı da kesirsel bütünleşme literatürü üzerinedir. Yeni bir kesirsel olarak dönüştürülmüş mevsimsel seri tipi tanıtılmıştır. Üçüncü makale ise değişen varyans probleminin parametrik olmayan mevsimsel birim kök testleri zerinde etkisini incelemektedir. Bu testlerin değişen varyans problemi altında gerçek boş mevsimsel birim kök hipotezini sıklıkla reddettikleri gözlemlenmiştir. Belirtilen büyüklük sapmalarını ortadan kaldırmak için parametrik olmayan bir bootstrap metodu kullanan mevsimsel birim kök testleri geliştirilmiştir. Bu testler mevsimsel zaman serilerinde gözlemlenen değişen varyanslar sınıfı altında büyüklük problemlerini düzeltme açısından başarılıdır. Aynı zamanda gösterilmiştir ki bu testler asimptotik olarak pivotaldir.

Anahtar Kelimeler: Bootstrap Metodu, Durağan Olmayan Varyans, Parametrik Olmayan Test, Düzenlenmiş Zaman Serisi, Mevsimsel Birim Kök.

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CHAPTER 1

INTRODUCTION

Over the last three decades, identifying long-run movements in financial and economic data has become a problem of great interest in time series econometrics literature. The main contributions in this area have been based on developing models and techniques to analyze the non-stationarity of the series. Since the seminal study of Nelson and Plosser (1982), many theoretical and applied studies have been pursued in the area of unit root[1] non-stationarity. The empirical evidence in this work show that many macroeconomic and financial variables are modeled as I(1) processes.

Researchers should be very cautious about I(1) variables since the realized shocks have permanent impacts on the evolution of the time series. This situation has undesirable influences on the statistical analysis. Presence of unit roots invades the necessary conditions to make use of the Law of Large Numbers and Central Limit Theorem. Consequently, widely utilized test statistics turn out to be invalid. Also, long-run forecasting methods deliver quietly poor results. In this

[1]Unit root time series are integrated of order 1, i.e., I(1)

regard, detection of unit roots still has tremendous importance for time series analysts.

Dickey and Fuller (1979) is the pioneering study in the unit root testing literature. After this study, many alternative testing mechanisms have been developed. The main motivations behind these new methods are to design tests which apply to a broad class of models and to improve the finite sample performance of the previous tests. Despite the flowing contributions in the area, there is still much to do as to improve the inference in testing for unit roots. This is the motivation for the three articles which constitutes this dissertation.

In the first article, we construct a unit root testing mechanism that provides robust results under the presence of range restrictions and significant volatility breaks in time series.Granger (2010) states that range restrictions (bounds) are observed frequently in time series data. These restrictions force the integrated series to look like stationary. Therefore, standard unit root tests become ineffective to detect the difference between stationarity and non-stationarity. In this context, Cavaliere (2005) and Cavaliere and Xu (2014) analyze the non-stationarity behaviour in bounded time series and develop new unit root tests. On the other hand, Bollen et al. (2000), Romer (1986) and Rodrigues and Rubia (2005) document that regulated time series can exhibit persistent volatility breaks. Neglecting this property of volatility in limited time series can result in spurious rejections of unit roots. To overcome this problem, we develop a new class of unit root tests. To conduct these tests, we also extend the asymptotic analysis required for integrated time series under permanent volatility shifts. The proposed tests are applicable to a wide class of models of range constraints and volatility changes.

Second article is related to the non-stationarity in seasonal time series. Integrated seasonal variables evolve by persistent aggregation of the seasonal shocks. If the seasonal integration is disregarded, the seasonal autoregressive modeling becomes statistically invalid. Therefore, detection of integration in seasonal time series has gained immense importance in the unit root literature. After the seminal study of Hylleberg et al. (1990), seasonal unit root testing literature has marked an explosive growth in theoretical contributions. However, the common feature of the proposed tests is the utilization of parametric lag augmentation to account for serial correlation. Taylor (2005) is the only exception who develops nonparametric seasonal variance ratio tests. However, the tests in this study suffer from low finite sample power. Therefore, in this article, we propose a new family of nonparametric seasonal unit root tests by utilizing the fractional integration operator. The tests in this family are free from parametric lag augmentation and improve the existing nonparametric seasonal unit root tests in terms of asymptotic local power. This chapter is a joint work with Assist. Prof. Burak Eroglu and Mirza Trokic and published in Economics Letters on June 2018 with following reference: Eroğlu, B. A., Göğebakan, K. Ç ., Trokic, M., 2018. Powerful nonparametric seasonal unit root tests. Economics Letters 167, 75-80. An earlier version of this paper was also circulated in 2017 under the title of “Fractional Seasonal Variance Ratio Tests” in CEFIS Working Paper Series of İstanbul Bilgi University.

In the last article, we improve the performance of nonparametric seasonal unit root tests developed in the second article under the combination of the two types of heteroscedasticity. The first one is periodic heteroscedasticity, where innovations in some seasons are more variable than others. The second one is non-stationary volatility which is related to the persistent volatility breaks in seasonal time series. We show that under these patterns of heteroscedasticity, standard nonparametric seasonal unit root tests suffer from severe size distortions. As a remedy, we propose nonparametric wild bootstrap seasonal unit root tests which replicate the patterns of heteroscedasticity and eliminate the aforementioned size distortions in the standard nonparametric tests. These tests are asymptotically valid and provide good approximations to the limiting distributions of each test statistics under periodic non-stationary volatility.

The dissertation is organized as follows: Chapter 2 introduces a new class of tests which is robust to both range constraints and non-stationary volatility. Chapter 3 is about the family of nonparametric seasonal unit root tests. Chapter 4 is about extending nonparametric seasonal unit root tests under periodic non-stationary volatility. All proofs are placed in the appendix.

CHAPTER 2

TESTING FOR UNIT ROOTS IN BOUNDED TIME SERIES UNDER NONSTATIONARY VOLATILITY

2.1 Introduction

In time series econometrics, modeling non-stationarity for bounded (limited) processes is an exciting avenue of research. A bounded process is defined in Granger (2010) as 'process that has bounds either below (at zero, say) or above (full capacity) or both'. Based on this definition, series such as expenditure shares, unemployment rates, nominal interest rates can inherently be categorized as limited processes. Also, series like minimum wages and target zone exchange rates are bounded by policy control. Granger (2010) asserts that existence of bounds compels integrated series to behave like stationary ones, as they approach the bounds. Neglecting this feature makes the inference of the standard unit root tests theoretically invalid. As a remedy, Cavaliere (2005) extends the asymptotic theory of the standard unit root tests to the domain of limited time series by introducing bounded unit root distribution to the literature.Cavaliere and Xu

(2014) improve the finite sample properties of the unit root tests defined in Cavaliere and Xu (2014) by accounting for potential autocorrelation. However, none of these studies take into account the possible permanent volatility shifts in the innovations.

Non-stationary volatility observed in bounded time series such as the U.S. short-term nominal interest rate, EMS target exchange rate and U.S. unemployment rate (see Romer (1986), Bollen et al. (2000) and Rodrigues and Rubia (2005)). The effect of significant volatility breaks on the standard unit root tests is examined by Cavaliere and Taylor (2007, 2008, 2009). These studies develop new unit root tests that are robust to a broad class of volatility breaks. Smooth volatility breaks and multiple volatility shifts are included in the aforementioned class. However, the tests they propose are not reliable under limited time series.

In this chapter, we aim to propose a new approach to unit root testing that is robust to both non-stationary volatility and the range constraints. To do so, we first improve the asymptotic theory for integrated time series under permanent volatility shifts. We introduce the variance shifted bounded Brownian motion and present the convergence of observed series to this process. Second, we develop generalizations of standard unit root tests to the limited time series with a variety of volatility changes. Therefore, we provide a general class of unit root tests. The tests in this class are asymptotically valid. Third, we use direct simulation methods -by consistently estimating bound parameters and variance profile- to obtain the critical values of our proposed tests.

This chapter is organized as follows. Section 2.2 introduces bounded integrated processes with non-stationary volatility and details initial assumptions. Bounded unit root distribution under permanent variance changes is introduced and convergence results are presented in Section 2.3. Section 2.4 presents the asymptotic results for ADF and M type unit root tests in the presence of range constraints and non-stationary volatility. Section 2.5 shows the unit root testing mechanism. The finite sample properties of new unit root tests are presented in Section 2.6. Section 2.7 concludes. Simulation result tables are placed at the end of the chapter.

2.2 The Model and Assumptions

In this section, we introduce the class of bounded non-stationary processes where the innovations display a general class of permanent volatility changes. Processes of this class can be referred to 'time transformed bounded unit root' processes which will be explained in detail in Section 2.3.

For the unit root case, a bounded time series $\{X_t\}_{t=0}^{T}$ with fixed bounds $\underline{b}, \overline{b} (\underline{b} < \overline{b})$, a constant deterministic part and non-stationary volatility can be represented as:

$$X_t = \theta' \delta_t + Y_t \tag{2.1}$$

$$Y_t = \rho Y_{t-1} + \varepsilon_t, \rho = 1 \tag{2.2}$$

initialized at $Y_0 = O_p(1)$. The term ε_t is decomposed as follows:

$$\varepsilon_t = u_t + \underline{\xi}_t - \overline{\xi}_t \tag{2.3}$$

$$u_t = \psi(L)v_t \tag{2.4}$$

where $\underline{\xi}_t$ and $\overline{\xi}_t$ are non-negative processes such that $\underline{\xi}_t > 0$ if and only if $Y_{t-1} + u_t < \underline{b} - \theta'\delta_t$ and $\overline{\xi}_t > 0$ if and only if $Y_{t-1} + u_t > \overline{b} - \theta'\delta_t$.

Since we consider non-stationary volatility in the innovations, we consider v_t as a process in the following form

$$v_t = \sigma_t z_t \tag{2.5}$$

where $z_t \sim iid(0, 1)$.

Remark 1. Notice that the innovation term $\{v_t\}$ in (2.5) is characterized by $\{\sigma_t\}$ and $\{z_t\}$. Since conditional on $\{\sigma_t\}$, $\{z_t\}$ has independent and identical distribution (i.i.d.), the innovation v_t has mean zero and time varying variance σ_t^2.

Throughout this chapter, we will utilize the following assumptions which will be taken to hold on (2.1) – (2.5):

Assumption. $\mathcal{A}$

1. The lag operator $\psi(L) = \sum_{j=0}^{\infty} \psi_j L^j$ satisfies $\psi(z) \neq 0$ for all $|z| \leq 1$ and $\sum\limits_{j=0}^{\infty} j^s |\psi_j| < \infty$ for some $s \geq 1$.
2. $\{z_t, \mathcal{F}_t\}$ is a MDS (martingale difference sequence) with respect to some filtration $\mathcal{F}_t$, such that $(a) E(z_t^2) = 1 < \infty$, $(b) T^{-1} \sum_{t=1}^{T} z_t^2 \xrightarrow{p} 1$, and $(c) E|z_t|^r < k < \infty$ for some $r \geq 4$.

3. $\sup_{t=1,..,T} E|\underline{\xi}_t|^r < \infty$ and $\sup_{t=1,..,T} E|\overline{\xi}_t|^r < \infty$ or $\max_{t=1,...T} |\underline{\xi}_t|$ and $\max_{t=1,...T} |\overline{\xi}_t|$ are of $o_p(T^{1/2})$

4. $\frac{\underline{b}-\theta'\delta_t}{\lambda T^{1/2}} = \underline{c}+o(1)$ and $\frac{\overline{b}-\theta'\delta_t}{\lambda T^{1/2}} = \overline{c}+o(1)$ where $\underline{c} \leq \overline{c}$ and $\lambda^2 = \psi(1)^2\bar{\omega}^2$ denotes the long run variance.

Assumption $\mathcal{A}1$ implies that $\{u_t\}$ is stable and has an invertible representation in terms of the v_t's. $\mathcal{A}2$ sets the sequence of the error terms $\{z_t\}$ to be martingale differences. These assumptions are quite standard in the time series econometrics literature. $\mathcal{A}3 - \mathcal{A}4$ give the conditions that bounds should satisfy to perform a proper analysis. Assumption $\mathcal{A}3$ presents a restrictive condition that prohibits $\{X_t\}$ excessive leaps at the bounds. Under Assumption $\mathcal{A}4$, the upper and lower bound locations are taken as nuisance parameters. Moreover, Assumption $\mathcal{A}4$ states that, under the presence of bounds, the $\sim T^{1/2}$ order of the random walk part of the series is preserved.

Remark 2. Under Assumption $\mathcal{A}1$, $\Psi(z)^{-1} =: \alpha(z) = 1 - \sum_{j=1}^{\infty} \alpha_j z^j$ is well defined. By allowing $\underline{\xi}_t^* := \alpha(L)\underline{\xi}_t$ and $\overline{\xi}_t^* := \alpha(L)\overline{\xi}_t$ we can conclude that

$$\varepsilon_t = \Psi(L)v_t + \underline{\xi}_t - \overline{\xi}_t = \Psi(L)v_t^*, \quad v_t^* = v_t + \underline{\xi}_t^* - \overline{\xi}_t^* \tag{2.6}$$

Therefore, Δx_t can be written in the LP representation form as $\Delta x_t = \Psi(L)v_t^*$

Remark 3. With the help of Assumption $\mathcal{A}4$ the location of the bounds $\underline{b}$ and $\overline{b}$ are linked to the sample size T. Granger (2010) states that "I(1) process has the strong relationship between now and the distant past, i.e. $corr(X_t, X_{tk}) = 1$ for any k". $\mathcal{A}4$ allows us this property to be preserved in the presence of the range constraints. Moreover, this assumption allows us to obtain the limiting distributions of unit root test statistics in the presence of range constraints, and to develop convenient unit root tests without imposing parametric assumptions on the behavior of X_t in the neighborhood of the bounds.

Remark 4. Following Cavaliere and Xu (2014), we can decompose X_t as

$$X_t = \theta'\delta_t + C(1)\sum_{i=1}^{t} v_i + \sum_{i=1}^{t}(\underline{\xi}_t - \overline{\xi}_t) + \tilde{u}_0 - \tilde{u}_t \tag{2.7}$$

In Eq. (2.7), it is seen that non-stationary part of the $BI(1)$ process can be broken down into a random walk, $\sum_{i=1}^{t} v_i$, and the cumulated regulators,$\sum_{i=1}^{t}(\underline{\xi}_t - \overline{\xi}_t)$. Assumption $\mathcal{A}4$ implies that the order of these two terms are same. As a result, regulators are allowed to affect both short run and long run dynamics of X_t.

The following assumption is given to capture the heteroscedasticity in the innovations:

Assumption. $\mathcal{B}$ The volatility term σ_t satisfies $\sigma_{\lfloor sT \rfloor} := \omega(s)$ for all $s \in [0, 1]$, where $\omega(.) \in \mathfrak{D}$ is non-stochastic and strictly positive. For $t < 0$,$\sigma_t \leq \sigma < \infty$

Assumption $\mathcal{B}$ is the critical assumption which permits the innovation variance dynamics to be considered in a general framework. Under this assumption, the variance of the error term needs only to be bounded and to exhibit a countable amount of jumps. Therefore, a broad family of a volatility process can be allowed. The conventional homoscedasticity assumption made in the unit root testing literature that is $\sigma_t = \sigma$ for all t, also satisfies Assumption $\mathcal{B}$.

We need a fundamental object to capture non-stationary volatility in the asymptotic analysis. This object is characterized by the following function in $\mathcal{C}$:

$$\eta(s) = (\int_0^1 \omega(r)^2 dr)^{-1} \int_0^s \omega(r)^2 dr \tag{2.8}$$

The object defined in (2.8) stands for the variance profile of the time series, since it is constructed by the time series pattern of the observed volatility. Further, Cavaliere and Taylor (2007) show that $\int_0^1 \omega(r)^2 dr = \overline{\omega}^2$ is the limit of $T^{-1}\sum_{t=1}^{T}\sigma_t^2$, where $\overline{\omega}^2$ represents the asymptotic average variance of the error term.

Remark 5. The assumption that the volatility function $\omega(.)$ is deterministic is given to simplify the theoretical setup. However, this assumption can be relaxed by allowing for the cases where the error terms $\{z_t\}$ and $\omega(.)$ are stochastically independent. In fact, in such cases, if the sample paths of the volatility process $\omega(.)$ satisfy Assumption $\mathcal{B}$, the results obtained in this chapter can be considered as conditional on the realization of $\omega(.)$.

2.3 The Variance Transformed Bounded Unit Root Distribution

This section presents how the presence of both range constraints and permanent volatility shifts modifies the asymptotic framework. Therefore, unit root distribution will be generalized to time transformed bounded unit root case.

First, we focus on the effect of the volatility pattern on the asymptotic theory. Throughout this chapter, one of the key roles on the limiting distributions is played by $B_\eta(s) = B(\eta(s))$, which is defined in Cavaliere and Taylor (2007). $B(.)$ is a standard Brownian motion and $\eta(.)$ is an increasing homeomorphism with $\eta(0) = 0$ and $\eta(1) = 1$. Consequently, $\{B_\eta(.)\}$ is referred to as the 'variance-transformed' Brownian motion.

Second, to account for the bounded nature of the time series, consider also the following result from Harrison (1985), which is very crucial in explaining the main arguments of this paper:

Lemma 1. Fix the limits $[\underline{c}, \overline{c}]$ with $c > 0$ and let C be the space of all continuous functions. Define C_0 as the set of all functions $x \in \mathrm{C}$ such that $x_0 \in [\underline{c}, \overline{c}]$. Then for each $x \in \mathrm{C}_0$, there is a unique pair of continuous functions (l, q) which satisfy $l_t = \sup_{0 \le s \le t} (x_s - q_s)^-$ and $q_t = \sup_{0 \le s \le t} (\overline{c} - x_s - l_s)^-$ and this same pair satisfies the following properties:

1. l_t and q_t are continous and increasing and satisfy $l_0 = q_0 = 0$
2. $h_t = (x_t + l_t - q_t) \in [\underline{c}, \overline{c}]$ for all $t \geq 0$
3. l_t and q_t increase only when $h_t = \underline{c}$ and $h_t = \overline{c}$ respectively.

Cavaliere and Taylor (2007) show that the variance transformed Brownian motion $B_\eta(.) \in \mathrm{C}$. This property coupled with the findings of Lemma 1 tell us that we can find a pair of functions $L(.)$ and $Q(.)$ such that $B_\eta(.) + L(.) - Q(.)$ is regulated process. Now, we introduce the following key definition:

Definition 1. Let $B_\eta(t)$ be variance shifted Brownian motion defined on C (see, Cavaliere and Taylor (2007)). Fix the bounds $\underline{c}$ and $\overline{c}$. If $B_\eta(0) \in [\underline{c}, \overline{c}]$, then there exists continuous functions $L(t)$ and $Q(t)$ satisfying the conditions of Lemma 1 such that $B_{\eta}^{\underline{c},\overline{c}}(t) = (B_{\eta}(t) + L(t) - Q(t)) \in [\underline{c}, \overline{c}]$. The functions $l(t)$ and $q(t)$ are called regulators and the function $B_{\eta}^{\underline{c},\overline{c}}(.)$ is called 'regulated variance transformed Brownian motion' with bounds at $[\underline{c}, \overline{c}]$.

Consider next the continuous-time approximation of a process $\{Y_t\}$ on the cadlag space $\mathcal{D}[0,1]$ to transform the original series through the broken line process.

$$Y_T(s) = (\lambda^2 T)^{-1/2}(Y_{\lfloor Ts \rfloor} - Y_0), s \in [0,1]$$

The main result of this section is now summarized in the following theorem:

Theorem 1. Let $\{Y_t\}$ be the process defined between (2.2)-(2.5) and fix the bounds $[\underline{b}, \overline{b}]$. If $\mathcal{D}[0,1]$ is endowed with uniform topology and $Y_0 \in [\underline{c}, \overline{c}]$ then $Y_T(s) \xrightarrow{w} B_{\eta}^{\underline{c},\overline{c}}(s)$.

2.3.1 Extension for the Deterministic Components

In the previous analysis, we only consider the pure random walk process Y_t, but in our initial characterization we observe the series X_t with deterministic components. To remove this deterministic component, we regress δ_t on x_t and obtain the residuals $\hat{X}_t = Y_t - (\hat{\theta} - \theta)'\delta_t$ where $\hat{\theta}$ is OLS estimate of θ. In Lemma 2 below, we now provide the representations for the OLS de-trended time transformed Brownian motions:

Lemma 2. Under assumption $\mathcal{A}$ and X_t is generated by (2.1) -(2.5), we define partial sum process for detrended process as $\hat{x}_T(s) = (\lambda^2 T)^{-1/2}(\hat{X}_{\lfloor Ts \rfloor} - \hat{X}_0), s \in [0,1]$. Then we have $\hat{X}_T(s) \xrightarrow{w} B_{j,\eta}^{\underline{c},\overline{c}}(s)$ for $j = 1, 2$. $j = 1$ when $\delta_t = 1$ (only mean) and $j = 2$ when $\delta_t = [1, t]'$, where

$$B_{j,\eta}^{c,\bar{c}}(s) = B_{\eta}^{c,\bar{c}}(s) - \left(\int_0^1 B_{\eta}^{c,\bar{c}}(r) D_j(r)' dr\right)\left(\int_0^1 D_j(r) D_j(r)' dr\right)^{-1} D_j(s)$$

and $D_j(s) = 1$ if $j = 1$ and $D_j(s) = [1, s]'$ if $j = 2$.

Remark 6. The OLS detrended Brownian motions represented in Lemma 2 will be used in characterizing the limiting distributions of the unit root test statistics which will be defined in the next section.

2.4 Large Sample Results for Unit Root Tests

In this section, we present the implementations of the limited time transformed unit root distribution for the well known unit root tests. Accordingly, we discuss the effect of the time-varying variances of the shocks on the asymptotic distributions of augmented Dickey-Fuller and M unit root tests when the series is regulated.

For given sample $\{X_t\}_{t=0}^{T}$, the ADF statistics are based on the OLS regression:

$$\hat{X}_t = \alpha \hat{X}_{t-1} + \sum_{i=1}^{k} \alpha_i \Delta \hat{X}_{t-i} + \varepsilon_{t,k} \tag{2.9}$$

and are defined as

$$ADF_\alpha = \frac{T(\hat{\alpha} - 1)}{\hat{\alpha}(1)}$$
$$ADF_t = \frac{\hat{\alpha} - 1}{s(\hat{\alpha})}$$

where $\hat{\alpha}(1) = 1 - \sum_{i=1}^{k} \hat{\alpha}_i$, with $\hat{\alpha}_i$ denoting OLS estimator of α_i in (2.9) and $s(\hat{\alpha})$ the OLS standard error of $\hat{\alpha}$. Here $\hat{X}_t$ is OLS residual of δ_t on X_t such that $\hat{X}_t = X_t - \hat{\theta}'\delta_t = Y_t - (\hat{\theta} - \theta)'\delta_t$.

The M statistics are defined as

$$MZ_\alpha = \frac{T^{-1}\hat{X}_T^2 - T^{-1}\hat{X}_0^2 - s_{AR}^2(k)}{2T^{-2}\sum_{t=1}^{T}\hat{X}_{t-1}^2}$$
$$MSB = (T^{-2}\sum_{t=1}^{T}\hat{X}_{t-1}^2 / s_{AR}^2(k))^{1/2}$$
$$MZ_t = MZ_\alpha \times MTB$$

where $s_{AR}^2(k)$ is an autoregressive estimator of the spectral density frequency zero of $\{\varepsilon_t\}$. Specifically,

$$s_{AR}^2(k) = \hat{\sigma}^2 / \hat{\alpha}(1)^2$$

where $\hat{\alpha}(1)$ is as defined above and $\hat{\sigma}^2$ is the OLS variance estimator from the regression 2.9. By following Lewis and Reinsel (1985), for the aforementioned tests, the lag truncation parameter is assumed to satisfy the Assumption below:

Assumption. $\mathcal{C}$ As $T \to \infty$, $1/k + k^2/T \to 0$.

In Theorem below, we introduce the representations for the limiting null distributions of the aforementioned test statistics in the presence of range constraints and non-stationary volatility.

Theorem 2. Let $\{X_t\}_{t=0}^{T}$ be generated as in (2.1)-(2.5) with $\rho = 1$. Under Assumptions $\mathcal{A}$ and $\mathcal{B}$,

(i) $s_{AR}^2(k) \xrightarrow{p} \bar{\omega}^2 C(1)^2$

(ii) $ADF_\alpha, MZ_\alpha \xrightarrow{w} 0.5(B^{\underline{c},\overline{c}}_{\eta,j}(1)^2 - B^{\underline{c},\overline{c}}_{\eta,j}(0)^2 - 1)(\int_0^1 B^{\underline{c},\overline{c}}_{\eta,j}(s)^2 ds)^{-1}$

(iii) $MSB \xrightarrow{w} (\int_0^1 B^{\underline{c},\overline{c}}_{\eta,j}(s)^2 ds)^{1/2}$

(iv)$ADF_t, MZ_t \xrightarrow{w} 0.5(B^{\underline{c},\overline{c}}_{\eta,j}(1)^2 - B^{\underline{c},\overline{c}}_{\eta,j}(0)^2 - 1)(\int_0^1 B^{\underline{c},\overline{c}}_{\eta,j}(s)^2 ds)^{-1/2}$

where $B^{\underline{c},\overline{c}}_{\eta,j}(s)$ is defined as in Lemma 2.

Remark 7. The limiting distributions used in ADF and M unit root tests in Theorem 2 differ from standard integrated of order one -I(1)- asymptotics. The sample paths of limiting process are bounded between $\underline{c}$ and $\overline{c}$ and time transformed by directing process $\eta(.)$. The well known case of $I(1)$ process has no bounds by setting $\underline{c}$ and $\overline{c}$ equal to infinity and $\eta(s) = s$.

2.5 Testing for Unit Root Tests under The Presence of Bounds and Non-stationary Volatility

As we have discussed in Section 2.4, the conventional unit root inference is influenced by the presence of bounds and permanent volatility shifts. Therefore the limiting null distributions of the frequently used test statistics become nonstandard. The application of the proposed unit root tests for bounded variance transformed processes requires the employment of three steps. First, Section 2.5.1 presents the consistent estimation of the variance profile $\eta(s)$. Second, Section 2.5.2 introduces the consistent estimators of the nuisance parameters $\underline{c}$ and $\overline{c}$. Last, in Section 2.5.3, we develop the simulation-based tests which can be performed to acquire asymptotically valid unit root tests in the presence of the bounds and non-stationary volatility.

2.5.1 Consistent estimation of the variance profile

In this section, we present the variance profile estimator, which is defined in Cavaliere and Taylor (2007). Let $\hat{X}_t$ be de-trended or demeaned version of observed series X_t, then we have

$$\hat{\eta}(s) = \frac{\sum_{t=1}^{\lfloor sT \rfloor} (\Delta \hat{X}_t)^2 + (sT - \lfloor sT \rfloor)(\Delta \hat{X}_{\lfloor sT \rfloor + 1})^2}{\sum_{t=1}^{T} (\Delta \hat{X}_t)^2} \tag{2.10}$$

In the following Lemma, we now demonstrate that $\hat{\eta}(.)$ is a uniformly consistent estimator for the variance profile $\eta(.)$ in the presence of bounds by generalizing Theorem 2 of Cavaliere and Taylor (2007) to the present framework.

Lemma 3. Under the assumptions $\mathcal{A}$ and $\mathcal{B}$, we have $\hat{\eta}(s) \xrightarrow{p} \eta(s)$ uniformly for all $s \in [0, 1]$.

2.5.2 Consistent estimation of the range parameters

In Section 2.2, we assume that the bounds $\underline{b}$, $\overline{b}$ are known. Therefore, there is a feasible way to estimate the nuisance parameters $\underline{c}$, $\overline{c}$ consistently. To this aim, in our setup, consider two estimators which are easy to implement:

$$\hat{\underline{c}} = \frac{\underline{b} - x_0}{s_{AR}(k) T^{1/2}}, \quad \hat{\overline{c}} = \frac{\overline{b} - x_0}{s_{AR}(k) T^{1/2}} \tag{2.11}$$

where $s^2_{AR}(k)$ is the spectral AR estimator of the long run variance as defined in Section 2.4. The result on the consistency of $\hat{\underline{c}}$ and $\hat{\overline{c}}$ is given in the next lemma:

Lemma 4. Let the assumptions of Theorem 2 hold. Then, $\hat{\underline{c}} \xrightarrow{p} \underline{c}$ and $\hat{\overline{c}} \xrightarrow{p} \overline{c}$.

Lemma 4 states that, the two nuisance parameters $\underline{c}$ and $\overline{c}$ appearing in the asymptotic distributions of Theorem 1 and 2, can be consistently estimated with the usage of $\hat{\underline{c}}$ and $\hat{\overline{c}}$ respectively. These estimators with the variance profile estimator are key ingredients for conducting simulation based tests.

2.5.3 Simulation based tests

Now, given the estimate $\hat{\eta}(s)$ and $\hat{\underline{c}}$ and $\hat{\overline{c}}$, we are able to approximate the quantiles from the asymptotic null distributions of the conventional M and ADF statistics given in Theorem 2, by utilizing direct simulation methods.

Let $\{e_t\}$ be iid(0,1) random variable. From Cavaliere and Taylor (2007) Theorem 3 we know that

$$B_{\hat{\eta},T}(s) := T^{-1/2} \sum_{t=1}^{\lfloor \hat{\eta}(\lfloor sT \rfloor/T)T \rfloor} e_t \xrightarrow{w} B_{\eta}(s) \text{for all} s \in [0,1] \tag{2.12}$$

Therefore, define $f_t = \Delta B_{\hat{\eta},T}(t/T) = B_{\hat{\eta},T}(t/T) - B_{\hat{\eta},T}((t-1)/T)$ for $t = 1, 2, ..., T$. Consider the recursive setup as follows:

$$\tilde{X}_t = \begin{cases} \hat{\underline{c}} & \text{if } \tilde{X}_{t-1} + f_t < \hat{\underline{c}} T^{1/2} \\ \hat{\overline{c}} & \text{if } \tilde{X}_{t-1} + f_t > \hat{\overline{c}} T^{1/2} \\ \tilde{X}_{t-1} + f_t & \text{otherwise} \end{cases} \tag{2.13}$$

Now, we define the following theorem as sample T approaches to infinity:

Theorem 3. Under the assumptions $\mathcal{A}$ and $\mathcal{B}$, we have $\tilde{X}_T(s) \xrightarrow{w} B_{\eta}^{\underline{c},\overline{c}}(s)$ for all $s \in [0,1]$ where $\tilde{X}_T(s) = T^{-1/2} X_{\lfloor sT \rfloor}$.

Remark 8. Theorem 3 shows that as T approaches to infinity, the simulated process $\tilde{X}_t$ is distributed as the variance transformed bounded Brownian motion $B_{\eta}^{\underline{c},\overline{c}}(.)$ of Theorem 1. We therefore approximate the quantiles from the non-pivotal limiting null distributions in Theorem 2 by utilizing numerical simulations based on estimating $\underline{c}$ and $\overline{c}$ and using the cadlag processes $\tilde{X}_t$ and f_t. Using the conventional ADF and M statistics in Section 2.4, we can now obtain the simulation based versions of the ADF and M tests.Testing for unit roots can be performed by simulated critical values for a given significance level.

In practice, our proposed tests, which are robust to presence of both non-stationary volatility and bounds can be applied as follows. First, by creating a sequence of $iid(0,1)$ random numbers, $B_{\hat{\eta}}(s)$ is generated. Then, with the recursive procedure defined in (2.13), $B_{\hat{\eta}}^{\hat{\underline{c}},\hat{\overline{c}}}(.)$ is constructed. Next, corresponding demeaned process $B_{\hat{\eta},1}^{\hat{\underline{c}},\hat{\overline{c}}}(.)$ is obtained by standard OLS projection. For illustration, let us take the test of ADF_t. The quantity $0.5(B_{\hat{\eta},j}^{\hat{\underline{c}},\hat{\overline{c}}}(j)^2 - B_{\hat{\eta},j}^{\hat{\underline{c}},\hat{\overline{c}}}(0)^2 - 1)(T^{-1}B_{\hat{\eta},j}^{\hat{\underline{c}},\hat{\overline{c}}}((t-1)/T)^2)^{-1/2}$ corresponding to discretized version of the limiting distribution is computed. This operation is needed to be employed for B independent replications. Next step is to construct order statistic of the B outcomes as $ADF_t^{(1)} \leq ADF_t^{(2)} \leq ... \leq ADF_t^{(B)}$. For a significance level ζ, an estimate of the desired quantile is calculated as $ADF_t^{(\lfloor \zeta B \rfloor)}$. The rejection decision of the unit root null at significance level ζ is given where ADF_t is lower than $ADF_t^{(\lfloor \zeta B \rfloor)}$. Note that, MZ_α, MZ_t and ADF_α tests can be performed with the same procedure.

Remark 9. It is noteworthy to state that, by construction, power functions of our tests closely approximate the size-adjusted powers of the corresponding standard unit root tests, heteroscedasticity robust unit root tests of Cavaliere and Taylor (2007, 2008) and bounded unit root tests of Cavaliere and Xu (2014).

2.6 Finite sample simulations

This section presents Monte Carlo simulation results for ADF_a and MZ_a unit root tests under a variety of volatility patterns and bound parameters. We compare the performance of our proposed unit root tests ADF_a^{bnv} and MZ_a^{bnv}, which are robust to the presence of both bounds and heteroscedasticity with (i) standard tests ADF_a^s and MZ_a^s, (ii) heteroscedasticity robust tests ADF_a^{nv} and MZ_a^{nv} of Cavaliere and Taylor (2007, 2008), and (iii) bounded unit root tests ADF_a^b and MZ_t^b of Cavaliere and Xu (2014).[1]

We generate data as in Equations (2.1)-(2.5) with the sample sizes $T = \{250, 500\}$ by setting $Y_0 = \theta = 0$. We consider the case of symmetric bounds ($\overline{c} = -\underline{c} =: c > 0$). Additionally, to investigate the performance of the tests under heteroscedasticity, we consider the following error term variance specifications [2] :

1. Constant Volatility (CV): $\omega_1(s) = 1$, for $s \in [0, 1]$.

2. Single Break in Volatility (SBV): $\omega_2(s) = 1 + (\frac{1}{3} - 1)^* I(s > 0.2T)$ for $s \in [0, 1]$.

3. Trending Volatility (TV): $\omega_3(s) = 1 + (\frac{1}{3} - 1)^* s$ for $s \in [0, 1]$.

[1] The performance of ADF_t, MZ_t and MSB tests are omitted since they exhibit similar performance to MZ_a and ADF_a tests and qualitatively their contribution is negligable to the reported results. They are available upon request.

[2] In the error term variance specifications, we follow Cavaliere and Taylor (2007).

4. Exponential Integrated Stochastic Volatility (EISV): $\omega_4(s) = exp(9B(s))$ for $s \in [0,1]$ and $B(s)$ is standard Brownian motion process.

All experiments are performed under 10.000 replications. All unit root tests are employed at a nominal asymptotic 0.05 level. Following Cavaliere and Taylor (2009) and Cavaliere and Xu (2014), we draw the innovations z_t from i.i.d. $N(0,1)$. We obtain the distribution of $\varepsilon_t = \Delta X_t$ by reflecting the distribution of $u_t = C(L)v_t$ at $\underline{b} - X_{t-1}$ and $\overline{b} - X_{t-1}$.[3] Critical values of our tests are obtained by direct simulation method described in Section 2.5.3 by fixing the step size to T. The variance profile and bound parameters are estimated as described in Sections 2.5.1 and 2.5.2. All of the test results are based on OLS demeaned data.

We report the performance of the aforementioned unit root tests for serially uncorrelated and correlated innovations. For the case where the innovations are serially uncorrelated, we set $\Psi(L) = 1$ and correspondingly $k = 0$ in (2.9). Then, we allow for weak dependence in u_t. Under this scenario, we choose the optimal lags in equation (2.9) by using MAIC lag length selection criterion of Ng and Perron (2001) with $k \leq \lfloor 12(T/100)^{0.25} \rfloor$. Finally, under weak dependence, by following Ferretti and Romo (1996), Chang and Park (2002) and Cavaliere and Xu (2014), to improve the finite sample performance of the tests, we use re-colored simulation-based tests. The procedure in these tests re-builds stationary serial correlation into MC innovations without changing the limiting distributions presented in Theorem 2. [4]

[3] We set $\underline{\xi}_t := 2(\underline{b} - (X_{t-1} + \varepsilon_t))I(X_{t-1} + \varepsilon_t < \underline{b})$ and $\overline{\xi}_t := 2(\overline{b} - (X_{t-1} + \varepsilon_t))I(X_{t-1} + \varepsilon_t < \overline{b})$. We also set various types of truncation mechanisms and obtain very similiar results. They are also available upon request.

[4] We refer the reader to Chang and Park (2002) for the detailed information about re-colored simulation based tests.

There are four serial correlation scenarios considered for the innovation sequence $\{u_t\}$. In the first scenario, innovations do not contain any serial correlation. In the second one, u_t follows MA(1) model, i.e. $u_t = v_t + 0.5v_{t-1}$. In the third one, u_t is a stationary AR(1) process, i.e. $u_t = 0.5u_{t-1} + v_t$. Finally, last one follows an ARMA(1,1) process, i.e. $u_t = -0.5u_{t-1} + v_t + 0.5v_{t-1}$[5]. We set $\rho \in \{1, 0.93, 0.86\}$, where $\rho = 1$ is for the size and other values are for the power evaluation of the tests.

2.6.1 Size Performance

In this section, we report the finite sample size performance of the aforementioned tests. Table 2.1 demonstrates the size performance under uncorrelated shocks. In Tables 2.2, 2.3 and 2.4, we report the size performance for weakly dependent shocks with the models of $MA(1)$, $AR(1)$ and $ARMA(1,1)$, respectively.

2.6.1.1 Uncorrelated errors

First, consider the constant volatility case, where the errors are homoscedastic. Under the case of no bounds, all of the tests have the same performance and are close to the nominal significance level. This is an expected result since it is a benchmark case for all of the tests. However, in the presence of bounds, standard tests ADF_a^s-MZ_a^s and heteroscedasticity robust tests ADF_a^{nv}-MZ_a^{nv} become significantly over-sized, since they are not robust to the presence of bounds. For instance, under T=250, for $c = 0.4$, empirical rejection frequency (ERF) of ADF_a^s

[5]Results for other serial correlation scenarios do not differ qualitatively from those reported here

is 0.32 and ERF of ADF_a^{nv} is 0.27. We do not see over-sizing in ADF_a^b and MZ_a^b, since by nature, they are robust to homoscedastic data with bounds. Our tests ADF_a^{bnv} and MZ_a^{bnv} exhibit great performance under this case.

Next, consider the single break in volatility case. Now under the case of no bounds, ADF_a^s-MZ_a^s and ADF_a^b-MZ_a^b are significantly oversized. We observe that under $T = 250$, with no bounds, ERFs of MZ_a^s and MZ_a^b are 0.16. These are expected results since these tests are not robust to non-stationary volatility. Under no bounds, as expected ADF_a^{nv}-MZ_a^{nv} and our tests ADF_a^{bnv}- MZ_a^{bnv} have correct size. However, as bounds become tight, ADF_a^{nv}, MZ_a^{nv} become size distorted, since as it is mentioned before these tests are not robust to the presence of bounds. For instance, under T=500 where $c = 0.6$, ERFs of ADF_a^s, ADF_a^{nv} and ADF_a^b are 0.33, 0.15 and 0.14 respectively. However, ERF of our proposed tests ADF_a^{bnv} and MZ_a^{bnv} are so close to nominal significance level, 0.05 under this case, for all bound specifications.

Now, consider the trending volatility case. In this case, we observe the similar performance to the single break volatility case. Under this case, ADF_a^s-MZ_a^s and ADF_a^b-MZ_a^b are still oversized, but the size distortion is not as severe as the single break volatility case. This result is consistent with the findings of Cavaliere and Taylor (2007). Again, under $T = 250$, with no bounds, ERFs of MZ_a^s and MZ_a^b are 0.07. We observe as bounds become tighter, ADF_a^{nv}, MZ_a^{nv} have nearly the same size distortion as in the case of the single break in volatility. It is important to note that, under this heteroscedasticity scenario with all bound parameters, our tests ADF_a^{bnv} and MZ_a^{bnv} are largely satisfactory and avoid the over-sizing problems associated with other tests.

Under uncorrelated shocks, finally consider the case of exponential integrated stochastic volatility. ADF_a^s-MZ_a^s and ADF_a^b-MZ_a^b have the most severe size distortion under this scenario. For instance, under $T = 500$ with $c = 0.4$, ERFs of ADF_a^s and MZ_a^b are 0.64 and 0.26. Also, we observe that under the same case, ERF of ADF_a^{nv} is 0.23. However, our tests ADF_t^{bnv} and MZ_t^{bnv} are always successful in controlling size under exponential integrated stochastic volatility.

Overall, we observe that size distortions observed in ADF_a^s-MZ_a^s, ADF_a^b-MZ_a^b and ADF_a^{nv}-MZ_a^{nv} tests do not start to disappear as sample size T increases, because the nuisance parameter problems arising from non-stationary volatility and bounds do not disappear in the long run.

2.6.1.2 Autocorrelated errors

The size performance of aforementioned tests are now investigated for u_t following a linear process. As it can be seen from Tables 2.2, 2.3 and 2.4, our tests ADF_a^{bnv} and MZ_a^{bnv} are successful in controlling size under weakly dependent shocks regardless of the bound parameter, volatility type and sample size.

Under weakly dependent errors, standard MZ_a^s and ADF_t^s have the similar over-sizing problem with the case of uncorrelated errors. For example, consider the results in Table 2.3, where the innovations follow $AR(1)$ process. We observe that in single break volatility case with $c = 0.6$ and $T = 250$, ERF of MZ_a^s is 0.23.

Under autocorrelated errors, the tests MZ_a^{nv} and ADF_a^{nv} are still oversized under tight bounds. For an evidence, let us take a closer look at Table 2.2, where the innovations follow $MA(1)$ process. For instance, under $T = 250$, data with single break volatility and bound parameter $c = 0.4$, ERF of ADF_a^{nv} is 0.11. Finally, the tests MZ_a^b and ADF_a^b still exhibit significant over-sizing in the presence of non-stationary volatility under weakly dependent shocks. Now, for instance consider Table 2.4, where shocks follow $ARMA(1,1)$ process. It can be seen that in exponential integrated stochastic volatility case with no bounds and $T = 500$, ERF of ADF_a^b is 0.28. Overall, we observe in Tables 2.2-2.4 that, in addition to being robust to non-stationary volatility and bounds, our tests ADF_a^{bnv} and MZ_a^{bnv} are also successful in controlling size in the presence of the serial correlation problem. Therefore, ERFs our tests are still so close to the nominal significance level except mild under-sizing problem in some cases.[6]

2.6.2 Power Performance

In this section, we investigate the empirical power against particular alternatives $\rho \in \{0.93, 0.86\}$. In the previous section, we observe that other unit root tests do not exhibit satisfactory size performance. Therefore, their empirical power results are not reliable. So, we need size-adjusted power of these tests to make a meaningful comparison. Moreover, as it is stated in Remark 9, by construction, our tests possess power functions that closely approximate the size adjusted power functions of the aforementioned tests in the literature. Therefore, following

[6] We observe in all tables the size distortions created by EISV are different. We explain these high variations by the stochastic feature of the volatility process and the simulation uncertainty.

Cavaliere and Taylor (2007), we only report power performance for our ADF_a^{bnv} and MZ_a^{bnv} tests which are so similar to size-adjusted power performance of other tests.

We start by looking through Table 2.5, where innovations are uncorrelated. First, under constant volatility, we can conclude that for both ADF_a^{bnv} and MZ_a^{bnv}, as bounds get tighter, finite sample power decrease, but we still have satisfactory performance. This result is important since Cavaliere and Xu (2014) do not analyze the power performance of their unit root tests. Second, under single break volatility, power is less, compared to constant volatility case. Here, again, as bounds become tight, power performance declines. Next, under trending volatility, we observe higher power compared to single break volatility case, but lower than constant volatility case. In this case, we see a pattern of decrease in power as bounds become tighter. Finally, under exponential integrated stochastic volatility, we see similar performance to trending volatility case. Overall, we observe that ADF_a^{bnv} is more powerful compared to MZ_a^{bnv}.

Next consider Tables 2.6-2.8, where weak dependence models are $MA(1)$, $AR(1)$ and $ARMA(1,1)$ respectively. Overall the pattern is similar to the uncorrelated case under various heteroscedasticity scenarios and range specifications. However, empirical powers compared to uncorrelated case are lower. Especially power performance of tests under single break volatility case is much lower than the uncorrelated case. Under serially correlated errors, we now observe that the power of ADF_a^{bnv} test is outperformed by MZ_a^{bnv} test.

2.7 Conclusion

In this chapter, we have shown that the presence of both non-stationary volatility and range constraints have severe implications on the validity of the unit root tests in the literature. These tests suffer from significant size distortions. As a remedy, we have proposed a unit root testing mechanism that provides reliable inference under a very general class of volatility dynamics and bound specifications. For construction of testing procedure, we introduce 'variance shifted bounded Brownian motion' process and provide the convergence of the observed series to this process.

This testing mechanism is based on direct simulation methods which obtain asymptotic null distributions of the unit root statistics. Since the asymptotic distributions depend on nuisance parameters arising from heteroscedasticity and bounds, we consistently estimate the variance profile and bound parameters. Finite sample simulations show that the testing approach outlined in this chapter is successful in eliminating size distortions observed in other unit root tests.

Table 2.1: Finite sample size performance of ADF_a and MZ_a tests for i.i.d. shocks

		ADF_a^s	ADF_a^{nv}	ADF_a^b	ADF_a^{bnv}	MZ_a^s	MZ_a^{nv}	MZ_a^b	MZ_a^{bnv}
Volatility Type	**c**				**T=250**				
CV	∞	0.05	0.05	0.05	0.05	0.04	0.04	0.05	0.05
	0.8	0.11	0.10	0.05	0.05	0.10	0.09	0.05	0.05
	0.6	0.17	0.16	0.05	0.05	0.16	0.15	0.04	0.04
	0.4	0.32	0.27	0.04	0.04	0.30	0.25	0.03	0.04
SBV	∞	0.16	0.05	0.16	0.05	0.15	0.04	0.15	0.05
	0.8	0.29	0.12	0.19	0.04	0.28	0.10	0.18	0.04
	0.6	0.32	0.14	0.14	0.04	0.30	0.12	0.12	0.04
	0.4	0.48	0.22	0.10	0.04	0.46	0.20	0.09	0.04
TV	∞	0.08	0.05	0.08	0.05	0.07	0.04	0.07	0.05
	0.8	0.19	0.12	0.10	0.05	0.17	0.11	0.08	0.04
	0.6	0.23	0.16	0.07	0.04	0.22	0.15	0.08	0.04
	0.4	0.35	0.23	0.07	0.04	0.32	0.21	0.07	0.04
EISV	∞	0.17	0.04	0.17	0.04	0.16	0.03	0.16	0.04
	0.8	0.28	0.06	0.21	0.04	0.27	0.05	0.19	0.04
	0.6	0.41	0.10	0.22	0.04	0.40	0.08	0.20	0.04
	0.4	0.60	0.21	0.26	0.04	0.58	0.18	0.24	0.03
					T=500				
CV	∞	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.8	0.11	0.11	0.05	0.05	0.10	0.11	0.05	0.05
	0.6	0.19	0.18	0.04	0.04	0.18	0.18	0.04	0.04
	0.4	0.32	0.30	0.04	0.05	0.31	0.29	0.04	0.04
SBV	∞	0.16	0.05	0.16	0.05	0.15	0.05	0.15	0.05
	0.8	0.30	0.12	0.18	0.05	0.29	0.11	0.17	0.04
	0.6	0.33	0.15	0.14	0.04	0.32	0.14	0.13	0.04
	0.4	0.53	0.23	0.10	0.04	0.52	0.22	0.09	0.04
TV	∞	0.08	0.05	0.07	0.05	0.07	0.04	0.07	0.04
	0.8	0.18	0.12	0.09	0.05	0.17	0.11	0.08	0.04
	0.6	0.23	0.17	0.08	0.04	0.22	0.16	0.07	0.04
	0.4	0.36	0.27	0.08	0.04	0.35	0.26	0.07	0.04
EISV	∞	0.21	0.04	0.20	0.04	0.17	0.03	0.17	0.04
	0.8	0.28	0.08	0.22	0.04	0.31	0.05	0.23	0.04
	0.6	0.46	0.11	0.26	0.04	0.43	0.08	0.23	0.04
	0.4	0.64	0.23	0.29	0.04	0.62	0.18	0.26	0.04

Notes: (i) Nominal 5% asymptotic level (ii) Tests based on OLS de-meaned data (iii)Tests with superscripts s, nv, b, bnv are for standard, nonstationary volatility robust, bounds robust, and both bounds and nonstationary volatility robust versions, respectively. (iv) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively.

Table 2.2: Finite sample size performance of ADF_a and MZ_a tests for MA(1) shocks

Volatility Type	c	ADF_a^s	ADF_a^{nv}	ADF_a^b	ADF_a^{bnv}	MZ_a^s	MZ_a^{nv}	MZ_a^b	MZ_a^{bnv}
					T=250				
CV	∞	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.05
	0.8	0.09	0.08	0.04	0.04	0.08	0.09	0.05	0.05
	0.6	0.14	0.13	0.05	0.05	0.13	0.13	0.05	0.05
	0.4	0.21	0.20	0.05	0.05	0.19	0.21	0.05	0.05
SBV	∞	0.17	0.04	0.13	0.04	0.12	0.04	0.10	0.05
	0.8	0.23	0.06	0.13	0.04	0.18	0.08	0.11	0.04
	0.6	0.23	0.09	0.10	0.04	0.18	0.11	0.08	0.05
	0.4	0.25	0.11	0.06	0.04	0.19	0.13	0.06	0.04
TV	∞	0.08	0.04	0.07	0.04	0.07	0.04	0.07	0.05
	0.8	0.15	0.08	0.08	0.04	0.13	0.09	0.09	0.05
	0.6	0.18	0.11	0.09	0.05	0.15	0.11	0.10	0.05
	0.4	0.23	0.14	0.10	0.05	0.20	0.15	0.10	0.05
EISV	∞	0.09	0.07	0.08	0.04	0.06	0.04	0.07	0.05
	0.8	0.12	0.07	0.12	0.03	0.09	0.07	0.11	0.04
	0.6	0.14	0.09	0.13	0.03	0.11	0.08	0.13	0.04
	0.4	0.15	0.11	0.24	0.03	0.11	0.12	0.22	0.04
					T=500				
CV	∞	0.05	0.05	0.05	0.04	0.05	0.05	0.05	0.05
	0.8	0.10	0.09	0.05	0.05	0.09	0.09	0.05	0.05
	0.6	0.15	0.15	0.06	0.05	0.14	0.15	0.05	0.05
	0.4	0.21	0.22	0.05	0.05	0.20	0.23	0.06	0.05
SBV	∞	0.17	0.04	0.14	0.04	0.15	0.04	0.13	0.04
	0.8	0.26	0.07	0.13	0.04	0.23	0.07	0.12	0.04
	0.6	0.25	0.09	0.12	0.04	0.22	0.08	0.10	0.04
	0.4	0.31	0.12	0.08	0.04	0.26	0.09	0.07	0.05
TV	∞	0.07	0.04	0.07	0.04	0.06	0.04	0.07	0.04
	0.8	0.15	0.10	0.09	0.05	0.14	0.10	0.09	0.05
	0.6	0.19	0.13	0.08	0.05	0.18	0.12	0.07	0.05
	0.4	0.26	0.19	0.06	0.05	0.24	0.18	0.07	0.05
EISV	∞	0.09	0.04	0.08	0.04	0.08	0.04	0.08	0.04
	0.8	0.13	0.06	0.09	0.04	0.11	0.07	0.11	0.05
	0.6	0.16	0.08	0.13	0.04	0.15	0.09	0.12	0.05
	0.4	0.39	0.14	0.26	0.04	0.32	0.15	0.24	0.03

Notes: (i) Nominal 5% asymptotic level (ii) Tests based on OLS de-meaned data (iii)Tests with superscripts s, nv, b, bnv are for standard, nonstationary volatility robust, bounds robust, and both bounds and nonstationary volatility robust versions, respectively. (iv) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively.

Table 2.3: Finite sample size performance of ADF_a and MZ_a tests for AR(1) shocks

Volatility Type	c	ADF_a^s	ADF_a^{nv}	ADF_a^b	ADF_a^{bnv}	MZ_a^s	MZ_a^{nv}	MZ_a^b	MZ_a^{bnv}
					T=250				
CV	∞	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
	0.8	0.08	0.08	0.04	0.05	0.07	0.08	0.04	0.05
	0.6	0.12	0.12	0.05	0.05	0.11	0.12	0.05	0.05
	0.4	0.19	0.17	0.05	0.05	0.18	0.18	0.05	0.05
SBV	∞	0.16	0.04	0.13	0.04	0.12	0.05	0.10	0.05
	0.8	0.23	0.08	0.15	0.04	0.19	0.08	0.13	0.05
	0.6	0.23	0.10	0.11	0.04	0.19	0.08	0.09	0.04
	0.4	0.24	0.12	0.08	0.03	0.19	0.09	0.06	0.04
TV	∞	0.07	0.04	0.07	0.05	0.06	0.04	0.06	0.05
	0.8	0.13	0.09	0.09	0.05	0.11	0.10	0.09	0.05
	0.6	0.15	0.11	0.07	0.05	0.14	0.12	0.07	0.05
	0.4	0.19	0.14	0.08	0.05	0.18	0.15	0.07	0.05
EISV	∞	0.39	0.04	0.26	0.03	0.24	0.04	0.16	0.04
	0.8	0.49	0.08	0.30	0.04	0.33	0.09	0.20	0.04
	0.6	0.50	0.10	0.29	0.03	0.35	0.12	0.20	0.04
	0.4	0.48	0.12	0.24	0.03	0.34	0.14	0.16	0.03
					T=500				
CV	∞	0.05	0.05	0.05	0.05	0.04	0.05	0.05	0.05
	0.8	0.09	0.09	0.05	0.05	0.09	0.09	0.05	0.05
	0.6	0.14	0.14	0.05	0.05	0.13	0.14	0.05	0.05
	0.4	0.20	0.21	0.05	0.05	0.19	0.21	0.05	0.05
SBV	∞	0.16	0.04	0.13	0.03	0.14	0.04	0.12	0.04
	0.8	0.26	0.07	0.15	0.04	0.23	0.08	0.14	0.04
	0.6	0.26	0.08	0.12	0.04	0.24	0.09	0.11	0.05
	0.4	0.29	0.09	0.08	0.04	0.25	0.10	0.07	0.04
TV	∞	0.08	0.05	0.07	0.05	0.07	0.05	0.07	0.05
	0.8	0.14	0.09	0.09	0.05	0.14	0.09	0.09	0.05
	0.6	0.18	0.12	0.08	0.05	0.18	0.13	0.08	0.05
	0.4	0.22	0.16	0.06	0.05	0.22	0.17	0.06	0.05
EISV	∞	0.38	0.04	0.28	0.04	0.24	0.04	0.18	0.05
	0.8	0.47	0.08	0.32	0.05	0.32	0.09	0.24	0.05
	0.6	0.51	0.10	0.26	0.05	0.37	0.12	0.25	0.05
	0.4	0.47	0.12	0.23	0.05	0.33	0.14	0.18	0.05

Notes: (i) Nominal 5% asymptotic level (ii) Tests based on OLS de-meaned data (iii)Tests with superscripts s, nv, b, bnv are for standard, nonstationary volatility robust, bounds robust, and both bounds and nonstationary volatility robust versions, respectively. (iv) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively.

Table 2.4: Finite sample size performance of ADF_a and MZ_a tests for ARMA(1,1) shocks

		ADF_a^s	ADF_a^{nv}	ADF_a^b	ADF_a^{bnv}	MZ_a^s	MZ_a^{nv}	MZ_a^b	MZ_a^{bnv}
Volatility Type	**c**				**T=250**				
CV	∞	0.04	0.05	0.05	0.05	0.03	0.05	0.05	0.05
	0.8	0.09	0.09	0.05	0.05	0.08	0.10	0.05	0.05
	0.6	0.14	0.16	0.06	0.05	0.12	0.16	0.06	0.05
	0.4	0.24	0.26	0.06	0.05	0.22	0.27	0.06	0.05
SBV	∞	0.16	0.03	0.13	0.03	0.12	0.04	0.11	0.04
	0.8	0.23	0.06	0.13	0.03	0.18	0.07	0.11	0.04
	0.6	0.24	0.08	0.11	0.04	0.18	0.07	0.09	0.04
	0.4	0.26	0.10	0.07	0.04	0.20	0.09	0.07	0.05
TV	∞	0.07	0.05	0.07	0.05	0.06	0.05	0.07	0.05
	0.8	0.14	0.10	0.09	0.05	0.12	0.11	0.09	0.05
	0.6	0.18	0.14	0.08	0.05	0.16	0.14	0.08	0.05
	0.4	0.26	0.18	0.06	0.05	0.23	0.18	0.06	0.06
EISV	∞	0.29	0.03	0.21	0.04	0.22	0.03	0.16	0.04
	0.8	0.39	0.08	0.25	0.04	0.30	0.08	0.20	0.04
	0.6	0.42	0.09	0.22	0.04	0.32	0.11	0.17	0.03
	0.4	0.47	0.10	0.22	0.04	0.34	0.12	0.17	0.03
					T=500				
CV	∞	0.04	0.05	0.05	0.05	0.04	0.05	0.05	0.05
	0.8	0.10	0.11	0.06	0.05	0.09	0.11	0.06	0.05
	0.6	0.17	0.17	0.06	0.05	0.16	0.17	0.06	0.05
	0.4	0.26	0.26	0.05	0.05	0.25	0.27	0.05	0.05
SBV	∞	0.16	0.03	0.14	0.03	0.14	0.04	0.12	0.04
	0.8	0.24	0.06	0.14	0.04	0.22	0.07	0.13	0.04
	0.6	0.26	0.07	0.10	0.04	0.23	0.08	0.10	0.04
	0.4	0.32	0.09	0.07	0.04	0.26	0.11	0.06	0.05
TV	∞	0.07	0.05	0.08	0.05	0.06	0.05	0.07	0.05
	0.8	0.15	0.10	0.09	0.05	0.14	0.11	0.10	0.05
	0.6	0.19	0.15	0.08	0.05	0.18	0.15	0.08	0.05
	0.4	0.29	0.20	0.06	0.05	0.27	0.20	0.06	0.05
EISV	∞	0.17	0.03	0.13	0.03	0.18	0.03	0.14	0.03
	0.8	0.25	0.06	0.13	0.03	0.25	0.07	0.15	0.03
	0.6	0.35	0.08	0.15	0.03	0.34	0.09	0.17	0.04
	0.4	0.61	0.14	0.22	0.04	0.56	0.14	0.23	0.03

Notes: (i) Nominal 5% asymptotic level (ii) Tests based on OLS de-meaned data (iii)Tests with superscripts s, nv, b, bnv are for standard, nonstationary volatility robust, bounds robust, and both bounds and nonstationary volatility robust versions, respectively. (iv) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively.

Table 2.5: Finite sample power performance of ADF_a and MZ_a tests for i.i.d. shocks

		ADF_a^{bnv}				MZ_a^{bnv}			
		T=250		**T=500**		**T=250**		**T=500**	
Volatility Type	**c \ ρ**	**0.93**	**0.86**	**0.93**	**0.86**	**0.93**	**0.86**	**0.93**	**0.86**
CV	∞	0.87	1.00	1.00	1.00	0.86	1.00	1.00	1.00
	0.8	0.71	1.00	1.00	1.00	0.68	1.00	1.00	1.00
	0.6	0.49	0.99	0.99	1.00	0.43	0.99	0.99	1.00
	0.4	0.19	0.91	0.90	1.00	0.14	0.86	0.88	1.00
SBV	∞	0.55	0.97	0.97	1.00	0.52	0.96	0.96	1.00
	0.8	0.31	0.87	0.86	1.00	0.26	0.83	0.84	1.00
	0.6	0.20	0.75	0.77	1.00	0.15	0.67	0.74	1.00
	0.4	0.16	0.73	0.75	1.00	0.09	0.60	0.70	1.00
TV	∞	0.77	1.00	1.00	1.00	0.74	1.00	1.00	1.00
	0.8	0.49	0.99	0.99	1.00	0.44	0.99	0.98	1.00
	0.6	0.31	0.95	0.95	1.00	0.26	0.93	0.94	1.00
	0.4	0.16	0.84	0.83	1.00	0.11	0.76	0.79	1.00
EISV	∞	0.33	0.78	0.31	0.61	0.29	0.74	0.27	0.56
	0.8	0.30	0.75	0.34	0.65	0.26	0.70	0.29	0.60
	0.6	0.27	0.66	0.38	0.70	0.21	0.57	0.32	0.64
	0.4	0.17	0.56	0.62	0.81	0.10	0.40	0.52	0.72

Notes: (i) Tests based on OLS de-meaned data (ii) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively. (iii) ρ denotes near integration parameter (iv)Tests with superscript *bnv* are for both bounds and nonstationary volatility robust versions, respectively.

Table 2.6: Finite sample power performance of ADF_a and MZ_a tests for MA(1) shocks

		ADF_a^{bnv}				MZ_a^{bnv}			
		T=250		**T=500**		**T=250**		**T=500**	
Volatility Type	**c \ ρ**	**0.93**	**0.86**	**0.93**	**0.86**	**0.93**	**0.86**	**0.93**	**0.86**
CV	∞	0.65	0.94	0.99	1.00	0.65	0.90	0.98	1.00
	0.8	0.50	0.91	0.97	1.00	0.52	0.87	0.96	0.99
	0.6	0.39	0.85	0.93	1.00	0.41	0.81	0.91	0.98
	0.4	0.26	0.72	0.84	0.99	0.26	0.69	0.82	0.96
SBV	∞	0.24	0.58	0.71	0.95	0.28	0.57	0.70	0.89
	0.8	0.18	0.50	0.59	0.92	0.21	0.49	0.58	0.85
	0.6	0.18	0.50	0.56	0.91	0.22	0.48	0.55	0.84
	0.4	0.20	0.52	0.60	0.90	0.22	0.48	0.57	0.81
TV	∞	0.50	0.88	0.96	1.00	0.51	0.83	0.94	0.99
	0.8	0.33	0.79	0.89	0.99	0.37	0.76	0.87	0.97
	0.6	0.27	0.74	0.84	0.99	0.29	0.71	0.82	0.96
	0.4	0.22	0.64	0.78	0.98	0.24	0.61	0.76	0.93
EISV	∞	0.33	0.52	0.82	0.97	0.44	0.59	0.80	0.94
	0.8	0.24	0.44	0.78	0.96	0.29	0.46	0.76	0.92
	0.6	0.22	0.38	0.72	0.94	0.28	0.40	0.71	0.88
	0.4	0.15	0.30	0.68	0.91	0.17	0.29	0.65	0.85

Notes: (i) Tests based on OLS de-meaned data (ii) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively. (iii) ρ denotes near integration parameter (iv)Tests with superscript *bnv* are for both bounds and nonstationary volatility robust versions, respectively.

Table 2.7: Finite sample power performance of ADF_a and MZ_a tests for AR(1) shocks

		ADF_a^{bnv}				MZ_a^{bnv}			
		T=250		**T=500**		**T=250**		**T=500**	
Volatility Type	**c \ ρ**	**0.93**	**0.86**	**0.93**	**0.86**	**0.93**	**0.86**	**0.93**	**0.86**
CV	∞	0.72	0.94	0.99	1.00	0.72	0.91	0.98	0.99
	0.8	0.59	0.92	0.97	1.00	0.62	0.89	0.96	0.99
	0.6	0.45	0.87	0.95	1.00	0.48	0.84	0.94	0.98
	0.4	0.33	0.78	0.87	0.99	0.35	0.77	0.86	0.96
SBV	∞	0.22	0.57	0.71	0.94	0.30	0.58	0.69	0.88
	0.8	0.17	0.49	0.58	0.90	0.22	0.50	0.58	0.83
	0.6	0.15	0.46	0.54	0.88	0.20	0.48	0.54	0.82
	0.4	0.17	0.46	0.54	0.87	0.20	0.46	0.55	0.79
TV	∞	0.56	0.88	0.96	1.00	0.57	0.84	0.94	0.99
	0.8	0.41	0.81	0.91	0.99	0.45	0.79	0.89	0.97
	0.6	0.32	0.77	0.84	0.99	0.35	0.74	0.83	0.96
	0.4	0.26	0.68	0.77	0.98	0.29	0.66	0.76	0.94
EISV	∞	0.37	0.54	0.84	0.97	0.46	0.60	0.82	0.95
	0.8	0.26	0.47	0.79	0.96	0.30	0.48	0.77	0.92
	0.6	0.23	0.39	0.74	0.92	0.29	0.42	0.73	0.88
	0.4	0.17	0.32	0.68	0.90	0.19	0.31	0.67	0.86

Notes: (i) Tests based on OLS de-meaned data (ii) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively. (iii) ρ denotes near integration parameter (iv)Tests with superscript *bnv* are for both bounds and nonstationary volatility robust versions, respectively.

Table 2.8: Finite sample power performance of ADF_a and MZ_a tests for ARMA(1,1) shocks

		ADF_a^{bnv}				MZ_a^{bnv}			
		T=250		**T=500**		**T=250**		**T=500**	
Volatility Type	**c \ ρ**	**0.93**	**0.86**	**0.93**	**0.86**	**0.93**	**0.86**	**0.93**	**0.86**
CV	∞	0.76	0.96	0.99	1.00	0.77	0.92	0.99	1.00
	0.8	0.63	0.93	0.98	1.00	0.64	0.89	0.96	0.99
	0.6	0.46	0.90	0.95	1.00	0.48	0.86	0.93	0.98
	0.4	0.38	0.82	0.89	0.99	0.39	0.78	0.87	0.96
SBV	∞	0.25	0.64	0.73	0.96	0.32	0.61	0.70	0.89
	0.8	0.18	0.57	0.61	0.93	0.22	0.55	0.61	0.85
	0.6	0.17	0.55	0.61	0.93	0.23	0.53	0.59	0.84
	0.4	0.26	0.62	0.66	0.92	0.30	0.57	0.63	0.82
TV	∞	0.63	0.91	0.97	1.00	0.64	0.86	0.96	0.99
	0.8	0.40	0.84	0.92	0.99	0.43	0.79	0.89	0.97
	0.6	0.33	0.81	0.87	0.99	0.37	0.76	0.84	0.96
	0.4	0.30	0.75	0.82	0.98	0.33	0.71	0.79	0.93
EISV	∞	0.33	0.76	0.29	0.59	0.27	0.72	0.26	0.53
	0.8	0.30	0.73	0.31	0.62	0.24	0.68	0.27	0.58
	0.6	0.26	0.64	0.35	0.68	0.20	0.56	0.30	0.63
	0.4	0.15	0.53	0.60	0.78	0.09	0.36	0.50	0.71

Notes: (i) Tests based on OLS de-meaned data (ii) CV, SBV, TV and EISV denote 'Constant volatility', 'Single Break Volatility', 'Trending Volatility' and 'Exponential Integrated Stochastic Volatility' respectively. (iii) ρ denotes near integration parameter (iv)Tests with superscript *bnv* are for both bounds and nonstationary volatility robust versions, respectively.

CHAPTER 3

POWERFUL NONPARAMETRIC SEASONAL UNIT ROOTS

3.1 Introduction

This chapter is a joint work with Assist. Prof. Burak Eroglu and Mirza Trokic and published in Economics Letters on June 2018 with following reference: Eroğlu, B. A., Göğebakan, K. Ç ., Trokic, M., 2018. Powerful nonparametric seasonal unit root tests. Economics Letters 167, 75-80. An earlier version of this paper was also circulated in 2017 under the title of "Fractional Seasonal Variance Ratio Tests" in CEFIS Working Paper Series of İstanbul Bilgi University.

After the seminal work of Hylleberg et al. (1990) [HEGY], seasonal unit root testing has attracted attention in theoretical and empirical studies. The HEGY tests are the seasonal generalization of the augmented Dickey-Fuller [ADF] unit root test and allow researchers to test for unit root behavior at each of the zero, Nyquist and seasonal frequencies. Recently, HEGYs framework has been extended and improved by many subsequent authors such as Rodrigues and Taylor

(2007), Smith et al. (2009) and del Barrio Castro et al. (2012) [BOT]. Despite these improvements, these tests still use parametric lag augmentation to account for the serial correlation. Therefore, their size and power performance highly depend on the selection of the lag length parameter in the augmented regression. Motivated by these issues, Taylor (2005) is the only serious attempt to construct non-parametric seasonal variance ratio unit root tests by extending unit root test of Breitung (2002) to the seasonal framework. In this regard, we aim to propose a new family of non-parametric seasonal unit root tests, which are free from parametric lag augmentation and improve the existing non-parametric seasonal unit root tests in terms of asymptotic local power.

This new testing design is an extension of Nielsen (2009) non-parametric variance ratio test to the seasonal unit root context. Consequently, the proposed seasonal non-parametric unit root tests use the fractional integration and the family of these tests is indexed by the fractional integration parameter d. In the econometrics literature, there is a growing attention to the fractional integration techniques. (see Robinson (2003)). We show that these techniques are useful to construct powerful unit root tests at the zero, Nyquist and seasonal frequencies.

To improve the power performance of the parametric seasonal unit root tests, Gregoir (2006) and Rodrigues and Taylor (2007) propose nearly efficient tests. These tests are designed to utilize GLS de-trending techniques in the seasonal context. More recently, Jansson and Nielsen (2011) develop nearly efficient likelihood ratio tests for the seasonal unit root hypothesis. Accordingly, we also present GLS de-trended versions of our non-parametric seasonal unit root tests. Besides these, we also propose non-parametric joint unit root tests for the seasonal frequencies. With this contribution, we extend the scope of the non-parametric seasonal unit root testing framework.

The rest of the paper is organized as follows. In Section 3.2, the seasonal unit root model, the seasonal unit root hypothesis and the augmented HEGY-type tests are presented. Our nonparametric FSVR testing framework is developed in Section 3.3, and the asymptotic results are given in Section 3.4, respectively. The finite sample simulation study is presented in Section 3.5. Section 3.6 concludes the paper. The tables containing the critical values and simulation results are placed at the end of the chapter.

3.2 The Seasonal Unit Root Framework

3.2.1 The Seasonal Unit Root Model

We consider the univariate model defining the seasonal time series $\{x_{St+s}\}$ with the following data generating process (DGP) [1]

$$\alpha(L)x_{St+s} = u_{St+s}, \qquad s = 1 - S, ..., 0, \qquad t = 1, 2, ..., N \tag{3.1}$$

$$u_{St+s} = \phi(L)\epsilon_{St+s} \tag{3.2}$$

where S is the number of seasons, $\alpha(L) = 1 - \sum_{i=1}^{S} \alpha_j L_j$ is an S order AR polynomial which determines the seasonal unit root. Further, we assume station-

[1]The notation closely follows del Barrio Castro et al. (2012)

ary innovations with the serial correlation structure governed by AR polynomial $\phi(L) = 1 - \sum_{i=1}^{\infty} \phi_j L_j$. The total number of observations is given by $T = SN$. N denotes the total number of observation in each season. For simplicity, we suppose same number of observations in these seasons. We assume that the initial conditions $x_{1-S}, ..., x_0$ are $o_p(T^{1/2})$. Following del Barrio Castro et al. (2012), we propose the following assumptions on the innovation sequence $\{\epsilon_{St+s}\}$ which further characterize the dynamics of seasonal model:

Assumption. $\mathcal{A}$ Let $(\varepsilon_{St+s}, \mathcal{F}_{St+s})$ be a martingale difference sequence with filtration $\mathcal{F}_{St+s}$, where $\mathcal{F}_{St+s} \subset \mathcal{F}_{St+s+1}$ for all s, t such that

$\mathcal{A}.1$ $\mathbb{E}[\epsilon^2_{St+s} | \mathcal{F}_{St+s}] = \sigma^2$

$\mathcal{A}.2$ $\frac{1}{N} \sum_{t=1}^{N} \epsilon^2_{St+s} \xrightarrow{p} \sigma^2$ for all $s = 0, 1, ..., 1 - S$

$\mathcal{A}.3$ $\mathbb{E}|\epsilon_{St+s}|^r < K < \infty$ for some $r \geq 4$.

Assumption. $\mathcal{B}$ The lag polynomial $\phi(L) \neq 0$ for all $|L| \leq 1$, and $\sum_{j=0}^{\infty} j^h |\phi_j| < \infty$ for some $h \geq 1$.

These assumptions are very standard in the seasonal unit root literature. $\mathcal{A}.1$ and $\mathcal{A}.2$ indicate that the error terms are homoscedastic. Note that these conditions are strong enough to exclude periodic or seasonal variance difference for the error terms. $\mathcal{A}.3$ allows the rth moment of ε_{St+s} to exist and to be bounded uniformly in t. The detailed discussion about the given assumption can be found in del Barrio Castro et al. (2012).

Assumption $\mathcal{B}$ implies stationarity of the innovation term ϵ_{St+s}. In other words, the only source of non-stationary dynamics is the seasonal AR polynomial $\alpha(L)$.

3.2.2 The Seasonal Unit Root Hypothesis

Under the null hypothesis, we assume a seasonal unit root at each season S as follows:

$$H_0 : \alpha(L) = (1 - L^S) =: \Delta_S \tag{3.3}$$

This hypothesis can be partitioned as $H_0 = \cap_{j=0}^{\lfloor S/2 \rfloor} H_{0,j}$, where

$$H_{0,i} : \alpha_i = 1, \quad i = 0, S/2$$

$$H_{0,j} : \alpha_j = 1 \ and \ \beta_j = 0 \quad j = 1, ..., S^*$$

where $S^* = \lfloor (S-1)/2 \rfloor$. The seasonal lag polynomial can be decomposed as $\alpha(L) = \prod_{j=0}^{\lfloor S/2 \rfloor} \omega_j(L)$ with $\omega_0 = (1 - \alpha_0 L)$ (unit root in zero frequency), $\omega_j(L) = [1 - 2(\alpha_j cos\omega_j - \beta_j sin\omega_j)L + (\alpha_j^2 + \beta_j^2)L^2]$ ($\omega_j = 2\pi j/S$ are seasonal frequencies) for $j = 1, ..., S^*$ and $\omega_{\lfloor S/2 \rfloor}(L) := \big(1 + \alpha_{\lfloor S/2 \rfloor} L\big)$.

The alternative hypothesis is $H_1 = \cup_{j=0}^{\lfloor S/2 \rfloor} H_{1,j}$ which ensures stationarity at one or more of the zero frequency or seasonal frequencies. We can explicitly write the each alternatives as:

$$H_{1,i} : |\alpha_i| < 1, \quad i = 0, S/2$$

$$H_{1,j} : \alpha_j^2 + \beta_j^2 < 1 \quad j = 1, ..., S^*$$

We can test each hypothesis separately to reveal the individual unit root characteristics at different frequencies. In addition, this setup also allows us to conduct joint testing for the above hypothesis.

3.2.3 The Augmented HEGY tests

We first discuss the parametric HEGY tests. Hylleberg et al. (1990) expand the $AR(p+S)$ polynomial $\phi^*(z) = \alpha(z)\phi(z)$ around the zero and seasonal frequency unit roots and obtain the auxiliary HEGY regression

$$\begin{aligned}\Delta_S x_{St+s} = \pi_0 x_{0,St+s-1} + \pi_{S/2} x_{S/2,St+s-1} + \sum_{i=1}^{S*}(\pi_i x_{i,St+s-1} + \pi_i^* x_{i,St+s-1}^*) \\ + \sum_{j=1}^{p} \phi_j^* \Delta_S x_{St+s-j} + \varepsilon_{St+s}\end{aligned} \tag{3.4}$$

where the independent variables are defined as

$$\begin{aligned} x_{0,St+s} &:= \sum_{j=0}^{S-1} x_{St+s-j} \\ x_{S/2,St+s} &:= \sum_{j=0}^{S-1} cos[(j+1)\pi] x_{St+s-j} \end{aligned}$$

$$\begin{aligned} x_{i,St+s} &:= \sum_{j=0}^{S-1} cos[(j+1)\omega_i] x_{St+s-j} \qquad \textit{for all } i = 1, ..., S^* \\ x_{i,St+s}^* &:= \sum_{j=0}^{S-1} sin[(j+1)\omega_i] x_{St+s-j} \qquad \textit{for all } i = 1, ..., S^* \end{aligned}$$

After estimating the model in equation (3.4), HEGY device the tests for the null hypotheses of unit roots at zero, Nyquist and harmonic seasonal frequencies which are stated respectively as:

$$H_{0,k} : \pi_k = 0, \;\; k = 0, S/2 \tag{3.5}$$

$$H_{0,i} : \pi_i = \pi_i^* = 0, i = 1, ..., S^* \tag{3.6}$$

The tests for the presence of a unit root defined in Hylleberg et al. (1990) consist of the regression t statistic t_k (left sided) for the exclusion of $x_{k,St+s-1}$ for $k = 0, S/2$ and F_i for the joint exclusion of $x_{i,St+s-1}$ and $x^*_{i,St+s-1}$ for all $i = 1, ..., S^*$ from (3.4). Moreover, Ghysels et al. (1994) improve the HEGY-type tests by proposing the joint frequency (upper-tail) F-tests by using the same regression model in (3.4). The F test $F_{1,...,\lfloor S/2 \rfloor}$ is used for testing the exclusion of $\{x_{i,St+s-1}\}_{i=1}^{S^*}$, $\{x^*_{i,St+s-1}\}_{i=1}^{S^*}$ and $x_{S/2,St+s-1}$, and $F_{0,...,\lfloor S/2 \rfloor}$ is designed for testing the exclusion of $x_{0,St+s-1}$, $\{x_{i,St+s-1}\}_{i=1}^{S^*}$, $\{x^*_{i,St+s-1}\}_{i=1}^{S^*}$ and $x_{S/2,St+s-1}$. The null hypothesis for these two F tests can be represented respectively, as following:

$$H_{0,\{1,...,\lfloor S/2 \rfloor\}} = \bigcap_{i=1}^{\lfloor S/2 \lfloor} H_{0,i} : \;\; \pi_1 = \cdots = \pi_{S^*} = \pi_{\lfloor S/2 \rfloor} = 0$$
$$H_{0,\{0,...,\lfloor S/2 \rfloor\}} = \bigcap_{i=0}^{\lfloor S/2 \lfloor} H_{0,i} : \;\; \pi_0 = \pi_1 = \cdots = \pi_{S^*} = \pi_{\lfloor S/2 \rfloor} = 0$$

Remark 10. Empirical implementation of HEGY-type tests forces the practitioners to choose lag order p. This procedure has a time complexity. Moreover, the finite sample performance depends on the choice of p, though the lag order is not reflected in the limiting distribution of the statistics. Although optimal lag length selection methods are discussed in the literature, such procedures may still work poorly in small samples. For a detailed discussion of lag length selection techniques, we refer the reader to del Barrio Castro et al. (2016). In our test, we will avoid these issues and propose fully nonparametric family of tests.

3.3 Fractional Seasonal Variance Ratio Tests

In this section, we propose a new method for testing seasonal unit roots which is a family of nonparametric and tuning parameter-free tests indexed by the fractional parameter d. This method is a seasonal generalization of the powerful test of Nielsen (2009).

We first define the S dimensional process X_t which separates the observed series x_{St+s} from (3.1) into sub-seasonal observations. That is, each vector in this process corresponds to the observation in each season:

$$X_t := [x_{St-(S-1)}, x_{St-(S-2)}, ..., x_{St}]' \quad t = 1, ..., N$$

In our analysis, we will utilize the fractional integration operator of type II. This operator takes the weighted sum of the past values of a time series process, say y_t, where weights are determined by the fractional binomial coefficients. The following equation summarizes this operator:

$$\tilde{y}_t := \Delta_+^{-d} y_t = (1-L)_+^{-d} y_t = \sum_{k=0}^{t-1} \frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)} y_{t-k} = \sum_{k=0}^{t-k} \pi_k(d) y_{t-k} \qquad (3.7)$$

where $1 \geq d > 0$. We will apply this operator to each element of the X_t separately to obtain:

$$\tilde{X}_t = [\Delta_+^{-d} x_{St-(S-1)}, \Delta_+^{-d} x_{St-(S-2)}, ..., \Delta_+^{-d} x_{St}]' \tag{3.8}$$

After obtaining $\tilde{X}_t$, we reconstruct the associated fractional transformation of the seasonal process from this process. In other words, we create a new variable, say $\tilde{x}_{St+s}$ from $\tilde{X}_t$ as each vector of $\tilde{X}_t$ corresponds to one season of the final process $\tilde{x}_{St+s}$.

Remark 11. Applying the fractional integration operator as discussed above is new to the literature. For instance, Nielsen (2009) applies this operator to observed series directly. We do not follow Nielsen (2009) in this study purely because of technical reasons arising from the seasonality.

Now consider the HEGY transformation of the fractionally integrated process $\tilde{x}_{St+s}$:

$$\tilde{x}_{0,St+s} := \sum_{j=0}^{S-1} \tilde{x}_{St+s-j} \tag{3.9}$$

$$\begin{aligned}
\tilde{x}_{S/2,St+s} &:= \sum_{j=0}^{S-1} cos[(j+1)\pi]\tilde{x}_{St+s-j} \\
\tilde{x}_{i,St+s} &:= \sum_{j=0}^{S-1} cos[(j+1)\omega_i]\tilde{x}_{St+s-j} \qquad \textit{for all } i = 1, ..., S^* \\
\tilde{x}^*_{i,St+s} &:= \sum_{j=0}^{S-1} sin[(j+1)\omega_i]\tilde{x}_{St+s-j} \qquad \textit{for all } i = 1, ..., S^*
\end{aligned}$$

With these objects we can define the test statistics for testing unit root hypothesis in different frequencies. First, consider the testing unit root on zero and Nyquist frequencies:

$$\tau_k(d) = N^{2d} \frac{\sum_{t=1}^{N} \sum_{s=1-S}^{0} x_{k,St+s}^2}{\sum_{t=1}^{N} \sum_{s=1-S}^{0} \tilde{x}_{k,St+s}^2} \; for \; k = 0, S/2 \tag{3.10}$$

Remark 12. This test corresponds to the test of HEGY with the null hypothesis $H_{0,k}$ for $k = 0, S/2$.

For the harmonic frequencies, we propose two variance ratio test statistics:

$$\tau_j(d) = N^{2d} \frac{\sum_{t=1}^{N} \sum_{s=1-S}^{0} x_{j,St+s}^2}{\sum_{t=1}^{N} \sum_{s=1-S}^{0} \tilde{x}_{j,St+s}^2} \; for \; j = 1, .., S^* \tag{3.11}$$

$$\tau_j^*(d) = N^{2d} \frac{\sum_{t=1}^{N} \sum_{s=1-S}^{0} x_{j,St+s}^{*2}}{\sum_{t=1}^{N} \sum_{s=1-S}^{0} \tilde{x}_{j,St+s}^{*2}} \; for \; j = 1, .., S^* \tag{3.12}$$

where $\tau_j(d)$ is the test statistic for the unit root hypothesis at the harmonic frequency ω_j for all $j = 1, ..., S^*$, $\tau_j^*(d)$ is the test statistic for the unit root hypothesis at the harmonic frequency $2\pi - \omega_j$ for all $j = 1, ..., S^*$.

Remark 13. These tests correspond to the tests of HEGY with the null hypothesis $H_{0,j}$ for $j = 1, ..., S^*$.

Notice that the unit roots in the harmonic frequencies are complex conjugates. Then, one can use the following statistic to jointly test the presence of the pairs of complex unit roots:

$$\tilde{\tau}_j(d) = (\tau_j(d) + \tau_j^*(d))/2 \; for \; j = 1, .., S^* \tag{3.13}$$

Finally, we derive the F type statistics for the joint test for the unit roots in the different frequencies:

$$\tau_{1,..,S/2}(d) = \frac{1}{S-1}\left(\tau_2(d) + \sum_{j=1}^{S^*}\tau_j(d) + \sum_{j=1}^{S^*}\tau_j^*(d)\right) \tag{3.14}$$

$$\tau_{0,..,S/2}(d) = \frac{1}{S}\left(\tau_0(d) + \tau_2(d) + \sum_{j=1}^{S^*}\tau_j(d) + \sum_{j=1}^{S^*}\tau_j^*(d)\right) \tag{3.15}$$

Remark 14. Remark that the test in equation (3.14) corresponds to the joint unit root tests at all frequencies except the zero frequency. Further, the test in equation (3.15) corresponds to the joint unit root tests at all frequencies.

Remark 15. By the proposed nonparametric fractional seasonal variance ratio (FSVR) tests, we bring some technical innovations to the nonparametric seasonal unit root testing methods. Besides proposing a new family of nonparametric seasonal unit root tests, we also improve the nonparametric tests of Taylor (2005) by extending quarterly data ($S = 4$) to the general seasonal index S. Therefore, these tests can be used for semiannually ($S = 2$), quarterly ($S = 4$), monthly ($S = 12$) and weekly ($S = 7$) observed data. We also introduce joint nonparametric seasonal unit root tests defined in (3.14)-(3.15) to be comparable with parametric seasonal unit root tests of HEGY. These types of seasonal unit root tests are not included in the study of Taylor (2005).

3.4 Asymptotic Results for the Fractional Seasonal Variance Ratio Test

In this section, we present the asymptotic results for the FSVR tests. To derive the asymptotic results, we adopt a similar structure as del Barrio Castro et al. (2012). Under the overall null hypothesis, we can write $\{x_{St+s}\}$ as in S dimensional vector form:

$$X_t = X_{t-1} + U_t \tag{3.16}$$

$$X_t := [x_{St-(S-1)}, x_{St-(S-2)}, ..., x_{St}]' \quad t = 1, ..., N \tag{3.17}$$

$$U_t := [u_{St-(S-1)}, u_{St-(S-2)}, ..., u_{St}]' \quad t = 1, ..., N \tag{3.18}$$

where the innovation process can be written as:

$$U_t = \sum_{j=0}^{\infty} \Phi_j E_{t-j}$$

$$E_t := [\epsilon_{St-(S-1)}, \epsilon_{St-(S-2)}, ..., \epsilon_{St}]' \quad t = 1, ..., N$$

Here Φ_0 and Φ_js are defined as in del Barrio Castro et al. (2012):

$$\Phi_0 := \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ \phi_1 & 1 & 0 & ... & 0 \\ \phi_2 & \phi_1 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{S-1} & \phi_{S-2} & \phi_{S-3} & ... & 1 \end{bmatrix}$$

$$\Phi_j := \begin{bmatrix} \phi_{jS} & \phi_{jS-1} & \phi_{jS-2} & \cdots & \phi_{jS-(S-1)} \\ \phi_{jS+1} & \phi_{jS} & \phi_{jS-2} & \cdots & \phi_{jS-(S-2)} \\ \phi_{jS+2} & \phi_{jS} & \phi_{jS-1} & \cdots & \phi_{jS-(S-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{jS+S-1} & \phi_{jS+S-2} & \phi_{jS+S-3} & \cdots & \phi_{jS} \end{bmatrix} \qquad j = 1, 2, ...$$

The following lemma can be found as Lemma 1 in del Barrio Castro et al. (2012):

Lemma 5. Let X_t is generated as in equations (3.16)-(3.18). Then under Assumptions $\mathcal{A}.1 - 4$,

$$N^{-1/2} X_{\lfloor rN \rfloor} \Rightarrow \sigma \Phi(1) W(t) \qquad r \in [0, 1]$$

where $W(r)$ is S dimensional standard Brownian motion process, $\Phi(1) := \sum_{j=0}^{\infty} \Phi_j$, the right element of above convergence can be expressed as

$$(1/S)\sigma \left[\phi(1) C_0 + \phi(-1) C_{S/2} + 2 \sum_{i=1}^{S^*} (b_i C_i + a_i C_i^*) \right] W(r)$$

where $a_i = Im(\phi[\exp(\mathrm{i}w_i)])$, $b_i = \mathcal{R}e(\phi[\exp(\mathrm{i}w_i)])$ for $i = 1, ..., S^*$, C_j for all $j = 0, ..., S/2$ and C_k^* for $k = 1, ...S^*$ are circulant matrices. Further information about these matrices can be found in Appendix B.

The following lemma demonstrates the limiting distribution of the fractionally transformed process:

Lemma 6. Let X_t is generated as in equations (3.16)-(3.18) and $\tilde{X}_t$ is obtained from equation (4.13). Under Assumptions $\mathcal{A}.1-4$, then

$$N^{-d-1/2}\tilde{X}_{\lfloor rN \rfloor} \Rightarrow \sigma\Phi(1)W_{d+1}(t), \quad r \in [0,1]$$

where

$$W_{d+1}(r) = \frac{1}{\Gamma(d+1)}\int_0^r (r-t)^d dW(t) \tag{3.19}$$

is S dimensional Fractional Brownian motion.

Using the object derived in Lemma 5 and Lemma 6, we reach the following asymptotic results for the FSVR test statistics:

Theorem 4. Let the assumptions of Lemma 5 hold. Consider the FSVR test statistics given by equations (4.4)-(4.9). As $N \to \infty$,

$$\begin{aligned}
\tau_i(d) &\Rightarrow \frac{\int_0^1 W_i(r)^2 dr}{\int_0^1 W_{d+1,i}(r)^2 dr} = \eta_i(d) \quad i = 0, S/2 \\
\tau_j(d) &\Rightarrow \frac{\int_0^1 W_j(r)^2 dr + \int_0^1 W_j^*(r)^2 dr}{\int_0^1 W_{d+1,j}(r)^2 dr + \int_0^1 W_{d+1,j}^*(r)^2 dr} = \eta_j(d) \ \textit{for}\ j = 1, ..., S^* \\
\tau_j^*(d) &\Rightarrow \eta_j(d) \ \textit{for}\ j = 1, ..., S^* \\
\tilde{\tau}_j(d) &\Rightarrow \eta_j(d) \ \textit{for}\ j = 1, ..., S^* \\
\tau_{1,..,S/2}(d) &\Rightarrow \frac{1}{S-1}\left(\eta_2(d) + 2\sum_{j=1}^{S^*}\eta_j(d)\right) \\
\tau_{0,..,S/2}(d) &\Rightarrow \frac{1}{S}\left(\eta_0(d) + \eta_2(d) + 2\sum_{j=1}^{S^*}\eta_j(d)\right)
\end{aligned}$$

where $W_0(r)$, $W_{S/2}(r)$, $W_j^*(r)$ and $W_j(r)$ for $j = 1, ..., S^*$ are the independent standard Brownian motions, $W_{d,0}(r)$, $W_{d,S/2}(r)$, $W_{d,j}^*(r)$ and $W_{d,j}(r)$ are the independent fractional Brownian motions for $j = 1, ..., S^*$, respectively.

Remark 16. The convergence results stated in Theorem 4 do not take the deterministic component into account. To adjust for the deterministic elements, we extend these results by the following generalization of the DGP in (3.1)-(3.2):

$$y_{St+s} = \mu_{St+s} + x_{St+s}, s = 1 - S, ..., 0, t = 1, ..., N \tag{3.20}$$

where x_{St+s} is defined in (3.1)-(3.2) and $\mu_{St+s} = \gamma' Z_{St+s}$ where Z_{St+s} are purely deterministic. Following Smith et al. (2009), we define the characterization of μ_{St+s} with a typology of three cases, namely: no deterministic component ($j = 0$), seasonal intercepts ($j = 1$), seasonal intercepts and seasonal trends ($j = 2$). When the deterministic component is involved, to obtain tests which are exact invariant to the elements of γ, our test statistics (3.10)-(3.15) should be redefined using de-trended series as $\hat{x}_{St+s} = x_{St+s} - \hat{\gamma}' Z_{St+s}$. The detrending in this paper is done by using OLS de-trending as in, Smith et al. (2009) and Nielsen (2009). For example, we have the demeaned or detrended analogues of the Brownian motion in the asymptotic distribution of unit root test at the zero frequency as follows:

$$\begin{aligned} B_{0,d+1,j}(r) = W_{0,d+1}(r) - &\left(\int_0^1 W_0(t) D_j(t) dt\right) \left(\int_0^1 D_j(t) D_j(t) dt\right)^{-1} \\ &\times \int_0^r \frac{1}{\Gamma(d)} (r-t)^{d-1} D_j(t) dt \quad \forall\ j = 1, 2 \end{aligned}$$

where $D_1(t) = 1$ and $D_2(t) = [1, t]'$. For other frequencies, this object can be obtained similarly. For the demeaning and the detrending procedures for the HEGY type tests, see Smith et al. (2009).

Next, instead of simple OLS de-trending, following Rodrigues and Taylor (2007), we obtain demeaned or de-trended series by using GLS de-trending algorithm. Let the observed series Y_t and deterministic term Z_t to be defined as follows:

$$Y_t = [y_{St-(S-1)}, y_{St-(S-2)}, ..., y_{St}]'$$

$$Z_t = [z_{St-(S-1)}, z_{St-(S-2)}, ..., z_{St}]'$$

We apply quasi-differencing to Y_t and Z_t where

$$Y_{\bar{c}} = \left[Y_1, Y_2 - \left(1 - \frac{\bar{c}}{N}\right)Y_1, ..., Y_N - \left(1 - \frac{\bar{c}}{N}\right)Y_{N-1}\right]$$
$$Z_{\bar{c}} = \left[Z_1, Z_2 - \left(1 - \frac{\bar{c}}{N}\right)Z_1, ..., Z_N - \left(1 - \frac{\bar{c}}{N}\right)Z_{N-1}\right]$$

We can obtain the coefficient $\hat{\gamma}_{GLS}$ by regressing $Y_{\bar{c}}$ on $Z_{\bar{c}}$. Then the GLS de-trended series are given by converting the vector process $\hat{X}_t = Y_t - \hat{\gamma}_{GLS} Z_t$ to the seasonal process $\hat{x}_{St+s}$ where $\hat{X}_t = [\hat{x}_{St-(S-1)}, \hat{x}_{St-(S-2)}, ..., \hat{x}_{St}]'$. We replace these GLS de-trended series with x_{St+s} in the testing procedure. Finally asymptotic distributions are replaced with the GLS demeaned or de-trended versions of Brownian motions and Fractional Brownian motions. The objects are same as described in Nielsen (2009).

Remark 17. The asymptotic results in Theorem 4 are based on the seasonal unit root null hypothesis of H_0 given at equation (3.3). Under the local alternatives, Rodrigues and Taylor (2007) present the decomposition of $\alpha(L)$ presented in Section 2.2 as $\alpha_i = (1 + \nu_i/T)$ for $i = 0, S/2$, and $\alpha_j = (1 + \nu_j/T), \beta_j = 0$ for $j = 1, ..., S^*$, where $\nu_0, \nu_1, ..., \nu_{S/2}$ are finite constants. Therefore, following Rodrigues and Taylor (2007) and Nielsen (2009), under the local alternatives, the Brownian motions in Theorem 4 can be replaced by the corresponding Ornstein-

Uhlenbeck processes which are given as follows:

$$J_{j,c}(r) = \int_0^r exp(-\nu_j(r-\lambda))dW_j(t)$$
$$\tilde{J}_{j,c,d+1}(r) = \int_0^r exp(-\nu_j(r-\lambda))dW_{j,d+1}(t)$$

where ν_j, $j = 0, ..., S/2$ are the frequency specific non-centrality parameters.

Remark 18. Let $CV_{\gamma,S}(\eta_j(d))$ be the γ-th quantile of the asymptotic distribution of the test statistic $\tau_j(d)$ for $j = 0, ..., S/2$. We reject the one sided t type test if $\tau_j(d) < CV_{\gamma,S}(\eta_j(d))$ for all $j = 0, ..., S/2$. For the joint tests, we reject the null hypothesis if $\tau_{0,...,S/2}(d) > CV_{(1-\gamma),S}(\eta_{0,...,S/2}(d))$ where $CV_{(1-\gamma),S}(\eta_{0,...,S/2}(d))$ is the $(1-\gamma)$-th quantile of the empirical bootstrap distribution of the test statistic $\tau_{0,...,S/2}(d)$.

The asymptotic critical values for the FSVR tests for two different parameter values for d and two different number of seasons S are given in Table 3.1. We report the critical values for $d \in \{0.1, 1\}$. Moreover, we consider the cases $S \in \{4, 12\}$ which are empirically the most relevant ones. $S = 4$ corresponds to quarterly data and $S = 12$ is used for monthly data. In this table, the critical values are also classified according to the deterministic structures given in Remark 16 and the nominal significance levels.

Remark 19. For $d = 1$, the nonparametric seasonal unit root tests of Taylor (2005) and inverse of our tests are equivalent. Moreover, it is important to note that our paper firstly introduces GLS de-trending procedure to the nonparametric seasonal unit root testing framework.

3.5 Finite Sample Simulations

In this section, we investigate the finite sample size and size-adjusted power of the conventional HEGY tests from section 3.2.3, the FSVR tests from section 3.3. The experiments are done using $10,000$ Monte Carlo replications. The reported finite sample simulations in this section are based on the following quarterly ($S = 4$) DGP:

$$(1 - \alpha_0 L)(1 + \alpha_2 L)(1 + \alpha_1^2 L^2)x_{4t+s} = u_{4t+s} \tag{3.21}$$

$$u_{4t+s} = \phi(L)\epsilon_{4t+s}, \quad \epsilon_{4t+s} \sim NIID(0, 1) \tag{3.22}$$

with $\epsilon_{4t+s} = 0$ for $t \leq 0$. For investigation of the finite sample size performance, we pick $\alpha_0 = \alpha_1 = \alpha_2 = 1$ and for the finite sample power performance, we pick $\alpha_j \in \{0.93, 0.86\}$

By expanding $\phi(L) = \theta(L)\psi(L)^{-1}$ with two finite lag operators $\theta(L)$ and $\psi(L)$, we consider 5 different types of serial correlation scenarios for u_{4t+s} as follows:

$$(1 - \psi_1 L - \psi_2 L^2 - \psi_4 L^4)u_{4t+s} = (1 + \theta_1 L + \theta_2 L^2 + \theta_4 L^4)\epsilon_{4t+s} \tag{3.23}$$

where ϵ_{4t+s} is a martingale difference sequence. For the Monte Carlo simulations, in addition to no serial correlation case, we consider the parametrisation in equation (3.23) as follows: **(i)**$ARMA(1, 1) : \theta_1 = -0.5$, $\theta_2 = \theta_4 = 0$,

$\psi_1 = 0.5, \psi_2 = \psi_4 = 0$, **(ii)**$ARMA(2,2) : \theta_2 = 0.5,\ \theta_1 = \theta_4 = 0,,\ \psi_2 = -0.5$, $\psi_1 = \phi_4 = 0$, **(iii)**$MA(4) : \theta_4 = 0.5,\ \theta_1 = \theta_2 = 0,\ \psi_1 = \psi_2 = \psi_4 = 0$, **(iv)**$ARMA(4,4) : \theta_4 = -0.5,\ \theta_1 = \theta_2 = 0,\ \psi_4 = 0.5\ \psi_1 = \psi_2 = 0$.

Results are reported for the t_0, $t_{S/2}$, F_1, $F_{0,\lfloor S/2 \rfloor}$ and $F_{1,\lfloor S/2 \rfloor}$ statistics of the HEGY tests, and $\tau_0(d)$, $\tau_{S/2}(d)$, $\tilde{\tau}_1(d)$, $\tau_{0,\lfloor S/2 \rfloor}(d)$ and $\tau_{1,\lfloor S/2 \rfloor}(d)$ of the FSVR tests. Table 3.2 contains OLS demeaned results. Corresponding results for GLS demeaned data are reported in Table 3.3. We set $\bar{c} = 13.5$.

Following Nielsen (2009), we investigate the impact of the fractional integration parameter d in the performance of FSVR tests. The best results in terms of power and size are obtained with $d = 0.1$. For comparison with tests of Taylor (2005), we also report the case with $d = 1$.[2]

The results in Table 3.2 indicate that the FSVR and HEGY tests are correctly sized in the OLS demeaning case. These results apparently suggest that in all frequencies the FSVR tests, useful and significant power gains are obtained compared to the tuning parameter free test of Taylor (2005). It is clear that the FSVR tests even rival the parametric HEGY tests regarding the power performance.

We observe in Table 3.3 that, when GLS demeaning is used, the size performance of FSVR tests is satisfactory. We demonstrate that GLS demeaning increases the power performance of our nonparametric tests. GLS demeaning do not increase the power of the nonparametric tests of Taylor (2005). It is important to note

[2]The results with other values of d are available upon request.

that these tests are clearly dominated by our FSVR tests. On the other hand, under the GLS demeaning case, HEGY tests have the advantage in the individual coefficient testing for i.i.d. and ARMA type errors. For the MA serial correlation case, consistent with findings of Nielsen (2009), individual coefficient FSVR-GLS tests are superior to the individual coefficient HEGY-GLS tests. [3] However, in the joint tests, the FSVR and HEGY methods enjoy similar power properties. In some cases of the joint tests, even FSVR-GLS is superior to the HEGY-GLS test.

3.6 Conclusion

This paper brings some technical innovations to the seasonal unit root testing literature. First, we have developed a new class of non-parametric seasonal unit root tests by making a seasonal generalization to Nielsen (2009). The proposed seasonal test statistics are constructed as a ratio of the sample variance of the seasonally transformed series and that of a fractional partial sum of the same transformed series. The test statistics at all frequencies are free of tuning parameters, such as lag length and bandwidth etc. Further, we extend the seasonal variance ratio framework by proposing joint frequency tests. This type of analysis is missing in Taylor (2005).

Second important contribution is on fractional integration literature. We introduce a new fractionally transformed seasonal series, $\tilde{x}_{St+s}$. Although these series are univariate, we utilize the theory for type II multivariate fractional Brownian motions to find the asymptotic distribution for these objects. Further, one can

[3]This puzzled result comes from the lag length selection procedure required for the HEGY tests.

assume the fractionally transformed seasonal series as data generation process and conduct test on the fractional integration parameter. We leave this issue for a future study.

Last but not least, in the Monte Carlo simulations, we investigate the effect of the fractional integration parameter d in our testing procedure. The best results for both size and size adjusted power exercises are obtained when $d = 0.1$. This result is consistent with the findings of Nielsen (2009) for standard unit root tests. The simulation results reveal that our proposed non-parametric tests show competitive power performance with the parametric HEGY tests by exhibiting desirable size properties in all scenarios considered.

Table 3.1: Asymptotic Critical Values for the Fractional Seasonal Variance Ratio Tests for $S = 4$ and $S = 12$

							$\boldsymbol{S = 4}$						
			τ_0			$\tilde{\tau}_j$			$\tau_{0S/2}$			$\tau_{1S/2}$	
	$\mathbf{(1-\gamma)}$	90%	95%	99%	90%	95%	99%	90%	95%	99%	90%	95%	99%
$\boldsymbol{d}$	Deterministic												
	0	1.54	1.62	1.77	1.40	1.46	1.58	1.38	1.41	1.49	1.38	1.43	1.52
0.1	1	1.76	1.82	1.93	1.66	1.70	1.79	1.64	1.67	1.72	1.65	1.68	1.75
	2	1.92	1.98	2.08	1.85	1.88	1.96	1.82	1.85	1.90	1.83	1.86	1.92
	0	33.82	49.79	102.80	15.55	22.60	41.63	21.20	27.03	44.00	20.09	26.56	44.39
1	1	69.25	98.44	181.72	39.29	51.68	82.18	45.81	56.05	82.01	44.62	55.95	85.55
	2	226.91	290.06	448.47	155.55	186.78	262.34	160.84	182.75	236.90	161.06	186.61	249.927
							$S = 12$						
	0	1.54	1.62	1.77	1.40	1.46	1.58	1.314	1.33	1.37	1.31	1.33	1.37
0.1	1	1.76	1.82	1.94	1.66	1.71	1.79	1.59	1.61	1.64	1.59	1.61	1.64
	2	1.92	1.98	2.08	1.85	1.88	1.96	1.78	1.79	1.82	1.78	1.79	1.82
	0	33.52	49.35	100.27	15.53	22.61	42.24	14.42	16.81	22.63	13.87	16.22	21.88
1	1	69.90	98.25	179.06	39.51	51.66	83.10	33.08	37.14	46.63	32.08	36.24	45.58
	2	227.07	290.41	457.30	155.37	186.26	264.55	129.09	139.04	160.26	127.47	137.99	159.390

Note: We drop the (d) notation. We use 100000 Monte Carlo simulations to obtain the critical values with N=1000. γ denotes the significance level. Deterministic=0 is for the case of no deterministic adjustment, Deterministic=1 denotes the case of only seasonal intercept adjustment and Deterministic=2 denotes the case of both seasonal intercept and trend adjustment. The critical values for other values of d is available upon request.

Table 3.2: Finite sample size and size-adjusted power for the fractional seasonal variance ratio (FSVR) and HEGY tests: OLS demeaning

		$\boldsymbol{\tau_0(d)}$		$\boldsymbol{t_0}$	$\boldsymbol{\tau_2(d)}$		$\boldsymbol{t_2}$	$\boldsymbol{\tilde{\tau}_1(d)}$		$\boldsymbol{F_1}$	$\boldsymbol{\tau_{12}(d)}$		$\boldsymbol{F_{12}}$	$\boldsymbol{\tau_{012}(d)}$		$\boldsymbol{F_{012}}$
Model	$\alpha \setminus \boldsymbol{d}$	**0.1**	**1**	**.**	**0.1**	**1**	**.**	**0.1**	**1**	**.**	**0.1**	**1**	**.**	**0.1**	**1**	**.**
iid	1	0.05	0.05	0.05	0.04	0.05	0.05	0.05	0.05	0.05	0.04	0.05	0.05	0.04	0.05	0.05
	0.93	0.24	0.22	0.18	0.24	0.21	0.19	0.49	0.36	0.29	0.76	0.48	0.52	0.65	0.43	0.42
	0.86	0.54	0.39	0.52	0.55	0.39	0.54	0.89	0.66	0.80	0.99	0.83	0.97	0.98	0.77	0.93
ARMA (1,1)	1	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.04	0.04	0.05	0.04	0.05	0.05
	0.93	0.24	0.21	0.18	0.24	0.22	0.18	0.45	0.38	0.30	0.75	0.51	0.51	0.63	0.44	0.41
	0.86	0.55	0.40	0.53	0.55	0.41	0.53	0.88	0.68	0.83	1.00	0.86	0.97	0.97	0.77	0.93
ARMA (2,2)	1	0.04	0.05	0.05	0.04	0.05	0.05	0.05	0.04	0.05	0.04	0.05	0.05	0.04	0.05	0.05
	0.93	0.25	0.20	0.18	0.24	0.22	0.17	0.47	0.38	0.29	0.75	0.49	0.50	0.63	0.42	0.39
	0.86	0.57	0.38	0.53	0.55	0.41	0.51	0.89	0.69	0.81	1.00	0.85	0.97	0.98	0.77	0.92
MA(4)	1	0.03	0.04	0.05	0.03	0.04	0.03	0.05	0.05	0.05	0.03	0.04	0.05	0.03	0.04	0.05
	0.93	0.22	0.21	0.12	0.22	0.21	0.12	0.45	0.35	0.20	0.72	0.48	0.32	0.60	0.41	0.27
	0.86	0.51	0.38	0.33	0.50	0.38	0.34	0.85	0.64	0.56	0.99	0.82	0.85	0.96	0.74	0.75
ARMA (4,4)	1	0.04	0.05	0.05	0.04	0.06	0.05	0.05	0.05	0.05	0.04	0.05	0.06	0.04	0.05	0.06
	0.93	0.24	0.23	0.19	0.24	0.20	0.18	0.47	0.36	0.30	0.75	0.49	0.50	0.62	0.42	0.40
	0.86	0.55	0.41	0.53	0.56	0.38	0.53	0.89	0.66	0.82	1.00	0.84	0.97	0.98	0.76	0.93

(i) Nominal 5% asymptotic level (ii) Tests $\tau_0(d)$, $\tau_2(d)$, $\tau_1(d)$ correspond to FSVR tests at zero, Nyquist and joint harmonic frequencies, respectively. $\tau_{12}(d)$ corresponds to frequency FSVR tests at all frequencies except zero frequency Further, $\tau_{012}(d)$ corresponds FSVR the joint unit root tests at all frequencies. (iii) Tests t_0, t_2, F_1 correspond to HEGY tests at zero, Nyquist and joint harmonic frequencies, respectively. F_{12} corresponds to frequency HEGY tests at all frequencies except zero frequency Further, F_{012} corresponds HEGY the joint unit root tests at all frequencies. (iv) The results for HEGY tests do not depend on fractional integration parameter d. For HEGY tests, we use the optimal lag length selection described in del Barrio Castro et al. (2016).

Table 3.3: Finite sample size and size-adjusted power for the fractional seasonal variance ratio (FSVR) and HEGY tests: GLS demeaning

		$\tau_0(d)$		t_0	$\tau_2(d)$		t_2	$\tilde{\tau}_1(d)$		F_1	$\tau_{12}(d)$		F_{12}	$\tau_{012}(d)$		F_{012}
Model	$\alpha \setminus d$	**0.1**	**1**	**.**	**0.1**	**1**	**.**	**0.1**	**1**	**.**	**0.1**	**1**	**.**	**0.1**	**1**	**.**
iid	1	0.05	0.05	0.05	0.05	0.05	0.06	0.06	0.05	0.05	0.05	0.05	0.06	0.06	0.05	0.05
	0.93	0.31	0.19	0.38	0.32	0.18	0.41	0.63	0.27	0.58	0.91	0.35	0.87	0.81	0.30	0.77
	0.86	0.66	0.31	0.81	0.66	0.32	0.82	0.96	0.47	0.96	1.00	0.63	1.00	0.99	0.55	0.99
ARMA (1,1)	1	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.06	0.05	0.05	0.05
	0.93	0.33	0.18	0.41	0.32	0.19	0.41	0.66	0.26	0.60	0.92	0.33	0.89	0.83	0.31	0.77
	0.86	0.67	0.30	0.82	0.66	0.33	0.81	0.96	0.47	0.96	1.00	0.64	1.00	1.00	0.57	0.99
ARMA (2,2)	1	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.06
	0.93	0.33	0.19	0.38	0.32	0.20	0.39	0.66	0.28	0.59	0.91	0.35	0.88	0.83	0.31	0.77
	0.86	0.66	0.32	0.80	0.67	0.33	0.82	0.96	0.48	0.96	1.00	0.63	1.00	1.00	0.55	0.99
MA(4)	1	0.04	0.05	0.04	0.04	0.05	0.04	0.05	0.05	0.05	0.04	0.05	0.05	0.04	0.05	0.04
	0.93	0.33	0.19	0.31	0.32	0.19	0.32	0.61	0.27	0.47	0.91	0.33	0.77	0.80	0.30	0.64
	0.86	0.67	0.33	0.67	0.66	0.35	0.67	0.95	0.51	0.86	1.00	0.67	0.99	1.00	0.59	0.97
ARMA (4,4)	1	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.05	0.06
	0.93	0.34	0.18	0.41	0.33	0.17	0.40	0.63	0.27	0.57	0.91	0.32	0.88	0.82	0.29	0.77
	0.86	0.69	0.31	0.81	0.66	0.31	0.82	0.95	0.49	0.95	1.00	0.62	1.00	0.99	0.55	0.99

(i) Nominal 5% asymptotic level (ii) Tests $\tau_0(d)$, $\tau_2(d)$, $\tau_1(d)$ correspond to FSVR tests at zero, Nyquist and joint harmonic frequencies, respectively. $\tau_{12}(d)$ corresponds to frequency FSVR tests at all frequencies except zero frequency Further, $\tau_{012}(d)$ corresponds FSVR the joint unit root tests at all frequencies. (iii) Tests t_0, t_2, F_1 correspond to HEGY tests at zero, Nyquist and joint harmonic frequencies, respectively. F_{12} corresponds to frequency HEGY tests at all frequencies except zero frequency Further, F_{012} corresponds HEGY the joint unit root tests at all frequencies. (iv) The results of HEGY tests do not depend on fractional integration parameter d. For HEGY tests, we use the optimal lag length selection described in del Barrio Castro et al. (2016).

CHAPTER 4

NONPARAMETRIC SEASONAL UNIT ROOT TESTS UNDER PERIODIC NONSTATIONARY VOLATILITY

4.1 Introduction

There is growing literature related to the patterns of heteroscedasticity in seasonal economic and financial variables. These patterns can be grouped into two categories. The first one is periodic heteroscedasticity (PH), where shocks in some seasons are more variable than others. It is documented that there is significant evidence of PH in variables such as manufacturing series, marginal cost curves, unemployment rates (see Beaulieu and Miron (1993); Franses et al. (1996); Proietti (1998)). The second one is nonstationary volatility, where there are permanent changes in volatility within years. Some variables which exhibit nonstationary volatility are energy consumption, core CPI, real GDP growth and exchange rates (see McConnell and Perez-Quiros (2000); Kahn et al. (2002); Van Dijk et al. (2002)). The empirical evidence of heteroscedasticity coupled with seasonal nonstationarity led the researchers to investigate the seasonal unit root tests under heteroscedasticity. For instance, Burridge and Taylor (2001, 2004) and Cavaliere et al. (2017) suggest heteroscedasticity robust versions of the parametric seasonal unit tests of Hylleberg et al. (1990)[HEGY]. However, nonparametric seasonal unit root tests are still based on the assumption of homoscedasticity.

In this paper, we aim to improve the performance of nonparametric seasonal unit root tests under both PH and nonstationary volatility. To accomplish this improvement, we first consider the asymptotic properties of the fractional seasonal variance ratio (FSVR) test statistics of Eroğlu et al. (2018) under the aforementioned types of heteroscedasticity. We demonstrate that the asymptotic distributions of FSVR test statistics depend on nuisance parameters arising from the heteroscedasticity present in the innovations. Especially, limiting distributions of joint frequency FSVR tests depend on seasonal heterogeneity parameters and can not be directly simulated. Therefore, we suggest a wild bootstrap based procedure which provides good approximations to the limiting distributions of each FSVR test statistics.

This new testing design has several advantages. First, it is fully nonparametric and does not necessitate the parametric estimation as in Burridge and Taylor (2001) and in Cavaliere et al. (2017). Second, the nonparametric wild bootstrap reduces severe size distortions in the presence of heteroscedasticity and maintains the tuning parameter-free nature of the proposed tests. Third, the wild bootstrap implemented in this paper replicates the pattern of heteroscedasticity in the seasonal innovations without estimating variance profile. Finally, bootstrap saves us from simulating the asymptotic distributions of the test statistics of interest. This feature is important since in some cases, it is not possible to simulate the asymptotic distributions of the test statistics.

The remainder of the paper is organized as follows. In Section 4.2, we introduce seasonal data generating process (DGP) with periodic nonstationary volatility and detail the assumptions under which we will work. In Section 4.3 we analyze the nonparametric seasonal unit root tests under periodic non-stationary volatility structure. The wild bootstrap algorithm of the FSVR unit root tests are defined, and the limiting null distributions of the associated wild bootstrap FSVR tests are given in Section 4.4. The finite sample properties of the wild bootstrap HEGY and FSVR unit root tests are explored through Monte Carlo simulation in Section 4.5. Section 4.6 concludes. The simulation result tables are placed at the end of the chapter.

4.2 The heteroskedastic seasonal model and assumptions

In this section, we firstly present the univariate heteroskedastic seasonal time series $\{x_{Sn+s}\}$ with the following data generating process (DGP):

$$\alpha(L)x_{Sn+s} = u_{Sn+s}, \quad s = 1 - S, ..., 0, n = 1, ..., N \tag{4.1}$$

$$u_{Sn+s} = \Psi(L)\varepsilon_{Sn+s} \tag{4.2}$$

$$\varepsilon_{Sn+s} = \sigma_{Sn+s} e_{Sn+s} \tag{4.3}$$

where S represents the number of seasons, $\alpha(L) = 1 - \sum_{j=1}^{S} \alpha_j L^j$ is an $AR(S)$ polynomial which determines the seasonal unit root. Furthermore, lag polynomial $\Psi(L) = 1 + \sum_{j=1}^{\infty} \psi_j L^j$ generates serial correlation in the innovations. The lag operator L is defined as $L^{Sj+k} y_{Sn+s} = y_{S(n-j)+s-k}$. The sample size is given by $T = SN$. such that N is the total number of cycles completed. Furthermore, following Eroğlu et al. (2018), we assume that initial conditions $x_1, ..., x_S$ are $o_p(T^{1/2})$.

In this framework, the vector of seasons innovation process $\{\mathcal{E}_n\}$ is the source of seasonal non-stationarity behaviour of the model. We define it as $\mathcal{E}_n = (\varepsilon_{Sn-(S-1)}, ..., \varepsilon_{Sn})$. Throughout the paper, the following set of assumptions will be taken to hold:

Assumption. $\mathcal{A}$.1 Let $\mathcal{E}_n = \Omega_n E_n$, such that $E_n = diag\{e_{Sn-(S-1)}, ..., e_{Sn}\}$ and $\Omega_n = diag\{\sigma_{Sn-(S-1)}, ..., \sigma_{Sn}\}$ is an $S \times S$ nonstochastic matrix.

$\mathcal{A}$.2 The volatility term Ω_n satisfies:

$$\Omega_n = \Omega(n/N) := diag\{\sigma_{(1-S)}(n/N), ..., \sigma_0(n/N)\}$$

for all $n = 1, ..., N$, where $\Omega(.) \in D_{R^{S\times S}}[0,1]$, where $D_{R^{S\times S}}[0,1]$ denotes the space of $m \times n$ real matrices of cadlag function on $[0,1]$.

$\mathcal{A}$.3 Time varying unconditional variance matrix $\Upsilon(u) = \Omega(u)\Omega(u)'$ is assumed to be positive definite for all $u \in [0,1]$.

$\mathcal{A}$.4 The power series $\Psi(z)$ satisfies $\Psi(z) \neq 0$ for all $|z| \leq 1$, and $\sum_{j=0}^{\infty} j^h |\psi_j| < \infty$.

Remark 20. Assumptions $\mathcal{A}.1 - \mathcal{A}.3$ are the conditions for the innovations to display periodic nonstationary unconditional volatility. They imply that the time varying pattern in the volatility can be observed in either some or the entire seasons. Moreover, the time varying variance of the seasonal vector of shocks $\mathcal{E}_n$ is bounded, deterministic and exhibits a countable amount of jumps, by permitting single or multiple variance shifts. Also, alteration of the PH behavior of the shocks is allowed through the sample. It is important to note that the pattern of PH where $\Omega_n = diag\{\sigma_{(1-S)}, ..., \sigma_0\}$ is the special case of the volatility setup defined in Assumption $\mathcal{A}$. For the standard homoscedastic case, we define $\Omega_n = \sigma I_S$.

Remark 21. Assumption $\mathcal{A}.4$ implies stationarity of the innovation term u_{Sn+s}. In other words, the only source of nonstationary dynamics is the seasonal AR polynomial $\alpha(L)$.

4.3 Asymptotic analysis of FSVR tests under periodic non-stationary volatility

In this section, we investigate the asymptotic properties of the nonparametric FSVR tests of Eroğlu et al. (2018) under the assumption of heteroscedasticity. Remember that the FSVR tests are defined in different frequencies. First, consider the test statistics of unit root on the zero and Nyquist frequencies:

$$\tau_k(d) = N^{2d} \frac{\sum_{n=1}^{N} \sum_{s=1-S}^{0} x_{k,Sn+s}^2}{\sum_{n=1}^{N} \sum_{s=1-S}^{0} \tilde{x}_{k,Sn+s}^2} \; for\ k = 0, \lfloor S/2 \rfloor \tag{4.4}$$

For the harmonic frequencies, two variance ratio test statistics are given as:

$$\tau_j(d) = N^{2d}\frac{\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,Sn+s}^2}{\sum_{n=1}^{N}\sum_{s=1-S}^{0} \tilde{x}_{j,Sn+s}^2} \; for \; j = 1, .., S^* \tag{4.5}$$

$$\tau_j^*(d) = N^{2d}\frac{\sum_{n=1}^{N}\sum_{s=1-S}^{0} x_{j,Sn+s}^{*2}}{\sum_{n=1}^{N}\sum_{s=1-S}^{0} \tilde{x}_{j,Sn+s}^{*2}} \; for \; j = 1, .., S^* \tag{4.6}$$

where $\tau_j(d)$ and $\tau_j^*(d)$ are the test statistics for the unit root hypothesis at the harmonic frequencies ω_j and $2\pi - \omega_j$ for all $j = 1, ..., S^*$. The following test statistic is defined to jointly test the presence of the pairs of complex unit roots:

$$\tilde{\tau}_j(d) = (\tau_j(d) + \tau_j^*(d))/2 \; for \; j = 1, .., S^* \tag{4.7}$$

Finally, the F type statistics for the joint test for the unit roots in the different frequencies are given as:

$$\tau_{1,..,\lfloor S/2\rfloor}(d) = \frac{1}{S-1}\left(\tau_{\lfloor S/2\rfloor}(d) + \sum_{j=1}^{S^*}\tau_j(d) + \sum_{j=1}^{S^*}\tau_j^*(d)\right) \tag{4.8}$$

$$\tau_{0,..,\lfloor S/2\rfloor}(d) = \frac{1}{S}\left(\tau_0(d) + \tau_{\lfloor S/2\rfloor}(d) + \sum_{j=1}^{S^*}\tau_j(d) + \sum_{j=1}^{S^*}\tau_j^*(d)\right) \tag{4.9}$$

Remark 22. The test statistic in equation (4.8) corresponds to the joint unit root tests at all frequencies except the zero frequency. Further, the test statistic in equation (4.9) corresponds to the joint unit root tests at all frequencies.

Seasonal unit root tests of Eroğlu et al. (2018) are constructed with the assumption of homoscedasticity. Therefore these tests do not take the periodic heteroscedasticity structure of Burridge and Taylor (2001) and nonstationary volatility structure of Cavaliere and Taylor (2008) into the account. This section aims to discuss the effect of the aforementioned volatility structures on the asymptotic null distributions of FSVR tests.

First, to obtain asymptotic null distributions of the FSVR tests outlined above when the volatility process satisfies Assumption $\mathcal{A}.1 - \mathcal{A}.3$, first, we rewrite the $\{x_{Sn+s}\}$ in the vector of seasons form as follows:

$$X_n = X_{n-1} + U_n, \quad n = 1, ..., N \tag{4.10}$$

where $X_n = [x_{Sn-(S-1)}, ..., x_{Sn}]'$ and $U_t = [u_{Sn-(S-1)}, ..., u_{Sn}]'$, $n = 1, ..., N$. Let us define the vector error process U_n in the vector $MA(\infty)$ representation as

$$U_n = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{n-j} \tag{4.11}$$

where Ψ_j are $S \times S$ matrices in the form

$$\Psi_0 := \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ \psi_1 & 1 & 0 & ... & 0 \\ \psi_2 & \psi_1 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{S-1} & \psi_{S-2} & \psi_{S-3} & ... & 1 \end{bmatrix}$$

and for $j = 1, 2, ...$

$$\Psi_j := \begin{bmatrix} \psi_{jS} & \psi_{jS-1} & \psi_{jS-2} & \cdots & \psi_{jS-(S-1)} \\ \psi_{jS+1} & \psi_{jS} & \psi_{jS-2} & \cdots & \psi_{jS-(S-2)} \\ \psi_{jS+2} & \psi_{jS} & \psi_{jS-1} & \cdots & \psi_{jS-(S-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{jS+S-1} & \psi_{jS+S-2} & \psi_{jS+S-3} & \cdots & \psi_{jS} \end{bmatrix}$$

where ψ_j, $j = 1, 2, ...$, are the MA coefficients from the inverse of $\psi(z)$.

Now, first consider the following lemma which provides the first fundamental convergence result in our setup:

Lemma 7. Let X_n be generated by (4.10)-(4.11) and Assumptions $\mathcal{A}$ is satisfied. Then in the space of $D_{R^{m\times n}}[0, 1]$,

$$N^{-1/2} X_{\lfloor .N \rfloor} \Rightarrow \Psi(1) M(.), \tag{4.12}$$

where $\Psi(1) = \sum_{j=0}^{\infty} \Psi_j$ and $M(.) = \int_0^{.} \Omega(u) dW(u)$ is an S-dimensional continuous martingale, with $W(.) = (W_{1-S}(.), ..., W_0(.))'$, an S-variate standard Brownian motion process. $M(.)$ is a continuous martingale vector with the integrated covariation equal to $\Sigma(.) = \int_0^{.} \Omega(u)\Omega(u)'$.

For the asymptotic analysis of the fractional scaled vector of seasons data, following Eroğlu et al. (2018), we define the vector form $\tilde{X}_n$ as follows:

$$\tilde{X}_n = [\Delta_+^{-d} x_{Sn-(S-1)}, \Delta_+^{-d} x_{Sn-(S-2)}, ..., \Delta_+^{-d} x_{St}]' \tag{4.13}$$

The following lemma presents the key convergence result of the fractional scaled vector of seasons data under the case where the error terms have non-stationary volatility:

Lemma 8. Let X_n be generated by (4.10)-(4.11) and Assumptions $\mathcal{A}.1$ and $\mathcal{A}.2$ are satisfied. Then in the space of $D_{R^{m\times n}}[0,1]$,

$$N^{-d-1/2}\tilde{X}_{\lfloor .N\rfloor} \Rightarrow \Psi(1)M_{d+1}(.), \tag{4.14}$$

where $\Psi(1)=\sum_{j=0}^{\infty}\Psi_j$ and $M_{d+1}(r)=\frac{1}{\Gamma(d+1)}\int_0^r(r-t)^d dM(t)$ is an S dimensional fractional continuous martingale, where $M(.)$ is an S dimensional continuous martingale defined in (4.12).

Remark 23. $M_{d+1}(.)$ can be represented as a vector variance-transformed type II fractional Brownian motion on $[0,1]$ with independent entries. Therefore, $M_{d+1}(.)$ can be represented as:

$$M_{d+1}(.)=[\bar{\sigma}_{1-S}W_{\eta_{1-S},1-S,d+1}(.),...,\bar{\sigma}_0 W_{\eta_0,0,d+1}(.)]'$$

where $W_{\eta_s,s,d+1}(.) = W_{s,d+1}(\eta_s(.))$, with the process $\eta_s(.) = \bar{\sigma}_s^{-2}\int_0^{.}\sigma_s(r)^2dr$, $\bar{\sigma}_s := (\int_0^1\sigma_s(r)^2dr)^{1/2}$, $s=1-S,...,0$. $\{W_{s,d+1}(.)\}_{s=1-S}^{0}$ are the independent type II fractional Brownian mortions. It is important to note that the limiting processes $\{W_{s,d+1}(\eta_s(.)\}_{s=1-S}^{0}$ are a set of S independent scalar variance-transformed type II fractional Brownian motions under a transformation of the time domain with the process $\eta_s(.)$. For discussion of the properties of variance-transformed type II fractional Brownian motion, see Eroğlu and Yiğit (2016).

Using the fundamental convergence results of Lemma 7 and Lemma 8 , we now present the asymptotic null distributions of the FSVR test statistics under periodic non-stationary volatility:

Theorem 5. Let the assumptions of Lemma 7 and Lemma 8 hold. The statistics $\tau_0(d)$, $\tau_{S/2}(d)$, $\tau_j(d)$,$\tau_j^*(d)$, $\tilde{\tau}_j(d)$, $j = 1, ..., S^*$, $\tau_{1,..,\lfloor S/2 \rfloor}(d)$ and $\tau_{0,..,\lfloor S/2 \rfloor}(d)$ then have the following asymptotic null distributions:

$$\tau_i(d) \Rightarrow \frac{\int_0^1 B_{\eta,i}(r)^2 dr}{\int_0^1 B_{d+1,\eta,i}(r)^2 dr} =: \xi_{\eta,i}(d) \quad i = 0, \lfloor S/2 \rfloor$$

$$\tau_j(d) \Rightarrow \frac{\{c_j'\Sigma(1)c_j\} \int_0^1 B_{\eta,j}(r)^2 dr + \{\tilde{c}_j'\Sigma(1)\tilde{c}_j\} \int_0^1 B_{\eta,j}^*(r)^2 dr}{\{c_j'\Sigma(1)c_j\} \int_0^1 B_{d+1,\eta,j}(r)^2 dr + \{\tilde{c}_j'\Sigma(1)\tilde{c}_j\} \int_0^1 B_{d+1\eta,j}^*(r)^2 dr} =: \xi_{\eta,j}(d)$$

$$\tau_j^*(d) \Rightarrow \xi_{\eta,j}(d)$$

$$\tilde{\tau}_j(d) \Rightarrow \xi_{\eta,j}(d)$$

$$\tau_{1,..,\lfloor S/2 \rfloor}(d) \Rightarrow \frac{1}{S-1} \left(\xi_{\eta,S/2}(d) + 2 \sum_{j=1}^{S^*} \xi_{\eta,j}(d) \right) =: \xi_{\eta,1,...,\lfloor S/2 \rfloor}(d)$$

$$\tau_{0,..,\lfloor S/2 \rfloor}(d) \Rightarrow \frac{1}{S} \left(\xi_{\eta,0}(d) + \xi_{\eta,S/2}(d) + 2 \sum_{j=1}^{S^*} \xi_{\eta,j}(d) \right) =: \xi_{\eta,0,...,\lfloor S/2 \rfloor}(d)$$

where

$$B_{\eta,i}(.) := (c_i'\Sigma(1)c_i)^{1/2} c_i' M(.), i = 0, S/2,$$

$$B_{\eta,j}(.) := (c_j'\Sigma(1)c_j)^{1/2} c_j' M(.), \quad B_{\eta,j}^*(.) := (\tilde{c}_j'\Sigma(1)\tilde{c}_j)^{1/2} \tilde{c}_j' M(.)$$

and

$$B_{d+1,\eta,i}(.) := (c_i'\Sigma(1)c_i)^{1/2} c_i' M_{d+1}(.), i = 0, S/2,$$

$$B_{d+1,\eta,j}(.) := (c_j'\Sigma(1)c_j)^{1/2} c_j' M_{d+1}(.), \quad B_{d+1,\eta,j}^*(.) := (\tilde{c}_j'\Sigma(1)\tilde{c}_j)^{1/2} \tilde{c}_j' M_{d+1}(.)$$

are normalised mutually independent variance transformed conventional Brownian motion processes and variance transformed type II fractional Brownian motion processes obtained by the selection vectors $c_0 = [1, 1, 1, ..., 1]'$, $c_{S/2} = [1, -1, 1, -1, .., 1]'$, $c_j = [cos(\omega_j[1-S]), cos(\omega_j[2-S]), ..., cos(0)]$, $\tilde{c}_j = [sin(\omega_j[1-S]), sin(\omega_j[2-S]), ..., sin(0)]$, $j = 1, ..., S^*$.

Remark 24. For given variance profiles, the representations for the limiting null distributions of the $\tau_0(d)$ and $\tau_{S/2}(d)$ given in Theorem 5 have the same functional form as limiting null distributions of nonseasonal variance ratio statistic of Eroğlu and Yiğit (2016). The limiting null distributions of the $\tau_0(d)$, $\tau_{S/2}(d)$, $\tilde{\tau}_j(d)$, $j = 1, ..., S^*$, $\tau_{1...\lfloor S/2 \rfloor}(d)$ and $\tau_{0...\lfloor S/2 \rfloor}(d)$ are all free from any weak dependence nuisance parameters.

Remark 25. In contrast to the limiting null distributions of $\tau_0(d)$ and $\tau_{S/2}(d)$, it is seen that the limiting null distributions of the harmonic frequency statistics $\tau_j(d)$, $\tau_j^*(d)$ and $\tilde{\tau}_j(d)$ and joint statistics $\tau_{1...\lfloor S/2 \rfloor}(d)$ and $\tau_{0...\lfloor S/2 \rfloor}(d)$ depend on the ratio of $\{c_j' \Sigma(1) c_j\}$ to $\{\tilde{c}_j' \Sigma(1) \tilde{c}_j\}$. Since this ratio is unknown and very difficult to estimate, simulation-based tests are not easy to implement. See Burridge and Taylor (2001) for detailed discussion.

Remark 26. Thus far we have considered the case where $\{x_{Sn+s}\}$ admits no deterministic component. We have a trend in our specification of the model:

$$y_{Sn+s} = \mu_{St+s} + x_{Sn+s}, s = 1 - S, ..., 0, n = 1, ..., N$$

where $\mu_{Sn+s} = \gamma' Z_{Sn+s}$ and Z_{Sn+s} is purely deterministic and linear combination of the spectral indicator variables corresponding to the zero and seasonal frequencies. To conduct FSVR testing under nonstationary volatility which will be exact invariant to the elements of γ, we can focus on the simplest such procedure through ordinary least squares (OLS) detrending or local generalized least squares (GLS) de-trending as in Eroğlu et al. (2018). If the deterministic component allowed, the resulting de-trended series is defined as:

$$\hat{x}_{Sn+s} = x_{Sn+s} - \hat{\gamma}' Z_{Sn+s} \tag{4.15}$$

and FSVR tests can be constructed with the help of $\hat{x}_{Sn+s}$ rather than x_{Sn+s}. Therefore, variance transformed standard Brownian and type II fractional Brownian motions on frequencies which are defined before, are re-defined as appropriate to the deterministic scenario of interest (see Eroğlu et al. (2018) for de-trending procedures). To illustrate, let us give an example for zero frequency demeaned analogue. $B_{\eta,0,d+1}$ will be replaced by the process $B_{\eta,0,d+1} - \int_0^1 B_{\eta,0,d+1}(u)du$.

In the next section, we propose wild bootstrap implementations of nonparametric seasonal unit root tests of Eroğlu et al. (2018) which are robust to periodic heteroscedasticity and permanent shifts in the variance of innovations.

4.4 Wild bootstrap FSVR tests

As we demonstrate in Theorem 5, seasonal non-stationary volatility adds a time deformation to the volatility, by way of the seasonal generalization of the variance profile defined in Cavaliere and Taylor (2007), to the limiting null distributions of the FSVR unit root test statistics. Additionally, it is seen that the limiting null distributions of the harmonic frequency statistics $\tau_j(d)$, $\tau_j^*(d)$ and $\tilde{\tau}_j(d)$ and joint statistics $\tau_{1 \ldots \lfloor S/2 \rfloor}(d)$ and $\tau_{0 \ldots \lfloor S/2 \rfloor}(d)$ depend on the ratio of $\{c_j' \Sigma(1) c_j\}$ to $\{\tilde{c}_j' \Sigma(1) \tilde{c}_j\}$. Constructing a pivotal statistics and performing a simulation-based test is not an easy task, because this ratio is unknown and very challenging to estimate. In the time series literature, bootstrapping is very useful when the simulation-based tests are not suitable.

In this section, we first propose the seasonal nonparametric wild bootstrap implementations of the FSVR tests. Basawa et al. (1991) recommend to perform bootstrapping under the null hypothesis. This recommendation should be followed since if it is neglected finite sample performance of the unit root tests are deteriorated. Therefore, also by following Cavaliere and Taylor (2008), we use seasonal first difference residual wild bootstrap. Next, we show that with this bootstrap, we obtain correct p-values in the presence of both periodic heteroscedasticity and nonstationary volatility.

4.4.1 The seasonal wild bootstrap algorithm

The proposed seasonal wild bootstrap algorithm is formalized in what follows:

1. Obtain the FSVR test statistics, $\tau_0(d)$, $\tau_{S/2}(d)$, $\tau_j(d)$, $j = 1, ..., S^*(d)$, $\tau_{1 \ldots \lfloor S/2 \rfloor}(d)$ and $\tau_{0 .. \lfloor S/2 \rfloor}(d)$ along with $\hat{x}_{Sn+s}$, where either OLS or GLS detrending is utilized to cope with the deterministic component μ_{St+s}. Then, define the first difference residual vector as $\Delta \hat{X}_n = \hat{U}_n$, where $\hat{U}_n = [\hat{u}_{Sn-S-1}, \hat{u}_{Sn-S-2}, ..., \hat{u}_{Sn}]'$, $n = 1, ..., N$

2. Generate $U_n^* = [u^*_{Sn-S-1}, u^*_{Sn-S-2}, ..., u^*_{Sn}]'$ by utilizing the device of randomization, $u^*_{Sn+s} = \hat{u}_{Sn+s} w_{Sn+s}$, where $\{w_{Sn+s}\}_{n=1}^{N}$ is an i.i.d sequence

with $E(w_{Sn+s}) = 0$, $E(w^2_{Sn+s}) = 1$ and $E(w_{Sn+s}) < \infty$

3. Construct the wild bootstrap time series by employing the following recursive procedure:

$$\Delta X^*_n = U^*_n, n = 1, ..., N,$$

where $X^*_n = [x^*_{Sn-S-1}, x^*_{Sn-S-2}, ..., x^*_{Sn}]'$.

4. By using the sample of bootstrap, obtain the bootstrap FSVR test statistics, τ^*_0, $\tau^*_{S/2}(d)$, $\tau^*_j(d)$, $j = 1, ..., S^*$, $\tau^*_1 ... \lfloor S/2 \rfloor(d)$ and $\tau^*_0 .. \lfloor S/2 \rfloor(d)$ along with $\hat{x}^*_{Sn+s}(d)$, where either OLS or GLS detrending is utilized to cope with the deterministic component μ_{St+s}.

5. Define p-values of the bootstrap as $P^*_{i,T} = G^*_{i,T}(\tau_i(d))$, $i = 0, S/2$, $P^*_{j,T} = G^*_{j,T}(\tau_j(d))$, $j = 1, ..., S^*$, $P^*_{1...\lfloor S/2 \rfloor,T} = G^*_{1...\lfloor S/2 \rfloor,T}(\tau_{1...\lfloor S/2 \rfloor,T}(d))$, and $P^*_{0...\lfloor S/2 \lfloor,T} = G^*_{0...\lfloor S/2 \lfloor,T}(\tau_{0...\lfloor S/2 \lfloor,T}(d))$, where $G^*_{k,T}(.)$, $k = 0, ..., \lfloor S/2 \rfloor$, $G^*_{1...\lfloor S/2 \rfloor,T}(.)$ and $G^*_{1...\lfloor S/2 \rfloor,T}(.)$ are the conditional (on the sample) cumulative distribution functions of τ^*_0, $\tau^*_{S/2}$, τ^*_j, $j = 1, ..., S^*$, $\tau^*_1 ... \lfloor S/2 \rfloor$ and $\tau^*_0 .. \lfloor S/2 \rfloor$, respectively.

Remark 27. As mentioned in Cavaliere and Taylor (2009), the wild bootstrap resampling strategy has the critical feature that, conditionally on the actual data, the bootstrap error terms, U^*_n from step 2 are serially uncorrelated. This feature is satisfied regardless of any serial correlation observed in the first difference residuals $\hat{U}_n$. Therefore, the bootstrap error terms, U^*_n replicate the heteroskedasticity behavior in the actual shocks. It is essential to note that, since the increments of bootstrap samples are (conditionally) serially uncorrelated, and actual FSVR tests with the corresponding asymptotic null distributions do not depend on serial correlation parameters, parametric or semi-parametric correction for serial correlation is not needed for the bootstrap test statistics.

Theorem 6. Let X_n be generated by (4.10)-(4.11) and Assumptions $\mathcal{A}.1$ and $\mathcal{A}.2$ are satisfied. Then $\tau_i^*(d) \Rightarrow \xi_{\eta,i}(d)$, $i = 0, S/2$, $\tilde{\tau}_j^*(d) \Rightarrow \xi_{\eta,j}(d)$, $j = 1, .., S^*$, $\tau^*_{1,\ldots,\lfloor S/2 \rfloor}(d) \Rightarrow \xi_{\eta,1,\ldots,\lfloor S/2 \rfloor}(d)$ and $\tau^*_{1,\ldots,\lfloor S/2 \rfloor}(d) \Rightarrow \xi_{\eta,1,\ldots,\lfloor S/2 \rfloor}(d)$, where $\Rightarrow$ denotes weak convergence in probability. Moreover, $P_T^* \xrightarrow{w} U[0,1]$, where P_T^* denotes the FSVR p-values defined in the seasonal nonparametric bootstrap algorithm, and $U[0,1]$ denotes a uniform distribution on $[0,1]$.

Remark 28. Theorem 6 displays the practicality of the nonparametric seasonal wild bootstrap under the null hypothesis of the seasonal unit root tests. As the sample size goes to infinity, the bootstrapped FSVR statistics have the same asymptotic null distribution as the actual FSVR test statistics under periodic non-stationary volatility in the innovations. As a result, the bootstrap FSVR p-values have (asymptotically) uniform distribution under the null hypothesis, thus rendering the tests (asymptotically) correct-sized. Notice that the results also hold under conditional homoscedasticity since that special case is contained within Assumption $\mathcal{A}$.

4.5 Finite Sample Simulations

In this section, we report the finite sample size and power performance of the FSVR tests, the corresponding wild bootstrap FSVR tests, and the wild bootstrap HEGY tests. We permit the innovations to exhibit various patterns of heteroscedasticity which are allowed under Assumption $\mathcal{A}$. All of the reported results are based on OLS and GLS demeaned data permitting seasonal intercepts. All experiments are based on $10,000$ Monte Carlo simulations. Following Eroğlu et al. (2018), we select $d = 0.1$. FSVR tests are performed with $B = 199$ bootstrap replications. Bootstrapped variables are formed using independent standard normal variables w_{Sn+s}. For the standard FSVR tests, critical values of Eroğlu et al. (2018) are used.

In Monte Carlo simulations, data is generated according to quarterly ($S = 4$) DGP as follows:

$$(1 - \alpha_0 L)(1 + \alpha_2 L)(1 + \alpha_1^2 L^2) x_{4n+s} = u_{4n+s} \tag{4.16}$$

$$u_{Sn+s} = \phi(L)\varepsilon_{4n+s} \tag{4.17}$$

$$\varepsilon_{4n+s} = \sigma_{4n+s} e_{4n+s}, \quad e_{Sn+s} \sim NIID(0,1) \tag{4.18}$$

where $e_{Sn+s} = 0$, for $t \leq 0$. All simulations are conducted with N=100 and 0.05 significance level, with the volatility process σ_{4n+s}. We define the notation $\Sigma_i := (\sigma_{i,-3}, \sigma_{i,-2}, \sigma_{i,-1}, \sigma_{i,0}), \quad i = 0, 1$. For investigation of the finite sample size performance, we pick $\alpha_0 = \alpha_1 = \alpha_2 = 1$ and for the finite sample power performance, we pick $\alpha_j \in \{0.93, 0.86\}$

We consider 2 different types of serial correlation scenarios for u_{4n+s} consistent with Eroğlu et al. (2018). By expanding $\phi(L) = \theta(L)\psi(L)^{-1}$ with two finite lag operators $\theta(L)$ and $\psi(L)$, we consider the following types of serial correlation scenarios for u_{4t+s} as follows:

$$(1 - \psi_1 L - \psi_2 L^2 - \psi_4 L^4) u_{4n+s} = (1 + \theta_1 L + \theta_2 L^2 + \theta_4 L^4)\epsilon_{4n+s} \tag{4.19}$$

where ϵ_{4n+s} is a martingale difference sequence. For the Monte Carlo simulations, in addition to no serial correlation case, we consider the parametrisation in equation (4.19) as follows[1]: **(i)**$MA(4)$: $\theta_4 = 0.5$, $\theta_1 = \theta_2 = 0$, $\psi_1 = \psi_2 = \psi_4 = 0$, **(ii)**$ARMA(4,4)$: $\theta_4 = -0.5$, $\theta_1 = \theta_2 = 0$, $\psi_4 = 0.5$ $\psi_1 = \psi_2 = 0$.

We consider six different models of heteroscedasticity as follows:

Model 1 Constant Unconditional Volatility: $\sigma^2_{Sn+s} = \sigma^2_0$, $\sigma_0 = 1$.

Model 2 Periodic Heteroscedasticity: $\sigma^2_{Sn+s} = \sigma^2_{0,s}$, with: Case 1, $\Sigma_0 = (3, 1, 3, 1)$; or Case 2, $\Sigma_0 = (30, 1, 1, 1)$.

Model 3 Single Volatility Shift: $\sigma^2_{Sn+s} = \sigma^2_0 + (\sigma^2_1 - \sigma^2_0)I(4n + s \geq \lfloor \kappa T \rfloor$, with $\delta := \sigma_0/\sigma_1$, $\sigma_0 = 1$.

Model 4 Single Periodic Volatility Shift: $\sigma^2_{Sn+s} = \sigma^2_{0,s} + (\sigma^2_{1,s} - \sigma^2_{0,s})I(4n + s \geq \lfloor \kappa T \rfloor$, with Case 1, $\{\Sigma_0 = (3, 1, 3, 1), \Sigma_1 = \Sigma_0/\delta\}$; Case 2, $\{\Sigma_0 = (30, 1, 1, 1), \Sigma_1 = \Sigma_0/\delta\}$; Case 3, $\{\Sigma_0 = (1, 1, 1, 1), \Sigma_1 = (1, 1/\delta, 1, 1/\delta)\}$;

Model 5 Trending Volatility: $\sigma_{Sn+s} = \sigma_0 + (\sigma_1 - \sigma_0)n$, $\delta := \sigma_0/\sigma_1$, $\sigma_0 = 1$.

Model 6 Exponential Integrated Stochastic Volatility $\sigma^2_{Sn+s} = \sigma^2_0 exp(vB(r))$, $r \in [0, 1]$.

[1]The other types of serial correlations in Eroğlu et al. (2018) are not considered to save space.

In Models 3,4 and 5, the ratio $\delta = \sigma_0/\sigma_1$ is chosen from the values $\{1/3, 3\}$, and the break fraction is chosen from $\kappa \in \{0.2, 0.8\}$. Therefore, early and late breaks, which can be positive ($\delta < 1$) or negative ($\delta > 1$) are permitted. In the case of Model 6, following Cavaliere and Taylor (2007)we take $v = 9$.

Notice that Model 4 uses a combination of the single volatility break (Model 3) of Cavaliere and Taylor (2008), and the PH pattern (Model 2) defined in Burridge and Taylor (2001). Therefore the pattern of PH shifts at the break fraction κ. Under Cases 1 and 2 of Model 4 the magnitude of the PH alters at the break time. In Case 3 of Model 4, the shocks are homoscedastic before the break but become periodically heteroscedastic afterward. Model 1, the constant volatility case, is the benchmark to make a comparison between the finite sample performance of the actual FSVR tests and their wild bootstrap analogues when there is no heteroscedasticity.

4.5.1 Size Performance

We first report the empirical size performance of the conventional FSVR τ_0, τ_2, τ_1, τ_{12}, and τ_{12} tests and their wild bootstrap analogues τ_0^*, τ_2^*, τ_1^*, τ_{12}^*, and τ_{12}^* under the heteroskedastic shocks. The results reported in Tables 4.1,4.2 and 4.3 depict the size performances of the conventional FSVR tests and their wild bootstrap analogues under OLS demeaning, with data generated according to i.i.d., ARMA(4,4) and MA(4) shocks, respectively. Additionally, the results reported in Tables 4.4, 4.5 and 4.6represent the corresponding results under GLS demeaning.

Consider first Tables 4.1-4.3 and 4.4-4.6 for the constant volatility case of Model 1 under different serial correlation scenarios. Here, under both GLS and OLS demeaning,as expected there are no considerable distinctions between the empirical rejection frequencies (ERFs) of the standard and wild bootstrap FSVR tests, particularly. Wild bootstrap is successful in replicating homoscedastic behavior of the data.

Consider next Tables 4.1-4.3 and 4.4-4.6 for Model 2 where there is standard PH in the innovations. The results pattern observed for the standard FSVR tests are consistent with those observed in Burridge and Taylor (2001). Similar with the results of Burridge and Taylor (2001), the ERFs of the FSVR τ_1, τ_{12},and τ_{12} tests rise as the PH ratio of $(\sigma^2_{0,-3} + \sigma^2_{0,-1})/(\sigma^2_{0,-2} + \sigma^2_{0,0})$ rises, while the ERFs of the τ_0 and τ_2 tests are not influenced regardless of the value of the aformentioned ratio. For example in Table 1, the size of τ_1 under Case 1, where PH degree is not strong, is 0.10. However, under Case 2, where PH degree is strong, it is 0.23. Under GLS demeaning, the results are similar. In contrast, the bootstrap FSVR tests are successful in controlling the size under both detrending procedures.

Consider next Tables 4.1-4.3 and 4.4-4.6 for Model 3, a single variance break. Here, significant over-sizing in the conventional FSVR tests is apparently observed in the results. The most pronounced patterns of size distortions is observed for the early negative ($\delta = 3$ and $\kappa = 0.2$) and late positive ($\delta = 1/3$ and $\kappa = 0.2$) breaks under OLS demeaning. Under GLS demeaning, the oversizing problem for early negative break annihilates. However, in this demeaning case, under late positive break, the ERFs are observed to exceed 30% for some of the tests (see Table 4.4. In contrast, under both de-trending procedures, the wild bootstrap FSVR tests show excellent control of size .

Consider next Tables 4.1-4.3 and 4.4-4.6 for Model 4, periodic volatility shift. We hereby obtain the results that yield significant consistency with the standard PH and single variance break cases which are introduced by Models 2 and 3, respectively. So, we still observe that early negative and late positive periodic volatility breaks create the highest size distortion under the OLS demeaning. In the meanwhile, late positive breaks create the highest size distortion on GLS demeaned tests. Under Case 3, the volatility break only influences the second and fourth quarters and this pattern of volatility shift lowers the distortion of the size performance compared to Cases 1 and 2 where the volatility shift in the innovations influences the entire seasons. Under this model, again the nonparametric wild bootstrap FSVR tests do a very successful job in finishing off the size distortions observed in standard FSVR tests.

Consider finally Tables 4.1-4.3 and 4.4-4.6 for Models 5 and 6, trending volatility and exponential integrated stochastic volatility, respectively. It can be seen that for trending volatility under OLS demeaning, standard FSVR tests do not display remarkable size distortions. However, under GLS demeaning, size distortions are observed especially for joint FSVR test when there is a positive ($\delta = 1/3$) trending volatility. The wild bootstrap FSVR tests are can handle the size control under this case. Under exponential integrated stochastic volatility, under both

OLS and GLS demeaned cases, for especially serially correlated error cases, there exist huge size distortions. For example in Table 4.3, OLS demeaned case for MA(4) serially correlated errors, the joint test τ_{012} has size of 34%. Again, wild bootstrap FSVR tests solve the size distortion problems under exponential integrated stochastic volatility.

We next report the empirical size performance of the nonparametric wild bootstrap FSVR tests τ_0^*, τ_2^*, τ_1^*, τ_{12}^*, and τ_{12}^* and the wild bootstrap HEGY tests t_0^*, t_2^*, t_1^*, t_{12}^*, and τ_{12}^* under the heteroskedastic innovations. The results reported in Tables 4.7, 4.8 and 4.9 represent the size performances of the wild bootstrap FSVR and HEGY tests under OLS demeaning, with data generated according to i.i.d., ARMA(4,4) and MA(4) shocks, respectively. Additionally, the results reported in Tables 4.10, 4.11 and 4.12 represent the corresponding results under GLS demeaning. We observe that under both OLS and GLS demeaned cases, wild bootstrap FSVR and HEGY tests display nearly similiar size performance.

4.5.2 Power Performance

We now switch to the power performance evaluation of the standard and wild bootstrap FSVR tests. Since standard and wild bootstrap FSVR tests have different size performance under periodic nonstationary volatility, we report size-adjusted powers for the standard FSVR tests for implementing a considerable comparison.

We first investigate the power properties of the standard FSVR τ_0, τ_2, τ_1, τ_{12}, and τ_{12} tests and their nonparametric wild bootstrap analogues τ_0^*, τ_2^*, τ_1^*, τ_{12}^*, and τ_{12}^* under the heteroskedastic shocks. The results reported in Tables 4.13, 4.14 and 4.15 represent the power performances of the standard FSVR tests and their wild analogues counterparts under OLS demeaning, with data generated according to i.i.d., ARMA(4,4) and MA(4) shocks, respectively. Additionally, the results reported in Tables 4.16, 4.17 and 4.18 represent the corresponding results under GLS demeaning.

The fundamental inferences that are obtained from the simulation results in Tables 4.13-4.15 and 4.16-4.18 can be easily summarized as follows. First, the finite sample power performance of the conventional FSVR tests and wild bootstrap counterparts are similar under both OLS and GLS demeaning. In the constant volatility case (Model 1) the reported results show that there is no meaningful loss in power performance with the usage of nonparametric wild bootstrap. At the same time, the wild bootstrap is very successful in solving the over-sizing problem caused by the conventional FSVR tests under heteroscedasticity. Second, the power performance of both the conventional and wild bootstrap FSVR tests depend on the heteroskedasticity present in the innovations. In some cases, when OLS de-trending is utilized, power performance of both tests are deteriorated, especially in the single volatility break case with $\delta = 3$ and $\kappa = 0.2$. However, under GLS demeaning, this problem disappears.

Next we report the power performance of the nonparametric wild bootstrap FSVR tests τ_0^*, τ_2^*, τ_1^*, τ_{12}^*, and τ_{12}^* and wild bootstrap HEGY tests t_0^*, t_2^*, t_1^*, t_{12}^*, and t_{12}^*

under the heteroskedastic innovations. The results reported in Tables 4.19, 4.20 and 4.21 represent the power performance of the wild bootstrap FSVR and HEGY tests under OLS demeaning, with data generated according to i.i.d., ARMA(4,4) and MA(4) shocks, respectively. Additionally, the results reported in Tables 4.22, 4.23 and 4.24 represent the corresponding results under GLS demeaning.

The power results are consistent with the results of Eroğlu et al. (2018). For the OLS demeaned data, under all serial correlation and heteroscedasticity scenarios, wild bootstrap FSVR tests are significantly more powerful compared to wild bootstrap HEGY tests. Under the GLS demeaning case, wild bootstrap HEGY tests have the advantage in the individual coefficient testing relative to the wild bootstrap FSVR tests under homoscedastic and PH errors. However, under non-stationary volatility, they have similar power performance. Additionally, under GLS demeaning, in general, FSVR tests outperforms HEGY tests in the joint tests.

4.6 Conclusion

In this chapter, we have explored the effect of a broad class of heteroscedastic models on the FSVR tests of Eroğlu et al. (2018). This class especially covers the well-known cases of periodic heteroscedasticity and non-stationary volatility. In this regard, first, we have analyzed the impact of the periodic non-stationary volatility on the asymptotic behavior of FSVR tests. We have shown that the obtained asymptotic null distributions of the FSVR test statistics are highly affected by the nuisance parameters arising from seasonal non-stationary volatility.

We have also documented the results of the standard FSVR tests under different patterns of heteroscedasticity. The results indicate that heteroscedastic shocks render these tests significantly oversized. We have also demonstrated that the derived asymptotic distributions of the FSVR test statistics are nonstandard and not suitable for performing simulation-based tests. As a remedy, we have proposed the nonparametric wild bootstrap analogues of the FSVR tests which are so useful in replicating the patterns of heteroscedasticity in the actual shocks. The bootstrap test statistics have also shown asymptotically pivotal inference. The proposed wild bootstrap FSVR tests are successful in solving the size distortion problem observed in the standard FSVR tests without losing finite sample power. Lastly, we have compared the finite sample properties of wild bootstrap FSVR and parametric HEGY tests. Simulation results reveal that wild bootstrap FSVR tests have desirable size and power properties.

Table 4.1: Size performance of the standard and wild bootstrap FSVR tests for i.i.d. shocks: OLS demeaning

Σ	δ	κ	τ_0	τ_2	τ_1	τ_{012}	τ_{12}	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*
Model 1: Homoscedasticity												
			0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.05	0.05	0.07
Model 2: Periodic Heteroscedasticity												
Case 1			0.05	0.05	0.10	0.10	0.09	0.04	0.05	0.05	0.05	0.06
Case 2			0.05	0.05	0.16	0.23	0.21	0.05	0.05	0.05	0.05	0.05
Model 3: Single Volatility Shift												
	3	0.2	0.11	0.12	0.16	0.24	0.21	0.05	0.05	0.05	0.05	0.07
		0.8	0.05	0.05	0.04	0.04	0.04	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.05	0.05	0.04	0.04	0.04	0.05	0.05	0.06	0.05	0.07
		0.8	0.12	0.11	0.18	0.26	0.22	0.05	0.05	0.05	0.04	0.06
Model 4: Single Periodic Volatility Shift												
Case 1	3	0.2	0.11	0.11	0.22	0.31	0.26	0.05	0.05	0.05	0.05	0.06
		0.8	0.05	0.05	0.07	0.07	0.06	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.04	0.05	0.07	0.06	0.06	0.04	0.05	0.05	0.05	0.06
		0.8	0.11	0.12	0.24	0.33	0.28	0.05	0.06	0.05	0.05	0.06
Case 2	3	0.2	0.11	0.10	0.27	0.36	0.33	0.04	0.05	0.05	0.05	0.05
		0.8	0.04	0.05	0.14	0.19	0.17	0.05	0.05	0.05	0.05	0.05
	0.33	0.2	0.05	0.05	0.14	0.20	0.18	0.05	0.05	0.05	0.05	0.05
		0.8	0.12	0.12	0.30	0.39	0.37	0.05	0.05	0.05	0.05	0.05
Case 3	3	0.2	0.06	0.06	0.11	0.12	0.11	0.05	0.06	0.05	0.05	0.06
		0.8	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.06	0.07
	0.33	0.2	0.04	0.05	0.08	0.07	0.07	0.05	0.05	0.05	0.05	0.06
		0.8	0.09	0.08	0.11	0.15	0.13	0.05	0.05	0.05	0.05	0.07
Model 5: Trending Volatility												
	3		0.05	0.04	0.05	0.05	0.05	0.05	0.04	0.05	0.05	0.06
	0.33		0.06	0.07	0.06	0.07	0.06	0.05	0.04	0.05	0.05	0.06
Model 6: Exponential Stochastic Integrated Volatility												
			0.12	0.12	0.09	0.11	0.11	0.05	0.05	0.05	0.05	0.06

Table 4.2: Size performance of the standard and wild bootstrap FSVR tests for ARMA (4,4) shocks: OLS demeaning

Σ	δ	κ	τ_0	τ_2	τ_1	τ_{012}	τ_{12}	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*
Model 1: Homoscedasticity												
			0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.05	0.05	0.06
Model 2: Periodic Heteroscedasticity												
Case 1			0.05	0.05	0.10	0.11	0.09	0.05	0.05	0.05	0.05	0.06
Case 2			0.05	0.05	0.16	0.23	0.19	0.05	0.05	0.05	0.05	0.05
Model 3: Single Volatility Shift												
	3	0.2	0.13	0.12	0.15	0.24	0.20	0.05	0.05	0.05	0.05	0.07
		0.8	0.05	0.05	0.05	0.04	0.04	0.05	0.05	0.06	0.05	0.06
	0.33	0.2	0.05	0.05	0.04	0.04	0.04	0.05	0.05	0.05	0.05	0.06
		0.8	0.11	0.12	0.16	0.27	0.21	0.04	0.05	0.05	0.05	0.06
Model 4: Single Periodic Volatility Shift												
Case 1	3	0.2	0.12	0.12	0.22	0.31	0.26	0.05	0.05	0.05	0.05	0.06
		0.8	0.05	0.05	0.07	0.07	0.07	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.05	0.05	0.07	0.07	0.06	0.05	0.05	0.05	0.05	0.06
		0.8	0.13	0.12	0.24	0.33	0.29	0.05	0.05	0.05	0.04	0.06
Case 2	3	0.2	0.11	0.11	0.30	0.38	0.34	0.05	0.05	0.05	0.05	0.05
		0.8	0.05	0.05	0.14	0.19	0.17	0.05	0.05	0.06	0.05	0.05
	0.33	0.2	0.04	0.05	0.14	0.20	0.18	0.05	0.05	0.05	0.05	0.05
		0.8	0.11	0.11	0.28	0.38	0.35	0.05	0.05	0.05	0.05	0.05
Case 3	3	0.2	0.06	0.06	0.10	0.12	0.11	0.05	0.05	0.05	0.05	0.06
		0.8	0.05	0.05	0.05	0.04	0.05	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.05	0.05	0.08	0.07	0.07	0.05	0.05	0.05	0.04	0.06
		0.8	0.10	0.09	0.12	0.16	0.14	0.06	0.05	0.05	0.04	0.06
Model 5: Trending Volatility												
	3		0.06	0.05	0.06	0.05	0.05	0.05	0.05	0.05	0.06	0.07
	0.33		0.06	0.06	0.07	0.07	0.07	0.05	0.05	0.05	0.06	0.07
Model 6: Exponential Stochastic Integrated Volatility												
			0.17	0.16	0.11	0.17	0.15	0.05	0.05	0.05	0.05	0.06

Table 4.3: Size performance of the standard and wild bootstrap FSVR tests for MA (4) shocks: OLS demeaning

$\boldsymbol{\Sigma}$	$\boldsymbol{\delta}$	$\boldsymbol{\kappa}$	$\boldsymbol{\tau_0}$	$\boldsymbol{\tau_2}$	$\boldsymbol{\tau_1}$	$\boldsymbol{\tau_{012}}$	$\boldsymbol{\tau_{12}}$	$\boldsymbol{\tau_0^*}$	$\boldsymbol{\tau_2^*}$	$\boldsymbol{\tau_1^*}$	$\boldsymbol{\tau_{012}^*}$	$\boldsymbol{\tau_{12}^*}$
Model 1: Homoscedasticity												
			0.04	0.03	0.04	0.03	0.03	0.04	0.03	0.03	0.03	0.04
Model 2: Periodic Heteroscedasticity												
Case 1			0.04	0.04	0.07	0.06	0.06	0.04	0.04	0.03	0.03	0.04
Case 2			0.04	0.04	0.13	0.19	0.17	0.04	0.04	0.04	0.04	0.04
Model 3: Single Volatility Shift												
	3	0.2	0.09	0.09	0.12	0.18	0.16	0.03	0.03	0.03	0.03	0.04
		0.8	0.04	0.03	0.03	0.02	0.03	0.04	0.03	0.04	0.03	0.04
	0.33	0.2	0.03	0.03	0.03	0.02	0.02	0.04	0.04	0.04	0.03	0.05
		0.8	0.08	0.08	0.12	0.17	0.15	0.03	0.03	0.03	0.02	0.03
Model 4: Single Periodic Volatility Shift												
Case 1	3	0.2	0.08	0.08	0.17	0.22	0.19	0.04	0.03	0.03	0.02	0.03
		0.8	0.04	0.03	0.05	0.04	0.04	0.04	0.04	0.03	0.03	0.04
	0.33	0.2	0.03	0.03	0.05	0.04	0.04	0.03	0.03	0.03	0.03	0.04
		0.8	0.08	0.08	0.19	0.25	0.21	0.03	0.03	0.03	0.02	0.03
Case 2	3	0.2	0.08	0.07	0.23	0.32	0.28	0.03	0.03	0.03	0.03	0.03
		0.8	0.03	0.04	0.12	0.16	0.15	0.04	0.04	0.04	0.04	0.04
	0.33	0.2	0.03	0.03	0.12	0.16	0.14	0.04	0.03	0.04	0.03	0.03
		0.8	0.08	0.09	0.26	0.35	0.31	0.03	0.03	0.03	0.03	0.03
Case 3	3	0.2	0.04	0.04	0.08	0.08	0.07	0.03	0.03	0.03	0.03	0.04
		0.8	0.03	0.04	0.03	0.03	0.03	0.03	0.04	0.04	0.03	0.05
	0.33	0.2	0.03	0.03	0.05	0.04	0.04	0.03	0.04	0.03	0.03	0.04
		0.8	0.06	0.06	0.08	0.11	0.09	0.03	0.03	0.03	0.02	0.04
Model 5: Trending Volatility												
	3		0.04	0.04	0.03	0.03	0.04	0.03	0.03	0.04	0.03	0.05
	0.33		0.04	0.04	0.04	0.04	0.04	0.04	0.03	0.03	0.03	0.04
Model 6: Exponential Stochastic Integrated Volatility												
			0.18	0.18	0.17	0.34	0.27	0.03	0.03	0.03	0.03	0.04

Table 4.4: Size performance of the standard and wild bootstrap FSVR tests for i.i.d. shocks: GLS demeaning

Σ	δ	κ	τ_0	τ_2	τ_1	τ_{012}	τ_{12}	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*
Model 1: Homoscedasticity												
			0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.05	0.04	0.06
Model 2: Periodic Heteroscedasticity												
Case 1			0.06	0.05	0.11	0.10	0.09	0.05	0.05	0.05	0.05	0.06
Case 2			0.05	0.05	0.16	0.21	0.19	0.05	0.05	0.05	0.05	0.05
Model 3: Single Volatility Shift												
	3	0.2	0.05	0.06	0.04	0.03	0.03	0.05	0.06	0.05	0.05	0.07
		0.8	0.04	0.04	0.03	0.03	0.03	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.08	0.07	0.07	0.11	0.09	0.05	0.05	0.05	0.04	0.06
		0.8	0.14	0.14	0.23	0.39	0.31	0.05	0.05	0.05	0.05	0.06
Model 4: Single Periodic Volatility Shift												
Case 1	3	0.2	0.05	0.06	0.07	0.06	0.06	0.05	0.05	0.05	0.05	0.06
		0.8	0.04	0.04	0.07	0.05	0.05	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.07	0.07	0.14	0.16	0.14	0.05	0.05	0.05	0.05	0.06
		0.8	0.14	0.14	0.29	0.43	0.36	0.05	0.05	0.05	0.04	0.06
Case 2	3	0.2	0.06	0.05	0.13	0.18	0.16	0.06	0.05	0.06	0.06	0.06
		0.8	0.03	0.04	0.12	0.16	0.15	0.05	0.05	0.05	0.05	0.05
	0.33	0.2	0.07	0.07	0.20	0.26	0.24	0.05	0.05	0.05	0.05	0.05
		0.8	0.14	0.13	0.35	0.45	0.41	0.04	0.05	0.04	0.05	0.05
Case 3	3	0.2	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.05	0.05	0.06
		0.8	0.05	0.05	0.04	0.04	0.04	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.07	0.07	0.12	0.15	0.12	0.05	0.05	0.05	0.05	0.06
		0.8	0.10	0.10	0.16	0.22	0.18	0.05	0.05	0.05	0.05	0.06
Model 5: Trending Volatility												
	3		0.04	0.05	0.04	0.04	0.04	0.05	0.05	0.05	0.05	0.06
	0.33		0.09	0.08	0.11	0.16	0.13	0.05	0.05	0.05	0.05	0.06
Model 6: Exponential Integrated Stochastic Volatility												
			0.09	0.09	0.09	0.10	0.11	0.05	0.05	0.05	0.05	0.06

Table 4.5: Size performance of the standard and wild bootstrap FSVR tests for ARMA (4,4) shocks: GLS demeaning

Σ	δ	κ	τ_0	τ_2	τ_1	τ_{012}	τ_{12}	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*
Model 1: Homoscedasticity												
			0.05	0.06	0.05	0.06	0.05	0.05	0.05	0.05	0.05	0.06
Model 2: Periodic Heteroscedasticity												
Case 1			0.06	0.05	0.10	0.10	0.08	0.05	0.05	0.05	0.05	0.06
Case 2			0.05	0.05	0.16	0.21	0.19	0.05	0.05	0.05	0.05	0.05
Model 3: Single Volatility Shift												
	3	0.2	0.05	0.05	0.04	0.03	0.03	0.05	0.05	0.05	0.05	0.07
		0.8	0.04	0.04	0.04	0.03	0.03	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.07	0.07	0.08	0.10	0.09	0.05	0.05	0.05	0.04	0.06
		0.8	0.14	0.13	0.25	0.38	0.32	0.06	0.05	0.05	0.05	0.06
Model 4: Single Periodic Volatility Shift												
Case 1	3	0.2	0.05	0.05	0.08	0.06	0.06	0.04	0.05	0.05	0.05	0.06
		0.8	0.04	0.04	0.06	0.06	0.05	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.08	0.08	0.14	0.16	0.14	0.05	0.05	0.06	0.05	0.06
		0.8	0.13	0.13	0.30	0.42	0.36	0.05	0.05	0.04	0.05	0.06
Case 2	3	0.2	0.06	0.05	0.14	0.19	0.17	0.05	0.06	0.06	0.06	0.06
		0.8	0.04	0.04	0.12	0.16	0.15	0.05	0.05	0.05	0.05	0.05
	0.33	0.2	0.07	0.07	0.19	0.26	0.24	0.05	0.05	0.05	0.05	0.05
		0.8	0.13	0.13	0.36	0.45	0.41	0.05	0.05	0.05	0.05	0.05
Case 3	3	0.2	0.04	0.05	0.07	0.06	0.06	0.05	0.05	0.05	0.06	0.07
		0.8	0.04	0.05	0.04	0.04	0.04	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.07	0.07	0.13	0.15	0.13	0.05	0.05	0.05	0.05	0.07
		0.8	0.10	0.11	0.14	0.20	0.17	0.05	0.05	0.05	0.04	0.06
Model 5: Trending Volatility												
	3		0.04	0.04	0.04	0.04	0.04	0.05	0.05	0.05	0.05	0.07
	0.33		0.09	0.09	0.12	0.16	0.14	0.05	0.05	0.05	0.05	0.07
Model 6: Exponential Stochastic Integrated Volatility												
			0.13	0.11	0.09	0.11	0.13	0.05	0.05	0.05	0.06	0.06

Table 4.6: Size performance of the standard and wild bootstrap FSVR tests for MA (4) shocks: GLS demeaning

Σ	δ	κ	τ_0	τ_2	τ_1	τ_{012}	τ_{12}	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*
Model 1: Homoscedasticity												
			0.04	0.04	0.04	0.03	0.04	0.04	0.04	0.04	0.03	0.05
Model 2: Periodic Heteroscedasticity												
Case 1			0.04	0.04	0.08	0.06	0.06	0.03	0.04	0.04	0.03	0.04
Case 2			0.04	0.04	0.14	0.19	0.17	0.04	0.04	0.04	0.04	0.04
Model 3: Single Volatility Shift												
	3	0.2	0.04	0.04	0.03	0.02	0.02	0.04	0.04	0.04	0.03	0.04
		0.8	0.03	0.03	0.02	0.02	0.02	0.03	0.03	0.04	0.03	0.04
	0.33	0.2	0.05	0.05	0.06	0.07	0.06	0.03	0.03	0.04	0.03	0.04
		0.8	0.11	0.10	0.20	0.30	0.24	0.03	0.03	0.03	0.03	0.04
Model 4: Single Periodic Volatility Shift												
Case 1	3	0.2	0.04	0.04	0.06	0.04	0.04	0.04	0.04	0.04	0.04	0.05
		0.8	0.03	0.03	0.05	0.04	0.03	0.04	0.04	0.04	0.04	0.04
	0.33	0.2	0.05	0.06	0.10	0.12	0.11	0.03	0.04	0.04	0.03	0.04
		0.8	0.10	0.11	0.27	0.36	0.31	0.03	0.04	0.03	0.03	0.04
Case 2	3	0.2	0.03	0.04	0.11	0.15	0.14	0.04	0.04	0.04	0.04	0.04
		0.8	0.02	0.02	0.10	0.14	0.12	0.04	0.04	0.04	0.04	0.04
	0.33	0.2	0.05	0.05	0.17	0.23	0.20	0.04	0.04	0.04	0.04	0.04
		0.8	0.11	0.12	0.31	0.39	0.36	0.03	0.03	0.03	0.03	0.03
Case 3	3	0.2	0.03	0.03	0.04	0.04	0.04	0.03	0.03	0.03	0.03	0.04
		0.8	0.04	0.03	0.03	0.02	0.02	0.04	0.04	0.04	0.03	0.05
	0.33	0.2	0.05	0.05	0.10	0.11	0.09	0.04	0.04	0.03	0.03	0.04
		0.8	0.08	0.08	0.12	0.17	0.14	0.04	0.03	0.03	0.03	0.04
Model 5: Trending Volatility												
	3		0.03	0.03	0.03	0.02	0.02	0.03	0.04	0.03	0.03	0.04
	0.33		0.06	0.06	0.09	0.11	0.10	0.03	0.03	0.04	0.03	0.05
Model 6: Exponential Stochastic Integrated Volatility												
			0.13	0.12	0.16	0.29	0.22	0.03	0.04	0.04	0.03	0.05

Table 4.7: Size performance of the wild bootstrap FSVR and HEGY tests for i.i.d. shocks: OLS demeaning

$\boldsymbol{\Sigma}$	$\boldsymbol{\delta}$	$\boldsymbol{\kappa}$	$\boldsymbol{\tau_0^*}$	$\boldsymbol{\tau_2^*}$	$\boldsymbol{\tau_1^*}$	$\boldsymbol{\tau_{012}^*}$	$\boldsymbol{\tau_{12}^*}$	$\boldsymbol{t_0^*}$	$\boldsymbol{t_2^*}$	$\boldsymbol{t_1^*}$	$\boldsymbol{t_{012}^*}$	$\boldsymbol{t_{12}^*}$
Model 1: Homoscedasticity												
			0.05	0.05	0.05	0.05	0.07	0.05	0.04	0.05	0.05	0.07
Model 2: Periodic Heteroscedasticity												
Case 1			0.04	0.05	0.05	0.05	0.06	0.05	0.04	0.05	0.05	0.06
Case 2			0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
Model 3: Single Volatility Shift												
	3	0.2	0.05	0.05	0.05	0.05	0.07	0.05	0.05	0.05	0.04	0.06
		0.8	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.04	0.07
	0.33	0.2	0.05	0.05	0.06	0.05	0.07	0.05	0.05	0.05	0.05	0.07
		0.8	0.05	0.05	0.05	0.04	0.06	0.05	0.05	0.05	0.05	0.08
Model 4: Single Periodic Volatility Shift												
Case 1	3	0.2	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.04	0.04	0.05
		0.8	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.04	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.06
		0.8	0.05	0.06	0.05	0.05	0.06	0.05	0.05	0.06	0.05	0.06
Case 2	3	0.2	0.04	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
		0.8	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.33	0.2	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
		0.8	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
Case 3	3	0.2	0.05	0.06	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.06
		0.8	0.05	0.05	0.05	0.06	0.07	0.05	0.05	0.04	0.04	0.06
	0.33	0.2	0.05	0.05	0.05	0.05	0.06	0.06	0.06	0.05	0.05	0.07
		0.8	0.05	0.05	0.05	0.05	0.07	0.05	0.05	0.05	0.05	0.07
Model 5: Trending Volatility												
	3		0.05	0.04	0.05	0.05	0.06	0.05	0.04	0.05	0.04	0.06
	0.33		0.05	0.04	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.07
Model 6: Exponential Stochastic Integrated Volatility												
			0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.04	0.05

Table 4.8: Size performance of the wild bootstrap FSVR and HEGY tests for ARMA (4,4) shocks: OLS demeaning

$\boldsymbol{\Sigma}$	$\boldsymbol{\delta}$	$\boldsymbol{\kappa}$	$\boldsymbol{\tau_0^*}$	$\boldsymbol{\tau_2^*}$	$\boldsymbol{\tau_1^*}$	$\boldsymbol{\tau_{012}^*}$	$\boldsymbol{\tau_{12}^*}$	$\boldsymbol{t_0^*}$	$\boldsymbol{t_2^*}$	$\boldsymbol{t_1^*}$	$\boldsymbol{t_{012}^*}$	$\boldsymbol{t_{12}^*}$
Model 1: Homoscedasticity												
			0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.04	0.07
Model 2: Periodic Heteroscedasticity												
Case 1			0.05	0.05	0.05	0.05	0.06	0.05	0.06	0.05	0.05	0.06
Case 2			0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
Model 3: Single Volatility Shift												
	3	0.2	0.05	0.05	0.05	0.05	0.07	0.05	0.05	0.05	0.04	0.06
		0.8	0.05	0.05	0.06	0.05	0.06	0.05	0.05	0.05	0.05	0.07
	0.33	0.2	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.07
		0.8	0.04	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.08
Model 4: Single Periodic Volatility Shift												
Case 1	3	0.2	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.06
		0.8	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.04	0.05
	0.33	0.2	0.05	0.05	0.05	0.05	0.06	0.05	0.04	0.05	0.05	0.06
		0.8	0.05	0.05	0.05	0.04	0.06	0.05	0.05	0.05	0.05	0.06
Case 2	3	0.2	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
		0.8	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.05	0.04	0.05
	0.33	0.2	0.05	0.05	0.05	0.05	0.05	0.06	0.06	0.05	0.05	0.05
		0.8	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
Case 3	3	0.2	0.05	0.05	0.05	0.05	0.06	0.05	0.04	0.05	0.04	0.06
		0.8	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.04	0.07
	0.33	0.2	0.05	0.05	0.05	0.04	0.06	0.05	0.05	0.05	0.05	0.06
		0.8	0.06	0.05	0.05	0.04	0.06	0.05	0.05	0.04	0.04	0.06
Model 5: Trending Volatility												
	3		0.05	0.05	0.05	0.06	0.07	0.05	0.05	0.05	0.05	0.06
	0.33		0.05	0.05	0.05	0.06	0.07	0.05	0.05	0.05	0.05	0.07
Model 6: Exponential Stochastic Integrated Volatility												
			0.05	0.05	0.05	0.05	0.06	0.06	0.05	0.04	0.04	0.05

Table 4.9: Size performance of the wild bootstrap FSVR and HEGY tests for MA (4) shocks: OLS demeaning

$\boldsymbol{\Sigma}$	$\boldsymbol{\delta}$	$\boldsymbol{\kappa}$	$\boldsymbol{\tau_0^*}$	$\boldsymbol{\tau_2^*}$	$\boldsymbol{\tau_1^*}$	$\boldsymbol{\tau_{012}^*}$	$\boldsymbol{\tau_{12}^*}$	$\boldsymbol{t_0^*}$	$\boldsymbol{t_2^*}$	$\boldsymbol{t_1^*}$	$\boldsymbol{t_{012}^*}$	$\boldsymbol{t_{12}^*}$
Model 1: Homoscedasticity												
			0.04	0.03	0.03	0.03	0.04	0.05	0.05	0.05	0.04	0.06
Model 2: Periodic Heteroscedasticity												
Case 1			0.04	0.04	0.03	0.03	0.04	0.05	0.05	0.05	0.04	0.05
Case 2			0.04	0.04	0.04	0.04	0.04	0.05	0.05	0.05	0.05	0.05
Model 3: Single Volatility Shift												
	3	0.2	0.03	0.03	0.03	0.03	0.04	0.05	0.05	0.05	0.04	0.07
		0.8	0.04	0.03	0.04	0.03	0.04	0.05	0.04	0.05	0.04	0.06
	0.33	0.2	0.04	0.04	0.04	0.03	0.05	0.05	0.05	0.05	0.04	0.06
		0.8	0.03	0.03	0.03	0.02	0.03	0.05	0.06	0.05	0.05	0.08
Model 4: Single Periodic Volatility Shift												
Case 1	3	0.2	0.04	0.03	0.03	0.02	0.03	0.05	0.05	0.06	0.05	0.06
		0.8	0.04	0.04	0.03	0.03	0.04	0.05	0.05	0.04	0.05	0.06
	0.33	0.2	0.03	0.03	0.03	0.03	0.04	0.04	0.05	0.04	0.04	0.05
		0.8	0.03	0.03	0.03	0.02	0.03	0.06	0.06	0.05	0.05	0.07
Case 2	3	0.2	0.03	0.03	0.03	0.03	0.03	0.05	0.05	0.05	0.05	0.05
		0.8	0.04	0.04	0.04	0.04	0.04	0.05	0.05	0.05	0.05	0.05
	0.33	0.2	0.04	0.03	0.04	0.03	0.03	0.06	0.06	0.06	0.05	0.06
		0.8	0.03	0.03	0.03	0.03	0.03	0.06	0.06	0.06	0.06	0.06
Case 3	3	0.2	0.03	0.03	0.03	0.03	0.04	0.05	0.05	0.05	0.04	0.06
		0.8	0.03	0.04	0.04	0.03	0.05	0.04	0.05	0.05	0.04	0.06
	0.33	0.2	0.03	0.04	0.03	0.03	0.04	0.04	0.05	0.04	0.04	0.05
		0.8	0.03	0.03	0.03	0.02	0.04	0.05	0.06	0.05	0.04	0.06
Model 5: Trending Volatility												
	3		0.03	0.03	0.04	0.03	0.05	0.05	0.05	0.05	0.04	0.07
	0.33		0.04	0.03	0.03	0.03	0.04	0.05	0.05	0.05	0.04	0.06
Model 6: Exponential Stochastic Integrated Volatility												
			0.03	0.03	0.03	0.03	0.04	0.05	0.05	0.04	0.04	0.06

Table 4.10: Size performance of the wild bootstrap FSVR and HEGY tests for i.i.d. shocks: GLS demeaning

$\boldsymbol{\Sigma}$	$\boldsymbol{\delta}$	$\boldsymbol{\kappa}$	$\boldsymbol{\tau_0^*}$	$\boldsymbol{\tau_2^*}$	$\boldsymbol{\tau_1^*}$	$\boldsymbol{\tau_{012}^*}$	$\boldsymbol{\tau_{12}^*}$	$\boldsymbol{t_0^*}$	$\boldsymbol{t_2^*}$	$\boldsymbol{t_1^*}$	$\boldsymbol{t_{012}^*}$	$\boldsymbol{t_{12}^*}$
Model 1: Homoscedasticity												
			0.05	0.05	0.05	0.04	0.06	0.05	0.05	0.05	0.05	0.08
Model 2: Periodic Heteroscedasticity												
Case 1			0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.06	0.05	0.06
Case 2			0.05	0.05	0.05	0.05	0.05	0.05	0.04	0.04	0.04	0.04
Model 3: Single Volatility Shift												
	3	0.2	0.05	0.06	0.05	0.05	0.07	0.05	0.05	0.05	0.05	0.08
		0.8	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.07
	0.33	0.2	0.05	0.05	0.05	0.04	0.06	0.06	0.05	0.05	0.05	0.08
		0.8	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.08
Model 4: Single Periodic Volatility Shift												
Case 1	3	0.2	0.05	0.05	0.05	0.05	0.06	0.05	0.04	0.05	0.05	0.07
		0.8	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.06	0.07
		0.8	0.05	0.05	0.05	0.04	0.06	0.06	0.05	0.05	0.05	0.07
Case 2	3	0.2	0.06	0.05	0.06	0.06	0.06	0.05	0.05	0.05	0.05	0.05
		0.8	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.33	0.2	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
		0.8	0.04	0.05	0.04	0.05	0.05	0.05	0.05	0.05	0.05	0.05
Case 3	3	0.2	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.07
		0.8	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.07
	0.33	0.2	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.06
		0.8	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.07
Model 5: Trending Volatility												
	3		0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.06	0.08
	0.33		0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.06	0.08
Model 6: Exponential Integrated Stochastic Volatility												
			0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.06	0.05	0.06

Table 4.11: Size performance of the wild bootstrap FSVR and HEGY tests for ARMA (4,4) shocks: GLS demeaning

$\boldsymbol{\Sigma}$	$\boldsymbol{\delta}$	$\boldsymbol{\kappa}$	$\boldsymbol{\tau_0^*}$	$\boldsymbol{\tau_2^*}$	$\boldsymbol{\tau_1^*}$	$\boldsymbol{\tau_{012}^*}$	$\boldsymbol{\tau_{12}^*}$	$\boldsymbol{t_0^*}$	$\boldsymbol{t_2^*}$	$\boldsymbol{t_1^*}$	$\boldsymbol{t_{012}^*}$	$\boldsymbol{t_{12}^*}$
Model 1: Homoscedasticity												
			0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.08
Model 2: Periodic Heteroscedasticity												
Case 1			0.05	0.05	0.05	0.05	0.06	0.05	0.06	0.05	0.05	0.06
Case 2			0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
Model 3: Single Volatility Shift												
	3	0.2	0.05	0.05	0.05	0.05	0.07	0.05	0.05	0.05	0.05	0.07
		0.8	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.06	0.06	0.08
	0.33	0.2	0.05	0.05	0.05	0.04	0.06	0.05	0.05	0.05	0.05	0.07
		0.8	0.06	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.08
Model 4: Single Periodic Volatility Shift												
Case 1	3	0.2	0.04	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.06
		0.8	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.06
	0.33	0.2	0.05	0.05	0.06	0.05	0.06	0.05	0.05	0.05	0.05	0.06
		0.8	0.05	0.05	0.04	0.05	0.06	0.05	0.05	0.06	0.05	0.06
Case 2	3	0.2	0.05	0.06	0.06	0.06	0.06	0.05	0.06	0.05	0.06	0.05
		0.8	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.33	0.2	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
		0.8	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.06	0.05	0.05
Case 3	3	0.2	0.05	0.05	0.05	0.06	0.07	0.05	0.05	0.05	0.05	0.06
		0.8	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.05	0.05	0.08
	0.33	0.2	0.05	0.05	0.05	0.05	0.07	0.06	0.05	0.05	0.05	0.06
		0.8	0.05	0.05	0.05	0.04	0.06	0.05	0.05	0.05	0.05	0.07
Model 5: Trending Volatility												
	3		0.05	0.05	0.05	0.05	0.07	0.05	0.05	0.05	0.05	0.08
	0.33		0.05	0.05	0.05	0.05	0.07	0.05	0.05	0.05	0.06	0.08
Model 6: Exponential Stochastic Integrated Volatility												
			0.05	0.05	0.05	0.06	0.06	0.06	0.05	0.05	0.06	0.07

Table 4.12: Size performance of the wild bootstrap FSVR and HEGY tests for MA (4) shocks: GLS demeaning

$\boldsymbol{\Sigma}$	$\boldsymbol{\delta}$	$\boldsymbol{\kappa}$	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*	t_0^*	t_2^*	t_1^*	t_{012}^*	t_{12}^*
Model 1: Homoscedasticity												
			0.04	0.04	0.04	0.03	0.05	0.04	0.04	0.05	0.04	0.07
Model 2: Periodic Heteroscedasticity												
Case 1			0.03	0.04	0.04	0.03	0.04	0.05	0.05	0.05	0.04	0.05
Case 2			0.04	0.04	0.04	0.04	0.04	0.06	0.05	0.05	0.05	0.05
Model 3: Single Volatility Shift												
	3	0.2	0.04	0.04	0.04	0.03	0.04	0.05	0.06	0.05	0.05	0.08
		0.8	0.03	0.03	0.04	0.03	0.04	0.04	0.04	0.05	0.04	0.07
	0.33	0.2	0.03	0.03	0.04	0.03	0.04	0.04	0.04	0.05	0.04	0.06
		0.8	0.03	0.03	0.03	0.03	0.04	0.05	0.05	0.06	0.05	0.08
Model 4: Single Periodic Volatility Shift												
Case 1	3	0.2	0.04	0.04	0.04	0.04	0.05	0.05	0.06	0.05	0.06	0.07
		0.8	0.04	0.04	0.04	0.04	0.04	0.05	0.04	0.05	0.04	0.05
	0.33	0.2	0.03	0.04	0.04	0.03	0.04	0.04	0.05	0.05	0.05	0.06
		0.8	0.03	0.04	0.03	0.03	0.04	0.06	0.06	0.06	0.06	0.07
Case 2	3	0.2	0.04	0.04	0.04	0.04	0.04	0.06	0.06	0.06	0.06	0.06
		0.8	0.04	0.04	0.04	0.04	0.04	0.05	0.06	0.06	0.06	0.06
	0.33	0.2	0.04	0.04	0.04	0.04	0.04	0.06	0.06	0.06	0.06	0.06
		0.8	0.03	0.03	0.03	0.03	0.03	0.06	0.06	0.06	0.06	0.06
Case 3	3	0.2	0.03	0.03	0.03	0.03	0.04	0.04	0.04	0.05	0.04	0.05
		0.8	0.04	0.04	0.04	0.03	0.05	0.04	0.04	0.04	0.04	0.06
	0.33	0.2	0.04	0.04	0.03	0.03	0.04	0.05	0.05	0.04	0.04	0.06
		0.8	0.04	0.03	0.03	0.03	0.04	0.05	0.05	0.05	0.05	0.07
Model 5: Trending Volatility												
	3		0.03	0.04	0.03	0.03	0.04	0.04	0.04	0.05	0.04	0.06
	0.33		0.03	0.03	0.04	0.03	0.05	0.04	0.04	0.05	0.04	0.07
Model 6: Exponential Stochastic Integrated Volatility												
			0.03	0.04	0.04	0.03	0.05	0.04	0.05	0.05	0.04	0.07

Table 4.13: Size adjusted power performance of the standard and wild bootstrap FSVR tests for i.i.d. shocks: OLS demeaning

Σ	δ	κ	α	τ_0	τ_2	τ_1	τ_{012}	τ_{12}	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*
Model 1: Homoscedasticity													
			0.93	0.23	0.25	0.46	0.74	0.62	0.24	0.24	0.45	0.74	0.69
			0.86	0.54	0.57	0.89	1.00	0.97	0.56	0.55	0.88	1.00	0.98
Model 2: Periodic Heteroscedasticity													
Case 1			0.93	0.24	0.24	0.39	0.61	0.56	0.22	0.23	0.40	0.62	0.60
			0.86	0.56	0.55	0.76	0.96	0.93	0.53	0.54	0.77	0.96	0.95
Case2			0.93	0.26	0.25	0.26	0.27	0.26	0.26	0.26	0.26	0.26	0.26
			0.86	0.58	0.57	0.59	0.60	0.59	0.58	0.58	0.58	0.59	0.59
Model 3: Single Volatility Shift													
	3	0.2	0.93	0.14	0.13	0.20	0.31	0.26	0.13	0.13	0.20	0.30	0.31
			0.86	0.36	0.34	0.57	0.84	0.72	0.36	0.35	0.57	0.83	0.78
		0.8	0.93	0.24	0.25	0.47	0.77	0.64	0.24	0.25	0.47	0.75	0.69
			0.86	0.56	0.58	0.91	1.00	0.98	0.56	0.58	0.90	1.00	0.99
	0.33	0.2	0.93	0.29	0.29	0.54	0.86	0.75	0.30	0.27	0.57	0.86	0.81
			0.86	0.62	0.63	0.93	1.00	0.99	0.63	0.62	0.94	1.00	1.00
		0.8	0.93	0.23	0.23	0.37	0.63	0.51	0.23	0.23	0.36	0.61	0.56
			0.86	0.49	0.47	0.74	0.96	0.89	0.49	0.48	0.73	0.95	0.91
Model 4: Single Periodic Volatility Shift													
Case 1	3	0.2	0.93	0.13	0.15	0.18	0.27	0.24	0.14	0.14	0.18	0.26	0.27
			0.86	0.35	0.38	0.49	0.73	0.67	0.36	0.36	0.49	0.73	0.70
		0.8	0.93	0.23	0.25	0.41	0.64	0.56	0.23	0.24	0.39	0.63	0.60
			0.86	0.55	0.58	0.80	0.97	0.94	0.56	0.56	0.78	0.96	0.95
	0.33	0.2	0.93	0.30	0.29	0.47	0.74	0.67	0.28	0.28	0.46	0.73	0.71
			0.86	0.64	0.64	0.84	0.98	0.97	0.61	0.62	0.83	0.98	0.97
		0.8	0.93	0.23	0.21	0.32	0.51	0.45	0.23	0.23	0.32	0.51	0.50
			0.86	0.48	0.47	0.64	0.88	0.82	0.49	0.50	0.63	0.88	0.85
Case 2	3	0.2	0.93	0.16	0.15	0.15	0.16	0.15	0.14	0.14	0.14	0.14	0.14
			0.86	0.38	0.38	0.38	0.39	0.37	0.35	0.36	0.36	0.36	0.36
		0.8	0.93	0.26	0.26	0.26	0.26	0.26	0.26	0.27	0.27	0.27	0.27
			0.86	0.59	0.58	0.59	0.59	0.59	0.59	0.59	0.60	0.60	0.60
	0.33	0.2	0.93	0.28	0.29	0.29	0.29	0.29	0.29	0.28	0.29	0.29	0.29
			0.86	0.61	0.62	0.63	0.63	0.63	0.62	0.62	0.63	0.63	0.63
		0.8	0.93	0.23	0.23	0.24	0.23	0.23	0.23	0.23	0.24	0.24	0.24
			0.86	0.47	0.47	0.49	0.48	0.48	0.48	0.47	0.50	0.49	0.49
Case 3	3	0.2	0.93	0.21	0.21	0.32	0.55	0.48	0.22	0.23	0.32	0.55	0.53
			0.86	0.51	0.51	0.74	0.96	0.92	0.52	0.53	0.74	0.96	0.94
		0.8	0.93	0.23	0.24	0.47	0.75	0.63	0.24	0.24	0.48	0.77	0.71
			0.86	0.54	0.57	0.89	1.00	0.98	0.55	0.57	0.89	1.00	0.99
	0.33	0.2	0.93	0.28	0.28	0.45	0.74	0.64	0.27	0.27	0.46	0.72	0.69
			0.86	0.61	0.61	0.83	0.99	0.97	0.61	0.61	0.83	0.98	0.98
		0.8	0.93	0.23	0.23	0.42	0.67	0.55	0.23	0.23	0.40	0.66	0.62
			0.86	0.51	0.51	0.81	0.97	0.93	0.51	0.51	0.80	0.97	0.95
Model 5: Trending Volatility													
	3		0.93	0.22	0.24	0.41	0.70	0.57	0.21	0.21	0.40	0.68	0.62
			0.86	0.53	0.56	0.87	0.99	0.97	0.51	0.53	0.87	0.99	0.98
	0.33		0.93	0.25	0.25	0.49	0.77	0.67	0.26	0.23	0.50	0.76	0.71
			0.86	0.56	0.57	0.89	1.00	0.97	0.57	0.54	0.89	1.00	0.98
Model 6: Exponential Stochastic Integrated Volatility													
			0.93	0.11	0.11	0.22	0.38	0.31	0.12	0.12	0.20	0.36	0.35
			0.86	0.29	0.28	0.64	0.93	0.85	0.30	0.30	0.62	0.93	0.88

Table 4.14: Size adjusted power performance of the standard and wild bootstrap FSVR tests for ARMA (4,4) shocks: OLS demeaning

Σ	δ	κ	α	τ_0	τ_2	τ_1	τ_{012}	τ_{12}	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*
Model 1: Homoscedasticity													
			0.93	0.25	0.25	0.46	0.77	0.63	0.25	0.24	0.46	0.75	0.70
			0.86	0.57	0.57	0.89	1.00	0.98	0.57	0.55	0.89	1.00	0.99
Model 2: Periodic Heteroscedasticity													
Case 1			0.93	0.25	0.23	0.39	0.61	0.55	0.26	0.25	0.38	0.62	0.59
			0.86	0.57	0.54	0.77	0.96	0.93	0.58	0.57	0.77	0.96	0.95
Case2			0.93	0.26	0.25	0.26	0.26	0.26	0.25	0.25	0.26	0.26	0.26
			0.86	0.59	0.56	0.58	0.58	0.58	0.58	0.57	0.58	0.58	0.58
Model 3: Single Volatility Shift													
	3	0.2	0.93	0.13	0.13	0.21	0.30	0.25	0.12	0.13	0.22	0.30	0.30
			0.86	0.34	0.35	0.58	0.83	0.74	0.33	0.35	0.59	0.84	0.78
		0.8	0.93	0.24	0.23	0.46	0.76	0.65	0.24	0.24	0.49	0.76	0.71
			0.86	0.56	0.55	0.89	1.00	0.98	0.57	0.56	0.91	1.00	0.99
	0.33	0.2	0.93	0.29	0.28	0.56	0.86	0.76	0.28	0.29	0.55	0.86	0.80
			0.86	0.63	0.62	0.93	1.00	0.99	0.62	0.63	0.93	1.00	1.00
		0.8	0.93	0.24	0.22	0.38	0.63	0.51	0.21	0.22	0.39	0.62	0.57
			0.86	0.51	0.47	0.75	0.96	0.90	0.47	0.48	0.76	0.96	0.92
Model 4: Single Periodic Volatility Shift													
Case 1	3	0.2	0.93	0.13	0.14	0.18	0.27	0.24	0.13	0.14	0.17	0.27	0.28
			0.86	0.35	0.36	0.51	0.73	0.66	0.35	0.37	0.50	0.73	0.71
		0.8	0.93	0.24	0.24	0.38	0.63	0.56	0.25	0.24	0.41	0.64	0.62
			0.86	0.55	0.57	0.78	0.97	0.94	0.57	0.56	0.80	0.97	0.96
	0.33	0.2	0.93	0.28	0.27	0.46	0.72	0.65	0.26	0.28	0.46	0.71	0.68
			0.86	0.61	0.62	0.84	0.98	0.97	0.60	0.62	0.83	0.98	0.97
		0.8	0.93	0.22	0.23	0.35	0.53	0.48	0.21	0.21	0.33	0.50	0.50
			0.86	0.47	0.49	0.65	0.88	0.83	0.45	0.47	0.64	0.87	0.84
Case 2	3	0.2	0.93	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15
			0.86	0.38	0.38	0.39	0.39	0.39	0.39	0.40	0.40	0.40	0.40
		0.8	0.93	0.25	0.25	0.25	0.26	0.26	0.26	0.26	0.28	0.27	0.27
			0.86	0.58	0.58	0.59	0.59	0.59	0.59	0.59	0.62	0.61	0.61
	0.33	0.2	0.93	0.29	0.28	0.30	0.29	0.29	0.28	0.28	0.28	0.28	0.28
			0.86	0.63	0.63	0.64	0.63	0.63	0.63	0.62	0.62	0.63	0.63
		0.8	0.93	0.24	0.24	0.25	0.24	0.25	0.23	0.23	0.24	0.24	0.24
			0.86	0.49	0.49	0.50	0.50	0.50	0.49	0.49	0.50	0.49	0.49
Case 3	3	0.2	0.93	0.23	0.20	0.32	0.54	0.46	0.22	0.21	0.32	0.54	0.52
			0.86	0.53	0.50	0.74	0.96	0.91	0.53	0.52	0.74	0.96	0.93
		0.8	0.93	0.23	0.25	0.48	0.78	0.66	0.24	0.24	0.49	0.77	0.71
			0.86	0.55	0.58	0.90	1.00	0.98	0.56	0.57	0.90	1.00	0.99
	0.33	0.2	0.93	0.29	0.29	0.46	0.73	0.65	0.28	0.29	0.45	0.71	0.68
			0.86	0.64	0.63	0.83	0.98	0.96	0.63	0.64	0.83	0.98	0.97
		0.8	0.93	0.23	0.24	0.41	0.68	0.56	0.25	0.23	0.41	0.65	0.61
			0.86	0.50	0.53	0.78	0.97	0.93	0.53	0.51	0.78	0.97	0.94
Model 5: Trending Volatility													
	3		0.93	0.21	0.22	0.41	0.69	0.58	0.22	0.22	0.42	0.71	0.66
			0.86	0.52	0.53	0.86	0.99	0.97	0.53	0.53	0.87	1.00	0.99
	0.33		0.93	0.25	0.26	0.47	0.75	0.64	0.24	0.26	0.48	0.77	0.72
			0.86	0.56	0.57	0.88	0.99	0.97	0.55	0.57	0.88	1.00	0.98
Model 6: Exponential Stochastic Integrated Volatility													
			0.93	0.05	0.05	0.15	0.31	0.24	0.06	0.05	0.14	0.30	0.29
			0.86	0.14	0.15	0.53	0.88	0.76	0.15	0.15	0.52	0.87	0.82

Table 4.15: Size adjusted power performance of the standard and wild bootstrap FSVR tests for MA (4) shocks: OLS demeaning

Σ	δ	κ	α	τ_0	τ_2	τ_1	τ_{012}	τ_{12}	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*
Model 1: Homoscedasticity													
			0.93	0.22	0.23	0.43	0.72	0.60	0.23	0.23	0.45	0.71	0.61
			0.86	0.50	0.51	0.86	0.99	0.96	0.50	0.51	0.86	0.99	0.96
Model 2: Periodic Heteroscedasticity													
Case 1			0.93	0.22	0.22	0.37	0.59	0.52	0.18	0.22	0.39	0.59	0.53
			0.86	0.51	0.50	0.73	0.94	0.91	0.53	0.51	0.74	0.94	0.93
Case2			0.93	0.24	0.23	0.24	0.25	0.25	0.25	0.21	0.25	0.24	0.27
			0.86	0.53	0.52	0.54	0.54	0.54	0.53	0.54	0.53	0.54	0.57
Model 3: Single Volatility Shift													
	3	0.2	0.93	0.10	0.11	0.19	0.26	0.23	0.11	0.11	0.21	0.26	0.23
			0.86	0.30	0.30	0.54	0.78	0.67	0.29	0.30	0.56	0.79	0.67
		0.8	0.93	0.23	0.24	0.46	0.75	0.62	0.22	0.22	0.48	0.74	0.62
			0.86	0.52	0.54	0.87	0.99	0.97	0.51	0.54	0.89	0.99	0.99
	0.33	0.2	0.93	0.25	0.27	0.53	0.84	0.72	0.23	0.24	0.55	0.82	0.74
			0.86	0.57	0.57	0.92	1.00	0.99	0.58	0.58	0.94	1.00	0.99
		0.8	0.93	0.20	0.21	0.36	0.61	0.49	0.21	0.21	0.38	0.62	0.48
			0.86	0.44	0.44	0.70	0.94	0.86	0.47	0.46	0.72	0.94	0.88
Model 4: Single Periodic Volatility Shift													
Case 1	3	0.2	0.93	0.11	0.12	0.17	0.23	0.22	0.11	0.12	0.17	0.23	0.21
			0.86	0.29	0.32	0.46	0.66	0.65	0.31	0.32	0.47	0.66	0.61
		0.8	0.93	0.22	0.23	0.41	0.63	0.62	0.22	0.23	0.41	0.63	0.56
			0.86	0.51	0.52	0.76	0.95	0.94	0.51	0.52	0.77	0.95	0.92
	0.33	0.2	0.93	0.28	0.28	0.46	0.72	0.71	0.27	0.28	0.46	0.72	0.65
			0.86	0.60	0.59	0.81	0.98	0.97	0.58	0.59	0.81	0.98	0.95
		0.8	0.93	0.21	0.21	0.31	0.47	0.46	0.20	0.21	0.31	0.47	0.42
			0.86	0.44	0.44	0.60	0.84	0.83	0.43	0.44	0.60	0.84	0.78
Case 2	3	0.2	0.93	0.13	0.13	0.14	0.14	0.13	0.12	0.13	0.14	0.14	0.14
			0.86	0.33	0.33	0.35	0.35	0.34	0.32	0.33	0.36	0.35	0.35
		0.8	0.93	0.22	0.24	0.24	0.24	0.23	0.24	0.23	0.24	0.24	0.24
			0.86	0.52	0.53	0.54	0.54	0.53	0.53	0.53	0.54	0.54	0.54
	0.33	0.2	0.93	0.27	0.28	0.28	0.28	0.27	0.28	0.27	0.28	0.28	0.28
			0.86	0.59	0.60	0.60	0.61	0.60	0.60	0.59	0.60	0.61	0.61
		0.8	0.93	0.21	0.21	0.20	0.21	0.20	0.21	0.19	0.20	0.21	0.21
			0.86	0.44	0.44	0.44	0.44	0.43	0.44	0.43	0.44	0.44	0.44
Case 3	3	0.2	0.93	0.20	0.20	0.30	0.53	0.52	0.20	0.20	0.30	0.53	0.45
			0.86	0.47	0.47	0.69	0.94	0.93	0.47	0.46	0.69	0.94	0.88
		0.8	0.93	0.23	0.23	0.43	0.72	0.71	0.23	0.23	0.43	0.72	0.59
			0.86	0.52	0.52	0.86	0.99	0.98	0.52	0.52	0.86	0.99	0.96
	0.33	0.2	0.93	0.28	0.27	0.43	0.71	0.70	0.27	0.25	0.43	0.71	0.64
			0.86	0.60	0.58	0.80	0.97	0.96	0.58	0.58	0.80	0.97	0.95
		0.8	0.93	0.22	0.21	0.39	0.63	0.62	0.21	0.21	0.39	0.63	0.52
			0.86	0.48	0.45	0.75	0.95	0.94	0.45	0.45	0.76	0.95	0.93
Model 5: Trending Volatility													
	3		0.93	0.19	0.20	0.38	0.64	0.52	0.20	0.23	0.31	0.60	0.50
			0.86	0.46	0.48	0.82	0.98	0.94	0.45	0.50	0.81	0.97	0.93
	0.33		0.93	0.25	0.25	0.46	0.74	0.62	0.25	0.27	0.36	0.74	0.59
			0.86	0.53	0.53	0.84	0.99	0.96	0.56	0.53	0.83	0.97	0.95
Model 6: Exponential Stochastic Integrated Volatility													
			0.93	0.12	0.12	0.16	0.24	0.20	0.17	0.14	0.19	0.24	0.22
			0.86	0.22	0.22	0.36	0.59	0.48	0.25	0.26	0.38	0.63	0.52

Table 4.16: Size adjusted power performance of the standard and wild bootstrap FSVR tests for i.i.d. shocks: GLS demeaning

Σ	δ	κ	α	τ_0	τ_2	τ_1	τ_{012}	τ_{12}	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*
Model 1: Homoscedasticity													
			0.93	0.33	0.33	0.63	0.92	0.82	0.32	0.31	0.65	0.91	0.86
			0.86	0.67	0.67	0.95	1.00	1.00	0.66	0.65	0.96	1.00	1.00
Model 2: Periodic Heteroscedasticity													
Case 1			0.93	0.32	0.32	0.49	0.77	0.69	0.34	0.33	0.51	0.78	0.75
			0.86	0.65	0.66	0.84	0.98	0.97	0.67	0.67	0.85	0.98	0.98
Case2			0.93	0.34	0.34	0.35	0.34	0.34	0.33	0.34	0.35	0.35	0.35
			0.86	0.67	0.68	0.68	0.68	0.68	0.67	0.68	0.68	0.69	0.69
Model 3: Single Volatility Shift													
	3	0.2	0.93	0.25	0.24	0.50	0.84	0.72	0.25	0.27	0.53	0.85	0.78
			0.86	0.47	0.47	0.81	0.98	0.94	0.48	0.49	0.83	0.98	0.96
		0.8	0.93	0.33	0.35	0.70	0.94	0.84	0.33	0.34	0.68	0.93	0.88
			0.86	0.68	0.70	0.97	1.00	1.00	0.68	0.69	0.97	1.00	1.00
	0.33	0.2	0.93	0.30	0.30	0.62	0.90	0.80	0.31	0.30	0.60	0.88	0.83
			0.86	0.66	0.66	0.97	1.00	1.00	0.67	0.66	0.96	1.00	1.00
		0.8	0.93	0.27	0.28	0.52	0.81	0.69	0.28	0.29	0.51	0.81	0.74
			0.86	0.55	0.56	0.87	0.99	0.97	0.56	0.58	0.87	0.99	0.98
Model 4: Single Periodic Volatility Shift													
Case 1	3	0.2	0.93	0.26	0.26	0.41	0.70	0.62	0.26	0.27	0.42	0.71	0.68
			0.86	0.48	0.47	0.68	0.92	0.88	0.47	0.49	0.69	0.93	0.91
		0.8	0.93	0.34	0.35	0.54	0.81	0.75	0.35	0.33	0.55	0.81	0.78
			0.86	0.67	0.69	0.88	0.99	0.98	0.68	0.67	0.89	0.99	0.98
	0.33	0.2	0.93	0.31	0.32	0.50	0.77	0.69	0.31	0.30	0.49	0.77	0.74
			0.86	0.67	0.68	0.86	0.99	0.98	0.66	0.66	0.86	0.99	0.99
		0.8	0.93	0.28	0.28	0.43	0.67	0.60	0.27	0.29	0.41	0.65	0.63
			0.86	0.55	0.55	0.75	0.95	0.92	0.54	0.56	0.73	0.94	0.93
Case 2	3	0.2	0.93	0.26	0.26	0.27	0.27	0.27	0.28	0.28	0.29	0.29	0.29
			0.86	0.48	0.47	0.48	0.48	0.48	0.49	0.49	0.51	0.51	0.51
		0.8	0.93	0.36	0.35	0.36	0.37	0.36	0.36	0.35	0.36	0.36	0.36
			0.86	0.69	0.68	0.69	0.70	0.69	0.69	0.68	0.69	0.69	0.69
	0.33	0.2	0.93	0.32	0.31	0.32	0.32	0.33	0.32	0.31	0.32	0.32	0.32
			0.86	0.68	0.67	0.68	0.69	0.69	0.68	0.67	0.68	0.68	0.68
		0.8	0.93	0.30	0.30	0.30	0.30	0.31	0.27	0.28	0.28	0.29	0.29
			0.86	0.58	0.58	0.59	0.59	0.59	0.55	0.56	0.56	0.57	0.57
Case 3	3	0.2	0.93	0.32	0.32	0.56	0.85	0.75	0.33	0.32	0.55	0.86	0.82
			0.86	0.65	0.63	0.90	0.99	0.98	0.65	0.63	0.90	0.99	0.99
		0.8	0.93	0.32	0.34	0.67	0.92	0.83	0.34	0.32	0.65	0.91	0.87
			0.86	0.66	0.68	0.96	1.00	1.00	0.67	0.67	0.96	1.00	1.00
	0.33	0.2	0.93	0.31	0.30	0.53	0.79	0.72	0.31	0.30	0.53	0.80	0.77
			0.86	0.67	0.67	0.89	0.99	0.98	0.67	0.67	0.89	0.99	0.99
		0.8	0.93	0.30	0.28	0.54	0.83	0.72	0.28	0.30	0.56	0.84	0.79
			0.86	0.60	0.59	0.89	1.00	0.98	0.58	0.61	0.91	1.00	0.99
Model 5: Trending Volatility													
	3		0.93	0.32	0.31	0.63	0.90	0.79	0.31	0.30	0.61	0.89	0.83
			0.86	0.63	0.62	0.92	1.00	0.98	0.62	0.61	0.92	1.00	0.99
	0.33		0.93	0.29	0.31	0.59	0.88	0.77	0.29	0.30	0.58	0.86	0.81
			0.86	0.62	0.65	0.95	1.00	0.99	0.63	0.64	0.94	1.00	1.00
Model 6: Exponential Stochastic Integrated Volatility													
			0.93	0.30	0.30	0.30	0.30	0.31	0.27	0.28	0.28	0.29	0.29
			0.86	0.58	0.58	0.59	0.59	0.59	0.55	0.56	0.56	0.57	0.57

Table 4.17: Size adjusted power performance of the standard and wild bootstrap FSVR tests for ARMA (4,4) shocks: GLS demeaning

$\boldsymbol{\Sigma}$	δ	κ	α	τ_0	τ_2	τ_1	τ_{012}	τ_{12}	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*
Model 1: Homoscedasticity													
			0.93	0.32	0.32	0.68	0.92	0.84	0.33	0.33	0.65	0.91	0.86
			0.86	0.65	0.65	0.96	1.00	1.00	0.66	0.66	0.96	1.00	1.00
Model 2: Periodic Heteroscedasticity													
Case 1			0.93	0.31	0.31	0.50	0.78	0.72	0.32	0.31	0.52	0.78	0.76
			0.86	0.65	0.65	0.85	0.99	0.98	0.67	0.66	0.86	0.99	0.98
Case2			0.93	0.33	0.33	0.35	0.35	0.35	0.34	0.33	0.34	0.34	0.34
			0.86	0.67	0.67	0.69	0.69	0.69	0.68	0.67	0.68	0.68	0.68
Model 3: Single Volatility Shift													
	3	0.2	0.93	0.25	0.26	0.49	0.83	0.70	0.26	0.26	0.50	0.83	0.78
			0.86	0.47	0.48	0.81	0.99	0.94	0.47	0.48	0.82	0.99	0.97
		0.8	0.93	0.32	0.35	0.68	0.93	0.84	0.35	0.35	0.68	0.93	0.88
			0.96	0.67	0.69	0.97	1.00	1.00	0.69	0.69	0.97	1.00	1.00
	0.33	0.2	0.93	0.30	0.32	0.62	0.90	0.79	0.30	0.30	0.61	0.89	0.83
			0.86	0.67	0.67	0.97	1.00	1.00	0.66	0.65	0.96	1.00	1.00
		0.8	0.93	0.26	0.27	0.52	0.82	0.71	0.29	0.26	0.51	0.81	0.75
			0.86	0.53	0.56	0.88	1.00	0.98	0.56	0.56	0.87	0.99	0.98
Model 4: Single Periodic Volatility Shift													
Case 1	3	0.2	0.93	0.26	0.25	0.41	0.68	0.61	0.24	0.25	0.41	0.68	0.66
			0.86	0.48	0.47	0.68	0.92	0.88	0.45	0.47	0.68	0.92	0.90
		0.8	0.93	0.34	0.34	0.55	0.81	0.74	0.35	0.35	0.57	0.81	0.78
			0.86	0.68	0.68	0.88	0.99	0.98	0.69	0.69	0.89	0.99	0.98
	0.33	0.2	0.93	0.30	0.30	0.49	0.76	0.70	0.31	0.30	0.51	0.76	0.73
			0.86	0.66	0.67	0.85	0.99	0.98	0.67	0.67	0.87	0.99	0.99
		0.8	0.93	0.27	0.28	0.42	0.67	0.60	0.28	0.28	0.40	0.66	0.64
			0.86	0.56	0.56	0.75	0.96	0.92	0.57	0.57	0.72	0.95	0.94
Case 2	3	0.2	0.93	0.25	0.25	0.26	0.26	0.26	0.27	0.28	0.28	0.28	0.28
			0.86	0.48	0.48	0.49	0.48	0.49	0.50	0.51	0.52	0.51	0.51
		0.8	0.93	0.36	0.36	0.38	0.37	0.37	0.36	0.37	0.38	0.38	0.38
			0.86	0.69	0.70	0.72	0.72	0.71	0.70	0.71	0.72	0.72	0.72
	0.33	0.2	0.93	0.33	0.32	0.33	0.33	0.33	0.31	0.31	0.33	0.33	0.33
			0.86	0.68	0.68	0.69	0.69	0.69	0.66	0.66	0.68	0.68	0.68
		0.8	0.93	0.29	0.28	0.29	0.29	0.29	0.28	0.29	0.29	0.29	0.29
			0.86	0.56	0.56	0.56	0.57	0.56	0.55	0.56	0.56	0.56	0.56
Case 3	3	0.2	0.93	0.32	0.30	0.55	0.85	0.75	0.34	0.31	0.53	0.86	0.82
			0.86	0.63	0.63	0.89	1.00	0.98	0.64	0.64	0.88	1.00	0.99
		0.8	0.93	0.33	0.33	0.66	0.93	0.83	0.33	0.34	0.67	0.92	0.88
			0.86	0.67	0.67	0.96	1.00	1.00	0.67	0.68	0.97	1.00	1.00
	0.33	0.2	0.93	0.32	0.29	0.52	0.78	0.70	0.32	0.31	0.52	0.80	0.76
			0.86	0.67	0.67	0.89	0.99	0.98	0.67	0.69	0.89	0.99	0.99
		0.8	0.93	0.29	0.29	0.56	0.85	0.75	0.30	0.28	0.55	0.82	0.78
			0.86	0.60	0.60	0.90	1.00	0.99	0.61	0.59	0.89	1.00	0.99
Model 5: Trending Volatility													
	3		0.93	0.31	0.29	0.62	0.89	0.79	0.31	0.29	0.62	0.89	0.84
			0.86	0.61	0.61	0.92	1.00	0.98	0.62	0.60	0.92	1.00	0.99
	0.33		0.93	0.29	0.31	0.56	0.87	0.76	0.30	0.30	0.58	0.87	0.82
			0.86	0.62	0.64	0.94	1.00	0.99	0.63	0.63	0.94	1.00	1.00
Model 6: Exponential Stochastic Integrated Volatility													
			0.93	0.27	0.28	0.28	0.29	0.29	0.30	0.30	0.30	0.30	0.31
			0.86	0.55	0.56	0.56	0.57	0.57	0.58	0.58	0.59	0.59	0.59

Table 4.18: Size adjusted power performance of the standard and wild bootstrap FSVR tests for MA (4) shocks: GLS demeaning

Σ	δ	κ	α	τ_0	τ_2	τ_1	τ_{012}	τ_{12}	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*
Model 1: Homoscedasticity													
			0.93	0.31	0.31	0.62	0.92	0.82	0.35	0.31	0.64	0.91	0.81
			0.86	0.66	0.64	0.96	1.00	1.00	0.65	0.65	0.96	0.98	1.00
Model 2: Periodic Heteroscedasticity													
Case 1			0.93	0.32	0.32	0.52	0.79	0.72	0.34	0.32	0.54	0.80	0.72
			0.86	0.67	0.66	0.86	0.99	0.98	0.66	0.64	0.85	0.99	0.97
Case2			0.93	0.32	0.33	0.34	0.34	0.34	0.35	0.37	0.35	0.35	0.35
			0.86	0.67	0.68	0.69	0.69	0.69	0.63	0.69	0.70	0.70	0.71
Model 3: Single Volatility Shift													
	3	0.2	0.93	0.28	0.28	0.60	0.91	0.80	0.28	0.28	0.61	0.91	0.84
			0.86	0.56	0.53	0.91	1.00	0.99	0.57	0.57	0.92	0.99	0.99
		0.8	0.93	0.35	0.37	0.69	0.93	0.88	0.35	0.37	0.71	0.93	0.90
			0.86	0.70	0.72	0.98	1.00	1.00	0.70	0.72	0.98	0.97	1.00
	0.33	0.2	0.93	0.30	0.31	0.57	0.87	0.77	0.36	0.37	0.57	0.88	0.79
			0.86	0.62	0.66	0.95	1.00	1.00	0.70	0.71	0.96	1.00	1.00
		0.8	0.93	0.28	0.27	0.50	0.82	0.68	0.31	0.31	0.52	0.82	0.68
			0.86	0.55	0.54	0.86	0.99	0.96	0.61	0.59	0.88	0.99	0.98
Model 4: Single Periodic Volatility Shift													
Case 1	3	0.2	0.93	0.27	0.28	0.47	0.76	0.68	0.28	0.29	0.48	0.78	0.69
			0.86	0.53	0.54	0.77	0.97	0.94	0.54	0.55	0.78	0.97	0.95
		0.8	0.93	0.34	0.36	0.53	0.81	0.75	0.36	0.35	0.54	0.83	0.76
			0.86	0.69	0.71	0.88	0.99	0.98	0.72	0.71	0.89	1.00	0.99
	0.33	0.2	0.93	0.30	0.30	0.46	0.73	0.66	0.30	0.32	0.47	0.75	0.67
			0.86	0.65	0.64	0.84	0.99	0.97	0.64	0.67	0.85	1.00	0.98
		0.8	0.93	0.27	0.27	0.40	0.66	0.58	0.27	0.29	0.41	0.68	0.59
			0.86	0.54	0.54	0.72	0.95	0.90	0.54	0.56	0.73	0.97	0.91
Case 2	3	0.2	0.93	0.30	0.30	0.31	0.31	0.31	0.30	0.32	0.32	0.33	0.32
			0.86	0.56	0.57	0.57	0.58	0.57	0.57	0.58	0.58	0.60	0.58
		0.8	0.93	0.38	0.38	0.38	0.38	0.38	0.38	0.40	0.39	0.40	0.39
			0.86	0.73	0.72	0.73	0.73	0.73	0.72	0.75	0.74	0.75	0.74
	0.33	0.2	0.93	0.30	0.30	0.30	0.31	0.31	0.30	0.32	0.31	0.33	0.32
			0.86	0.65	0.65	0.65	0.66	0.66	0.65	0.67	0.66	0.68	0.67
		0.8	0.93	0.29	0.28	0.30	0.30	0.29	0.28	0.30	0.31	0.32	0.30
			0.86	0.54	0.54	0.56	0.56	0.55	0.54	0.56	0.57	0.58	0.56
Case 3	3	0.2	0.93	0.33	0.34	0.58	0.87	0.79	0.34	0.36	0.59	0.87	0.80
			0.86	0.66	0.65	0.92	1.00	0.99	0.65	0.67	0.93	1.00	1.00
		0.8	0.93	0.33	0.34	0.66	0.93	0.82	0.27	0.29	0.59	0.89	0.83
			0.86	0.67	0.68	0.96	1.00	1.00	0.59	0.61	0.95	1.00	1.01
	0.33	0.2	0.93	0.30	0.30	0.51	0.78	0.70	0.23	0.25	0.43	0.70	0.71
			0.86	0.64	0.64	0.87	0.99	0.98	0.56	0.58	0.83	1.00	0.99
		0.8	0.93	0.27	0.29	0.54	0.84	0.74	0.20	0.22	0.46	0.77	0.75
			0.86	0.57	0.59	0.89	1.00	0.98	0.47	0.49	0.85	1.01	0.99
Model 5: Trending Volatility													
	3		0.93	0.34	0.32	0.66	0.92	0.82	0.32	0.33	0.67	0.91	0.84
			0.86	0.68	0.65	0.96	1.00	1.00	0.66	0.70	0.95	1.00	1.00
	0.33		0.93	0.30	0.30	0.56	0.85	0.74	0.31	0.33	0.59	0.86	0.76
			0.86	0.63	0.63	0.93	1.00	0.99	0.63	0.64	0.93	1.00	0.99
Model 6: Exponential Stochastic Integrated Volatility													
			0.93	0.30	0.30	0.30	0.30	0.31	0.31	0.32	0.45	0.63	0.62
			0.86	0.58	0.58	0.59	0.59	0.59	0.64	0.65	0.78	0.93	0.91

Table 4.19: Size adjusted power performance of the wild bootstrap FSVR and HEGY tests for i.i.d. shocks: OLS demeaning

Σ	δ	κ	α	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*	t_0^*	t_2^*	t_1^*	t_{012}^*	t_{12}^*
Model 1: Homoscedasticity													
			0.93	0.24	0.24	0.45	0.74	0.69	0.19	0.17	0.29	0.50	0.49
			0.86	0.56	0.55	0.88	1.00	0.98	0.54	0.51	0.80	0.97	0.95
Model 2: Periodic Heteroscedasticity													
Case 1			0.93	0.22	0.23	0.40	0.62	0.60	0.18	0.17	0.25	0.38	0.39
			0.86	0.53	0.54	0.77	0.96	0.95	0.52	0.50	0.66	0.88	0.86
Case2			0.93	0.26	0.26	0.26	0.26	0.26	0.16	0.16	0.16	0.16	0.16
			0.86	0.58	0.58	0.58	0.59	0.59	0.44	0.44	0.45	0.44	0.44
Model 3: Single Volatility Shift													
	3	0.2	0.93	0.13	0.13	0.20	0.30	0.31	0.01	0.01	0.01	0.00	0.01
			0.86	0.36	0.35	0.57	0.83	0.78	0.05	0.05	0.06	0.10	0.12
		0.8	0.93	0.24	0.25	0.47	0.75	0.69	0.14	0.14	0.25	0.41	0.40
			0.86	0.56	0.58	0.90	1.00	0.99	0.46	0.45	0.75	0.94	0.92
	0.33	0.2	0.93	0.30	0.27	0.57	0.86	0.81	0.33	0.33	0.50	0.78	0.74
			0.86	0.63	0.62	0.94	1.00	1.00	0.71	0.72	0.91	0.99	0.99
		0.8	0.93	0.23	0.23	0.36	0.61	0.56	0.20	0.21	0.26	0.41	0.42
			0.86	0.49	0.48	0.73	0.95	0.91	0.47	0.46	0.60	0.82	0.80
Model 4: Single Periodic Volatility Shift													
Case 1	3	0.2	0.93	0.14	0.14	0.18	0.26	0.27	0.01	0.01	0.01	0.01	0.01
			0.86	0.36	0.36	0.49	0.73	0.70	0.05	0.05	0.05	0.07	0.08
		0.8	0.93	0.23	0.24	0.39	0.63	0.60	0.14	0.14	0.20	0.30	0.31
			0.86	0.56	0.56	0.78	0.96	0.95	0.43	0.44	0.60	0.82	0.81
	0.33	0.2	0.93	0.28	0.28	0.46	0.73	0.71	0.33	0.34	0.45	0.66	0.64
			0.86	0.61	0.62	0.83	0.98	0.97	0.71	0.72	0.83	0.96	0.95
		0.8	0.93	0.23	0.23	0.32	0.51	0.50	0.20	0.20	0.27	0.36	0.36
			0.86	0.49	0.50	0.63	0.88	0.85	0.44	0.44	0.57	0.73	0.72
Case 2	3	0.2	0.93	0.14	0.14	0.14	0.14	0.14	0.01	0.01	0.01	0.01	0.01
			0.86	0.35	0.36	0.36	0.36	0.36	0.04	0.04	0.04	0.04	0.04
		0.8	0.93	0.26	0.27	0.27	0.27	0.27	0.11	0.11	0.11	0.11	0.11
			0.86	0.59	0.59	0.60	0.60	0.60	0.34	0.34	0.35	0.35	0.35
	0.33	0.2	0.93	0.29	0.28	0.29	0.29	0.29	0.29	0.29	0.29	0.29	0.29
			0.86	0.62	0.62	0.63	0.63	0.63	0.63	0.63	0.64	0.63	0.63
		0.8	0.93	0.23	0.23	0.24	0.24	0.24	0.19	0.19	0.19	0.19	0.19
			0.86	0.48	0.47	0.50	0.49	0.49	0.39	0.40	0.40	0.40	0.40
Case 3	3	0.2	0.93	0.22	0.23	0.32	0.55	0.53	0.11	0.09	0.13	0.22	0.23
			0.86	0.52	0.53	0.74	0.96	0.94	0.39	0.34	0.53	0.78	0.77
		0.8	0.93	0.24	0.24	0.48	0.77	0.71	0.17	0.16	0.25	0.45	0.45
			0.86	0.55	0.57	0.89	1.00	0.99	0.51	0.51	0.76	0.95	0.94
	0.33	0.2	0.93	0.27	0.27	0.46	0.72	0.69	0.33	0.32	0.44	0.65	0.63
			0.86	0.61	0.61	0.83	0.98	0.98	0.72	0.70	0.83	0.96	0.95
		0.8	0.93	0.23	0.23	0.40	0.66	0.62	0.22	0.20	0.31	0.45	0.45
			0.86	0.51	0.51	0.80	0.97	0.95	0.50	0.48	0.67	0.85	0.83
Model 5: Trending Volatility													
	3		0.93	0.21	0.21	0.40	0.68	0.62	0.12	0.11	0.19	0.31	0.32
			0.86	0.51	0.53	0.87	0.99	0.98	0.41	0.39	0.70	0.92	0.90
	0.33		0.93	0.26	0.23	0.50	0.76	0.71	0.29	0.27	0.43	0.66	0.64
			0.86	0.57	0.54	0.89	1.00	0.98	0.65	0.64	0.85	0.97	0.96
Model 6: Exponential Stochastic Integrated Volatility													
			0.93	0.12	0.12	0.20	0.36	0.35	0.04	0.04	0.08	0.09	0.11
			0.86	0.30	0.30	0.62	0.93	0.88	0.11	0.12	0.32	0.46	0.45

Table 4.20: Size adjusted power performance of the wild bootstrap FSVR and HEGY tests for ARMA (4,4) shocks: OLS demeaning

$\boldsymbol{\Sigma}$	δ	κ	α	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*	t_0^*	t_2^*	t_1^*	t_{012}^*	t_{12}^*
Model 1: Homoscedasticity													
			0.93	0.25	0.24	0.46	0.75	0.70	0.18	0.18	0.31	0.49	0.49
			0.86	0.57	0.55	0.89	1.00	0.99	0.54	0.52	0.81	0.96	0.95
Model 2: Periodic Heteroscedasticity													
Case 1			0.93	0.26	0.25	0.38	0.62	0.59	0.18	0.18	0.25	0.38	0.38
			0.86	0.58	0.57	0.77	0.96	0.95	0.52	0.52	0.67	0.88	0.87
Case2			0.93	0.25	0.25	0.26	0.26	0.26	0.14	0.14	0.14	0.15	0.15
			0.86	0.58	0.57	0.58	0.58	0.58	0.42	0.43	0.42	0.43	0.43
Model 3: Single Volatility Shift													
	3	0.2	0.93	0.12	0.13	0.22	0.30	0.30	0.01	0.01	0.01	0.01	0.01
			0.86	0.33	0.35	0.59	0.84	0.78	0.05	0.05	0.06	0.10	0.12
		0.8	0.93	0.24	0.24	0.49	0.76	0.71	0.14	0.15	0.25	0.42	0.43
			0.86	0.57	0.56	0.91	1.00	0.99	0.43	0.46	0.75	0.95	0.93
	0.33	0.2	0.93	0.28	0.29	0.55	0.86	0.80	0.33	0.33	0.52	0.78	0.75
			0.86	0.62	0.63	0.93	1.00	1.00	0.73	0.73	0.92	0.99	0.99
		0.8	0.93	0.21	0.22	0.39	0.62	0.57	0.20	0.20	0.27	0.42	0.43
			0.86	0.47	0.48	0.76	0.96	0.92	0.45	0.46	0.62	0.83	0.81
Model 4: Single Periodic Volatility Shift													
Case 1	3	0.2	0.93	0.13	0.14	0.17	0.27	0.28	0.01	0.01	0.01	0.01	0.01
			0.86	0.35	0.37	0.50	0.73	0.71	0.04	0.05	0.05	0.08	0.09
		0.8	0.93	0.25	0.24	0.41	0.64	0.62	0.13	0.14	0.20	0.29	0.30
			0.86	0.57	0.56	0.80	0.97	0.96	0.40	0.43	0.61	0.81	0.80
	0.33	0.2	0.93	0.26	0.28	0.46	0.71	0.68	0.31	0.31	0.47	0.66	0.65
			0.86	0.60	0.62	0.83	0.98	0.97	0.68	0.69	0.85	0.96	0.96
		0.8	0.93	0.21	0.21	0.33	0.50	0.50	0.19	0.20	0.26	0.36	0.38
			0.86	0.45	0.47	0.64	0.87	0.84	0.43	0.45	0.54	0.73	0.72
Case 2	3	0.2	0.93	0.15	0.15	0.15	0.15	0.15	0.01	0.01	0.01	0.01	0.01
			0.86	0.39	0.40	0.40	0.40	0.40	0.04	0.04	0.04	0.04	0.04
		0.8	0.93	0.26	0.26	0.28	0.27	0.27	0.11	0.11	0.11	0.11	0.11
			0.86	0.59	0.59	0.62	0.61	0.61	0.34	0.34	0.35	0.35	0.35
	0.33	0.2	0.93	0.28	0.28	0.28	0.28	0.28	0.31	0.31	0.31	0.31	0.31
			0.86	0.63	0.62	0.62	0.63	0.63	0.65	0.65	0.66	0.66	0.66
		0.8	0.93	0.23	0.23	0.24	0.24	0.24	0.18	0.18	0.18	0.17	0.18
			0.86	0.49	0.49	0.50	0.49	0.49	0.40	0.39	0.39	0.39	0.39
Case 3	3	0.2	0.93	0.22	0.21	0.32	0.54	0.52	0.11	0.09	0.13	0.21	0.21
			0.86	0.53	0.52	0.74	0.96	0.93	0.38	0.35	0.53	0.78	0.76
		0.8	0.93	0.24	0.24	0.49	0.77	0.71	0.17	0.18	0.29	0.47	0.46
			0.86	0.56	0.57	0.90	1.00	0.99	0.51	0.53	0.80	0.96	0.95
	0.33	0.2	0.93	0.28	0.29	0.45	0.71	0.68	0.30	0.30	0.44	0.66	0.65
			0.86	0.63	0.64	0.83	0.98	0.97	0.70	0.69	0.83	0.96	0.96
		0.8	0.93	0.25	0.23	0.41	0.65	0.61	0.19	0.20	0.30	0.43	0.43
			0.86	0.53	0.51	0.78	0.97	0.94	0.46	0.48	0.67	0.84	0.82
Model 5: Trending Volatility													
	3		0.93	0.22	0.22	0.42	0.71	0.66	0.12	0.11	0.21	0.35	0.35
			0.86	0.53	0.53	0.87	1.00	0.99	0.40	0.39	0.72	0.93	0.90
	0.33		0.93	0.24	0.26	0.48	0.77	0.72	0.28	0.26	0.43	0.66	0.63
			0.86	0.55	0.57	0.88	1.00	0.98	0.65	0.63	0.86	0.98	0.96
Model 6: Exponential Stochastic Integrated Volatility													
			0.93	0.06	0.05	0.14	0.30	0.29	0.01	0.01	0.01	0.01	0.01
			0.86	0.15	0.15	0.52	0.87	0.82	0.04	0.03	0.06	0.06	0.07

Table 4.21: Size adjusted power performance of the wild bootstrap FSVR and HEGY tests for MA (4) shocks: OLS demeaning

$\boldsymbol{\Sigma}$	$\boldsymbol{\delta}$	$\boldsymbol{\kappa}$	$\boldsymbol{\alpha}$	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*	t_0^*	t_2^*	t_1^*	t_{012}^*	t_{12}^*
Model 1: Homoscedasticity													
			0.93	0.23	0.23	0.45	0.71	0.61	0.11	0.12	0.18	0.28	0.30
			0.86	0.50	0.51	0.86	0.99	0.96	0.30	0.34	0.56	0.82	0.79
Model 2: Periodic Heteroscedasticity													
Case 1			0.93	0.18	0.22	0.39	0.59	0.53	0.11	0.11	0.17	0.24	0.25
			0.86	0.53	0.51	0.74	0.94	0.93	0.31	0.31	0.48	0.70	0.68
Case2			0.93	0.25	0.21	0.25	0.24	0.27	0.11	0.11	0.11	0.11	0.11
			0.86	0.53	0.54	0.53	0.54	0.57	0.27	0.26	0.26	0.27	0.27
Model 3: Single Volatility Shift													
	3	0.2	0.93	0.11	0.11	0.21	0.26	0.23	0.00	0.01	0.00	0.00	0.00
			0.86	0.29	0.30	0.56	0.79	0.67	0.02	0.02	0.02	0.02	0.03
		0.8	0.93	0.22	0.22	0.48	0.74	0.62	0.08	0.10	0.15	0.24	0.27
			0.86	0.51	0.54	0.89	0.99	0.99	0.25	0.27	0.49	0.78	0.74
	0.33	0.2	0.93	0.23	0.24	0.55	0.82	0.74	0.24	0.27	0.39	0.62	0.61
			0.86	0.58	0.58	0.94	1.00	0.99	0.55	0.56	0.79	0.97	0.95
		0.8	0.93	0.21	0.21	0.38	0.62	0.48	0.18	0.18	0.22	0.35	0.36
			0.86	0.47	0.46	0.72	0.94	0.88	0.35	0.34	0.46	0.72	0.69
Model 4: Single Periodic Volatility Shift													
Case 1	3	0.2	0.93	0.11	0.12	0.17	0.23	0.21	0.01	0.01	0.00	0.00	0.00
			0.86	0.31	0.32	0.47	0.66	0.61	0.02	0.02	0.02	0.02	0.02
		0.8	0.93	0.22	0.23	0.41	0.63	0.56	0.09	0.09	0.12	0.19	0.21
			0.86	0.51	0.52	0.77	0.95	0.92	0.25	0.25	0.38	0.61	0.59
	0.33	0.2	0.93	0.27	0.28	0.46	0.72	0.65	0.25	0.26	0.39	0.54	0.54
			0.86	0.58	0.59	0.81	0.98	0.95	0.55	0.55	0.73	0.90	0.89
		0.8	0.93	0.20	0.21	0.31	0.47	0.42	0.19	0.18	0.22	0.30	0.31
			0.86	0.43	0.44	0.60	0.84	0.78	0.34	0.34	0.43	0.61	0.60
Case 2	3	0.2	0.93	0.12	0.13	0.14	0.14	0.14	0.01	0.01	0.01	0.01	0.01
			0.86	0.32	0.33	0.36	0.35	0.35	0.01	0.01	0.01	0.01	0.01
		0.8	0.93	0.24	-0.01	0.24	0.24	0.24	0.08	0.08	0.09	0.08	0.08
			0.86	0.53	-0.01	0.54	0.54	0.54	0.20	0.20	0.21	0.21	0.21
	0.33	0.2	0.93	0.28	2.99	0.28	0.28	0.28	0.28	0.27	0.28	0.28	0.28
			0.86	0.60	-0.01	0.60	0.61	0.61	0.55	0.55	0.56	0.56	0.56
		0.8	0.93	0.21	-0.01	0.20	0.21	0.21	0.17	0.17	0.17	0.17	0.17
			0.86	0.44	-0.01	0.44	0.44	0.44	0.31	0.30	0.31	0.31	0.31
Case 3	3	0.2	0.93	0.20	0.20	0.30	0.53	0.45	0.06	0.06	0.07	0.10	0.12
			0.86	0.47	0.46	0.69	0.94	0.88	0.18	0.20	0.29	0.48	0.47
		0.8	0.93	0.23	0.23	0.43	0.72	0.59	0.10	0.11	0.17	0.28	0.30
			0.86	0.52	0.52	0.86	0.99	0.96	0.27	0.30	0.54	0.82	0.79
	0.33	0.2	0.93	0.27	0.25	0.43	0.71	0.64	0.24	0.24	0.33	0.49	0.49
			0.86	0.58	0.58	0.80	0.97	0.95	0.53	0.52	0.68	0.88	0.87
		0.8	0.93	0.21	0.21	0.39	0.63	0.52	0.16	0.17	0.24	0.34	0.35
			0.86	0.45	0.45	0.76	0.95	0.93	0.36	0.36	0.53	0.72	0.71
Model 5: Trending Volatility													
	3		0.93	0.14	0.14	0.31	0.52	0.50	0.07	0.07	0.12	0.17	0.20
			0.86	0.38	0.38	0.77	0.97	0.93	0.22	0.22	0.46	0.73	0.70
	0.33		0.93	0.19	0.17	0.36	0.63	0.59	0.22	0.21	0.32	0.51	0.51
			0.86	0.45	0.42	0.76	0.97	0.95	0.47	0.45	0.69	0.91	0.88
Model 6: Exponential Stochastic Integrated Volatility													
			0.93	0.08	0.07	0.12	0.16	0.17	0.13	0.13	0.22	0.35	0.34
			0.86	0.15	0.13	0.28	0.46	0.43	0.24	0.24	0.47	0.74	0.70

Table 4.22: Size adjusted power performance of the wild bootstrap FSVR and HEGY tests for i.i.d. shocks: GLS demeaning

Σ	δ	κ	α	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*	t_0^*	t_2^*	t_1^*	t_{012}^*	t_{12}^*
Model 1: Homoscedasticity													
			0.93	0.32	0.31	0.65	0.91	0.86	0.41	0.41	0.59	0.89	0.85
			0.86	0.66	0.65	0.96	1.00	1.00	0.83	0.82	0.96	1.00	1.00
Model 2: Periodic Heteroscedasticity													
Case 1			0.93	0.34	0.33	0.51	0.78	0.75	0.40	0.40	0.55	0.76	0.75
			0.86	0.67	0.67	0.85	0.98	0.98	0.80	0.81	0.91	0.99	0.98
Case2			0.93	0.33	0.34	0.35	0.35	0.35	0.36	0.35	0.36	0.36	0.36
			0.86	0.67	0.68	0.68	0.69	0.69	0.74	0.73	0.74	0.73	0.73
Model 3: Single Volatility Shift													
	3	0.2	0.93	0.25	0.27	0.53	0.85	0.78	0.27	0.28	0.39	0.62	0.61
			0.86	0.48	0.49	0.83	0.98	0.96	0.58	0.58	0.78	0.96	0.94
		0.8	0.93	0.33	0.34	0.68	0.93	0.88	0.39	0.41	0.64	0.90	0.87
			0.86	0.68	0.69	0.97	1.00	1.00	0.81	0.82	0.97	1.00	1.00
	0.33	0.2	0.93	0.31	0.30	0.60	0.88	0.83	0.39	0.39	0.60	0.86	0.83
			0.93	0.67	0.66	0.96	1.00	1.00	0.80	0.80	0.95	1.00	1.00
		0.8	0.86	0.28	0.29	0.51	0.81	0.74	0.30	0.28	0.33	0.53	0.52
			0.93	0.56	0.58	0.87	0.99	0.98	0.58	0.57	0.70	0.90	0.86
Model 4: Single Periodic Volatility Shift													
Case 1	3	0.2	0.93	0.26	0.27	0.42	0.71	0.68	0.27	0.27	0.36	0.54	0.53
			0.86	0.47	0.49	0.69	0.93	0.91	0.56	0.55	0.69	0.90	0.88
		0.8	0.93	0.35	0.33	0.55	0.81	0.78	0.39	0.38	0.56	0.80	0.78
			0.86	0.68	0.67	0.89	0.99	0.98	0.78	0.78	0.92	0.99	0.99
	0.33	0.2	0.93	0.31	0.30	0.49	0.77	0.74	0.38	0.39	0.55	0.78	0.76
			0.86	0.66	0.66	0.86	0.99	0.99	0.78	0.77	0.90	0.98	0.98
		0.8	0.93	0.27	0.29	0.41	0.65	0.63	0.30	0.28	0.35	0.48	0.49
			0.86	0.54	0.56	0.73	0.94	0.93	0.57	0.55	0.65	0.83	0.82
Case 2	3	0.2	0.93	0.28	0.28	0.29	0.29	0.29	0.26	0.26	0.27	0.26	0.26
			0.86	0.49	0.49	0.51	0.51	0.51	0.52	0.52	0.52	0.52	0.52
		0.8	0.93	0.36	0.35	0.36	0.36	0.36	0.35	0.36	0.36	0.36	0.36
			0.86	0.69	0.68	0.69	0.69	0.69	0.72	0.74	0.73	0.73	0.73
	0.33	0.2	0.93	0.32	0.31	0.32	0.32	0.32	0.34	0.33	0.35	0.35	0.35
			0.86	0.68	0.67	0.68	0.68	0.68	0.71	0.70	0.73	0.72	0.72
		0.8	0.93	0.27	0.28	0.28	0.29	0.29	0.27	0.27	0.25	0.26	0.26
			0.86	0.55	0.56	0.56	0.57	0.57	0.52	0.52	0.50	0.50	0.50
Case 3	3	0.2	0.93	0.33	0.32	0.55	0.86	0.82	0.37	0.39	0.54	0.78	0.76
			0.86	0.65	0.63	0.90	0.99	0.99	0.79	0.80	0.92	0.99	0.99
		0.8	0.93	0.34	0.32	0.65	0.91	0.87	0.40	0.39	0.60	0.89	0.86
			0.86	0.67	0.67	0.96	1.00	1.00	0.82	0.82	0.96	1.00	1.00
	0.33	0.2	0.93	0.31	0.30	0.53	0.80	0.77	0.38	0.37	0.56	0.78	0.76
			0.86	0.67	0.67	0.89	0.99	0.99	0.78	0.77	0.91	0.99	0.98
		0.8	0.93	0.28	0.30	0.56	0.84	0.79	0.32	0.31	0.47	0.62	0.62
			0.86	0.58	0.61	0.91	1.00	0.99	0.64	0.63	0.79	0.92	0.91
Model 5: Trending Volatility													
	3		0.93	0.31	0.30	0.61	0.89	0.83	0.40	0.40	0.59	0.89	0.86
			0.86	0.62	0.61	0.92	1.00	0.99	0.81	0.82	0.96	1.00	1.00
	0.33		0.93	0.29	0.30	0.58	0.86	0.81	0.37	0.38	0.49	0.79	0.76
			0.86	0.63	0.64	0.94	1.00	1.00	0.75	0.75	0.89	0.99	0.98
Model 6: Exponential Stochastic Integrated Volatility													
			0.93	0.27	0.28	0.28	0.29	0.29	0.27	0.27	0.25	0.26	0.26
			0.86	0.55	0.56	0.56	0.57	0.57	0.52	0.52	0.50	0.50	0.50

Table 4.23: Size adjusted power performance of the wild bootstrap FSVR and HEGY tests for ARMA (4,4) shocks: GLS demeaning

Σ	δ	κ	α	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*	t_0^*	t_2^*	t_1^*	t_{012}^*	t_{12}^*
Model 1: Homoscedasticity													
			0.93	0.33	0.33	0.65	0.91	0.86	0.41	0.40	0.58	0.89	0.86
			0.86	0.66	0.66	0.96	1.00	1.00	0.83	0.82	0.96	1.00	1.00
Model 2: Periodic Heteroscedasticity													
Case 1			0.93	0.32	0.31	0.52	0.78	0.76	0.40	0.40	0.55	0.77	0.76
			0.86	0.67	0.66	0.86	0.99	0.98	0.81	0.81	0.91	0.99	0.98
Case2			0.93	0.34	0.33	0.34	0.34	0.34	0.36	0.36	0.37	0.37	0.37
			0.86	0.68	0.67	0.68	0.68	0.68	0.75	0.75	0.76	0.76	0.76
Model 3: Single Volatility Shift													
	3	0.2	0.93	0.26	0.26	0.50	0.83	0.78	0.28	0.28	0.40	0.62	0.60
			0.86	0.47	0.48	0.82	0.99	0.97	0.58	0.58	0.78	0.96	0.94
		0.8	0.93	0.35	0.35	0.68	0.93	0.88	0.40	0.41	0.64	0.92	0.88
			0.86	0.69	0.69	0.97	1.00	1.00	0.82	0.82	0.97	1.00	1.00
	0.33	0.2	0.93	0.30	0.30	0.61	0.89	0.83	0.39	0.39	0.57	0.86	0.83
			0.86	0.66	0.65	0.96	1.00	1.00	0.80	0.79	0.94	1.00	0.99
		0.8	0.93	0.29	0.26	0.51	0.81	0.75	0.30	0.29	0.33	0.53	0.52
			0.86	0.56	0.56	0.87	0.99	0.98	0.57	0.56	0.67	0.89	0.86
Model 4: Single Periodic Volatility Shift													
Case 1	3	0.2	0.93	0.24	0.25	0.41	0.68	0.66	0.26	0.27	0.37	0.53	0.53
			0.86	0.45	0.47	0.68	0.92	0.90	0.54	0.55	0.70	0.89	0.88
		0.8	0.93	0.35	0.35	0.57	0.81	0.78	0.38	0.37	0.56	0.79	0.78
			0.86	0.69	0.69	0.89	0.99	0.98	0.79	0.78	0.91	0.99	0.99
	0.33	0.2	0.93	0.31	0.30	0.51	0.76	0.73	0.38	0.37	0.54	0.76	0.75
			0.86	0.67	0.67	0.87	0.99	0.99	0.77	0.77	0.90	0.98	0.98
		0.8	0.93	0.28	0.28	0.40	0.66	0.64	0.28	0.27	0.36	0.49	0.49
			0.86	0.57	0.57	0.72	0.95	0.94	0.56	0.55	0.67	0.83	0.81
Case 2	3	0.2	0.93	0.27	0.28	0.28	0.28	0.28	0.27	0.26	0.27	0.27	0.27
			0.86	0.50	0.51	0.52	0.51	0.51	0.54	0.55	0.55	0.55	0.55
		0.8	0.93	0.36	0.37	0.38	0.38	0.38	0.35	0.35	0.36	0.36	0.36
			0.86	0.70	0.71	0.72	0.72	0.72	0.73	0.72	0.74	0.73	0.73
	0.33	0.2	0.93	0.31	0.31	0.33	0.33	0.33	0.36	0.36	0.36	0.36	0.36
			0.86	0.66	0.66	0.68	0.68	0.68	0.73	0.72	0.72	0.73	0.73
		0.8	0.93	0.28	0.29	0.29	0.29	0.29	0.28	0.28	0.27	0.27	0.27
			0.86	0.55	0.56	0.56	0.56	0.56	0.52	0.52	0.51	0.51	0.51
Case 3	3	0.2	0.93	0.34	0.31	0.53	0.86	0.82	0.38	0.39	0.54	0.77	0.75
			0.86	0.64	0.64	0.88	1.00	0.99	0.79	0.81	0.92	0.99	0.99
		0.8	0.93	0.33	0.34	0.67	0.92	0.88	0.41	0.41	0.61	0.90	0.87
			0.86	0.67	0.68	0.97	1.00	1.00	0.84	0.83	0.97	1.00	1.00
	0.33	0.2	0.93	0.32	0.31	0.52	0.80	0.76	0.40	0.37	0.53	0.76	0.75
			0.86	0.67	0.69	0.89	0.99	0.99	0.79	0.79	0.90	0.98	0.98
		0.8	0.93	0.30	0.28	0.55	0.82	0.78	0.31	0.32	0.45	0.63	0.62
			0.86	0.61	0.59	0.89	1.00	0.99	0.64	0.65	0.78	0.93	0.91
Model 5: Trending Volatility													
	3		0.93	0.31	0.29	0.62	0.89	0.84	0.39	0.38	0.57	0.88	0.84
			0.86	0.62	0.60	0.92	1.00	0.99	0.80	0.79	0.96	1.00	1.00
	0.33		0.93	0.30	0.30	0.58	0.87	0.82	0.35	0.37	0.48	0.77	0.74
			0.86	0.63	0.63	0.94	1.00	1.00	0.73	0.74	0.89	0.99	0.98
Model 6: Exponential Stochastic Integrated Volatility													
			0.93	0.30	0.30	0.30	0.30	0.31	0.31	0.32	0.45	0.63	0.62
			0.86	0.58	0.58	0.59	0.59	0.59	0.64	0.65	0.78	0.93	0.91

Table 4.24: Size adjusted power performance of the wild bootstrap FSVR and HEGY tests for MA (4) shocks: GLS demeaning

Σ	δ	κ	α	τ_0^*	τ_2^*	τ_1^*	τ_{012}^*	τ_{12}^*	t_0^*	t_2^*	t_1^*	t_{012}^*	t_{12}^*
Model 1: Homoscedasticity													
			0.93	0.33	0.31	0.62	0.92	0.82	0.29	0.29	0.47	0.74	0.72
			0.86	0.65	0.64	0.96	1.00	1.00	0.64	0.64	0.88	0.99	0.98
Model 2: Periodic Heteroscedasticity													
Case 1			0.93	0.34	0.32	0.54	0.80	0.72	0.32	0.31	0.46	0.66	0.65
			0.86	0.66	0.64	0.85	0.99	0.97	0.67	0.65	0.82	0.97	0.95
Case2			0.93	0.35	0.37	0.35	0.35	0.35	0.33	0.34	0.34	0.34	0.34
			0.86	0.63	0.69	0.70	0.70	0.71	0.66	0.66	0.66	0.66	0.66
Model 3: Single Volatility Shift													
	3	0.2	0.93	0.28	0.28	0.61	0.91	0.84	0.24	0.25	0.35	0.57	0.57
			0.86	0.57	0.57	0.92	0.99	0.99	0.51	0.51	0.70	0.94	0.91
		0.8	0.93	0.35	0.37	0.71	0.93	0.90	0.29	0.30	0.50	0.81	0.78
			0.86	0.70	0.72	0.98	0.97	1.00	0.65	0.64	0.89	1.00	0.99
	0.33	0.2	0.93	0.36	0.37	0.57	0.88	0.79	0.27	0.28	0.45	0.72	0.70
			0.86	0.70	0.71	0.96	1.00	1.00	0.59	0.62	0.84	0.99	0.98
		0.8	0.93	0.31	0.31	0.52	0.82	0.68	0.24	0.24	0.29	0.46	0.46
			0.86	0.61	0.59	0.88	0.99	0.98	0.45	0.45	0.57	0.81	0.78
Model 4: Single Periodic Volatility Shift													
Case 1	3	0.2	0.93	0.28	0.29	0.48	0.78	0.69	0.26	0.27	0.34	0.51	0.51
			0.86	0.54	0.55	0.78	0.97	0.95	0.52	0.52	0.65	0.87	0.85
		0.8	0.93	0.36	0.35	0.54	0.83	0.76	0.32	0.30	0.47	0.69	0.67
			0.86	0.72	0.71	0.89	1.00	0.99	0.66	0.64	0.83	0.97	0.96
	0.33	0.2	0.93	0.30	0.32	0.47	0.75	0.67	0.28	0.29	0.45	0.65	0.64
			0.86	0.64	0.67	0.85	1.00	0.98	0.60	0.63	0.81	0.96	0.95
		0.8	0.93	0.27	0.29	0.41	0.68	0.59	0.26	0.28	0.32	0.46	0.46
			0.86	0.54	0.56	0.73	0.97	0.91	0.47	0.48	0.56	0.77	0.75
Case 2	3	0.2	0.93	0.30	0.32	0.32	0.33	0.32	0.28	0.28	0.28	0.28	0.28
			0.86	0.57	0.58	0.58	0.60	0.58	0.51	0.52	0.53	0.52	0.52
		0.8	0.93	0.38	0.40	0.39	0.40	0.39	0.35	0.36	0.37	0.37	0.37
			0.86	0.72	0.75	0.74	0.75	0.74	0.66	0.68	0.69	0.69	0.69
	0.33	0.2	0.93	0.30	0.32	0.31	0.33	0.32	0.33	0.32	0.33	0.33	0.33
			0.86	0.65	0.67	0.66	0.68	0.67	0.64	0.63	0.64	0.64	0.64
		0.8	0.93	0.28	0.30	0.31	0.32	0.30	0.28	0.28	0.27	0.27	0.27
			0.86	0.54	0.56	0.57	0.58	0.56	0.46	0.46	0.45	0.45	0.45
Case 3	3	0.2	0.93	0.34	0.36	0.59	0.87	0.80	0.29	0.29	0.45	0.65	0.64
			0.86	0.65	0.67	0.93	1.00	1.00	0.62	0.62	0.83	0.96	0.95
		0.8	0.93	0.27	0.29	0.59	0.89	0.83	0.29	0.30	0.46	0.75	0.72
			0.86	0.59	0.61	0.95	1.00	1.01	0.63	0.65	0.87	0.99	0.98
	0.33	0.2	0.93	0.23	0.25	0.43	0.70	0.71	0.33	0.31	0.43	0.67	0.66
			0.86	0.56	0.58	0.83	1.00	0.99	0.66	0.64	0.79	0.96	0.95
		0.8	0.93	0.20	0.22	0.46	0.77	0.75	0.27	0.29	0.38	0.55	0.54
			0.86	0.47	0.49	0.85	1.01	0.99	0.52	0.53	0.68	0.87	0.85
Model 5: Trending Volatility													
	3		0.93	0.32	0.33	0.67	0.91	0.84	0.27	0.30	0.46	0.74	0.71
			0.86	0.66	0.70	0.95	1.00	1.00	0.62	0.65	0.89	1.00	0.99
	0.33		0.93	0.31	0.33	0.59	0.86	0.76	0.27	0.27	0.37	0.63	0.62
			0.86	0.63	0.64	0.93	1.00	0.99	0.57	0.57	0.75	0.96	0.94
Model 6: Exponential Stochastic Integrated Volatility													
			0.93	0.31	0.32	0.45	0.63	0.62	0.33	0.31	0.43	0.67	0.66
			0.86	0.64	0.65	0.78	0.93	0.91	0.66	0.64	0.79	0.96	0.95

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APPENDICES

A Proofs for Chapter 2

Proof of Lemma 1. Lemma 1 is Proposition 2.4.6 in Harrison (1985).

Proof of Theorem 1. This proof follows the strategy outlined in Cavaliere (2005). In this regard, first Let $v_t = \sigma_t \epsilon_t$ be innovation process where ϵ_t is iid white noise process. Consider the following process:

$$\tilde{y}_t = \begin{cases} \underline{b} & \text{if } \tilde{y}_{t-1} + v_t < \underline{b} \\ \overline{b} & \text{if } \tilde{y}_{t-1} + v_t > \overline{b} \\ \tilde{y}_{t-1} + v_t & \text{otherwise} \end{cases}$$

and assume that $\tilde{y}_0 = y_0 = 0$. This implies the process can be recursively defined as follows:

$$\begin{aligned} \tilde{y}_t &= \tilde{y}_0 + \sum_{i=1}^{t} v_i + \sum_{i=1}^{t} l_i + \sum_{i=1}^{t} q_i \\ &= \tilde{y}_0 + S_t + L_t - Q_t \end{aligned}$$

The regulators l_t and q_t are defined as follows:

$$l_t = (\lambda \underline{c} T^{1/2} - (\tilde{y}_{t-1} + v_t))\mathrm{I}(\tilde{y}_{t-1} + v_t < \underline{c}\lambda T^{1/2})$$

$$q_t = (\tilde{y}_{t-1} + v_t - \overline{c}\lambda T^{1/2})\mathrm{I}(\tilde{y}_{t-1} + v_t > \overline{c}\lambda T^{1/2})$$

Note that the given sequence is indeed the Harrison (1985) construction of a bounded stochastic process, but it is not continuous. Thus, if we can obtain a continuous approximation of the given sequence on C$[0,1]$ space with a uniform metric and show convergence, convergence in the original D$[0,1]$ space will follow by Theorem 4.1 in Billingsley (2013).

Now apply the broken line process to $\tilde{y}_t$ and create its continuous approximant on C$[0,1]$ as follows:

$$\tilde{y}^*_T(s) = (\lambda^2 T)^{-1/2} \tilde{y}_{\lfloor sT \rfloor}$$

For the partial sum S_t, we can set:

$$S^*_T(s) = (\lambda^2 T)^{-1/2} \sum_{i=1}^{\lfloor Ts \rfloor} v_i + v_{\lfloor Ts \rfloor} + 1 \left(\frac{Ts - \lfloor Ts \rfloor}{\lambda T^{1/2}} \right) \tag{20}$$

With regulators l_t and q_t defined in above, the continous approximants of our regulators are

$$L^*_T(s) = \begin{cases} (\lambda^2 T)^{-1/2} \sum_{t=1}^{\lfloor sT \rfloor} l_t & \text{if } l_{\lfloor sT \rfloor + 1} = 0 \\ (\lambda^2 T)^{-1/2} \sum_{t=1}^{\lfloor sT \rfloor} l_t + & \\ (\lambda^2 T)^{-1/2} \left(\frac{sT - \lfloor sT \rfloor - \frac{l_{\lfloor sT \rfloor+1} - v_{\lfloor sT \rfloor+1}}{l_{\lfloor sT \rfloor+1}}}{1 - \frac{l_{\lfloor sT \rfloor+1} - v_{\lfloor sT \rfloor+1}}{l_{\lfloor sT \rfloor+1}}} \right). & \mathrm{I} \left(sT \geq \lfloor sT + \frac{l_{\lfloor sT \rfloor+1} - v_{\lfloor sT \rfloor+1}}{l_{\lfloor sT \rfloor+1}} \rfloor \right) \\ & \text{if } l_{\lfloor sT \rfloor + 1} > 0 \end{cases}$$

$$Q^*_T(s) = \begin{cases} (\lambda^2 T)^{-1/2} \sum_{t=1}^{\lfloor sT \rfloor} q_t & \text{if } q_{\lfloor sT \rfloor + 1} = 0 \\ (\lambda^2 T)^{-1/2} \sum_{t=1}^{\lfloor sT \rfloor} q_t + & \\ (\lambda^2 T)^{-1/2} \left(\frac{sT - \lfloor sT \rfloor - \frac{v_{\lfloor sT \rfloor+1} - q_{\lfloor sT \rfloor+1}}{q_{\lfloor sT \rfloor+1}}}{1 - \frac{v_{\lfloor sT \rfloor+1} - q_{\lfloor sT \rfloor+1}}{q_{\lfloor sT \rfloor+1}}} \right). & \mathrm{I} \left(sT \geq \lfloor sT + \frac{v_{\lfloor sT \rfloor+1} - q_{\lfloor sT \rfloor+1}}{q_{\rfloor sT \rfloor+1}} \rfloor \right) \\ & \text{if } q_{\lfloor sT \rfloor + 1} > 0 \end{cases}$$

Aforementioned construction is continuous version of Harrison (1985) construction of a bounded process. This construction satisfies the following properties:

1. $L_T^*(s)$ and $Q_T^*(s)$ are increasing continuous with $L_T^*(0) = 0$ and $Q_T^*(0) = 0$
2. $\tilde{y}_T^*(s) \in [\underline{c}, \overline{c}]$, all $s \in [0, 1]$.
3. $L_T^*(s)$ and $Q_T^*(s)$ increase only when $\tilde{y}_T^*(s) = \underline{c}$ and $\tilde{y}_T^*(s) = \overline{c}$, respectively.

By Cavaliere and Taylor (2007) and continuous mapping theorem (CMT), $S_T^*(s) \Rightarrow B_\eta(s)$, where $B_\eta(s)$ is variance transformed Brownian motion. By applying CMT to $\tilde{y}_T^*(s)$, the limiting result will be $\tilde{y}_T(s) = B_\eta(s) + L_T(s) - Q_T(s)$.

By Proposition 2.4.6 in Harrison (1985), this limiting result is the 'unique' such function that guarantees that $\tilde{y}_T^*(s) \in [\underline{c}, \overline{c}]$. In other words, the limiting distribution of $\tilde{y}_T^*(s)$ is regulated variance transformed Brownian motion.

It is left to show is that the aformentioned construction converges in $\mathrm{D}[0, 1]$. Consider

$$\tilde{y}_T^*(s) - \tilde{y}_T(s) = (L_T^*(s) - \frac{L_{\lfloor sT \rfloor}}{\lambda T^{1/2}})) - (Q_T^*(s) - \frac{Q_{\lfloor sT \rfloor}}{\lambda T^{1/2}}) + \lambda T^{-1/2} v_{\lfloor sT \rfloor + 1}(sT - \lfloor sT \rfloor)$$

Both $|L_T^*(s) - \frac{L_{\lfloor sT \rfloor}}{\lambda T^{1/2}})|$ and $|Q_T^*(T_s) - \frac{Q_{\lfloor sT \rfloor}}{\lambda T^{1/2}}|$ are smaller than $\lambda T^{-1/2} v_{\lfloor sT \rfloor + 1}(sT - \lfloor sT \rfloor)$, it follows that

$$|\tilde{y}_T^*(s) - \tilde{y}_T(s)| \leq 2\lambda T^{-1/2} v_{\lfloor sT \rfloor + 1}(sT - \lfloor T_s T \rfloor)$$

Thus,

$$\sup_{s \in [0,1]} |\tilde{y}_T^*(s) - \tilde{y}_T(s)| \leq 2\lambda T^{-1/2} |\max_{t=1,..,T} u_{T_t}|$$

The proof of Theorem 1 in Wang et al. (2002) show that under the conditions given, $|max_{t=1,..,T} u_{T_t}|$ is $O_p(1)$. Thus,

$$\sup_{s\in[0,1]} |\tilde{y}^*_T(s) - \tilde{y}_T(s)| \xrightarrow{p} 0.$$

This implies $\tilde{y}_T(s) \xrightarrow{p} B^{c,\bar{c}}_\eta(s)$.

Lemma 9. $\{y_t\}$ and $\{\tilde{y}_t\}$ satisfy the relation $\max\limits_{Tt=0,..,T} |y_t - \tilde{y}_t| \leq \max\limits_{t=0,..,T} \overline{\xi}_t + \max\limits_{t=0,..,T} \underline{\xi}_t$

Proof of Lemma 9. To see this first consider the observed process:

$$y_t = \sum_{i=1}^{t} \psi(L)u_i + \sum_{i=1}^{t} \underline{\xi}_i - \sum_{i=1}^{t} \overline{\xi}_i$$

Remark that we can apply Beveridge Nelson decomposition to first factor (see proof of Theorem 1 in Cavaliere and Taylor (2007)):

$$\sum_{i=1}^{t} \psi(L)u_i = \psi(1)\sum_{i=1}^{t} u_i + \tilde{u}_T - \tilde{u}_0$$

Here $\tilde{u}_T - \tilde{u}_0$ will be asymptotically unimportant. Thus we can remove it from the difference of y_t and $\tilde{y}_t$. Rest of the proof is same as in Lemma 7 of Cavaliere (2005), since all conditions on regulators $\underline{\xi}_t$ and $\overline{\xi}_t$ are same as in Cavaliere (2005).

Since under assumption $\mathcal{A}3$ we have $\max\limits_{t=1,...T} |\underline{\xi}_t|$ and $\max\limits_{t=1,...T} |\overline{\xi}_t|$ are of $o_p(T^{1/2})$, $\max\limits_{Tt=0,..,T} |y_t - \tilde{y}_t| \leq o_p(T^{1/2})$. In this case $sup_{s\in[0,1]}|y_T(s) - \tilde{y}_T(s)| \xrightarrow{p} 0$. As a result we have $y(s) \xrightarrow{w} B^{c,\bar{c}}_\eta(s)$. □

Proof of Lemma 2. We know that $\hat{X}_t = Y_t - (\theta - \hat{\theta})'\delta_t$ then the partial sum process can be expressed as:

$$\hat{X}_T(s) = (\lambda^2 T)^{-1/2} Y_{\lfloor Ts \rfloor} - (\lambda^2 T)^{-1/2} (\hat{\theta} - \theta)'\delta_{\lfloor Ts \rfloor}$$

First factor converges according to Theorem (1), and second factor is very similar to object in Theorem 1 of Nielsen (2009) except the limit of Y_t. While in Nielsen (2009), the partial sum of process Y_t converges to $B(s)$, in our case it will converge to $B_{\eta}^{c,\bar{c}}$. Using CMT and results of Nielsen (2009) we obtain the result. $\square$

Proof of Theorem 2. Throughout, to simplify the proof, it is assumed wlog that no deterministics are included in the model and in the estimation.

Under Assumption $\mathcal{A}$, let us define $\psi(z)^{-1} = \alpha(z) = 1 - \sum_{j=1}^{\infty} \alpha_j z^j$. By letting $\underline{\xi}_t^* = \alpha(L)\underline{\xi}_t$ and $\overline{\xi}_t^* = \alpha(L)\overline{\xi}_t$, define $u_t^* = u_t + \underline{\xi}_t^* - \overline{\xi}_t^*$, where $\varepsilon_t = \psi(L)u_t + \underline{\xi}_t - \overline{\xi}_t = \psi(L)u_t^*$, $w_t^* = \sum_{i=1}^{t} u_i^*$ and $r_t^* = \underline{\xi}_t^* - \overline{\xi}_t^*$

We will present the proof for ADF_t and ADF_α statistics. The proof for other statistics follow similiarly.

Let $Z_{t,k} = (\Delta X_{t-1}, ..., \Delta X_{t-k})'$, $\beta := (\alpha_1, ..., \alpha_k)'$ and ADF regression is

$$X_t = \alpha X_{t-1} + \beta' Z_{t,k} + u_{t,k}$$

with $u_{t,k} = u_t^* + \sum_{i=k+1}^{\infty} \alpha_i \Delta X_{t-i} = u_t^* + \sum_{i=k+1}^{\infty} \alpha_i \varepsilon_{t-i}$.

The proof for ADF_t is as follows. Under null hypothesis $\rho = 1$, by using Lemma A4 in Cavaliere and Xu (2014), we can conclude that (a) $T^{-1}\sum_{t=1}^{T} X_{t-1}u_{t,k} = C(1)T^{-1}\sum_{t=1}^{T} w_{t-1}^* u_t^* o_p(1)$, (b) $T^{-2}\sum_{t=1}^{T} X_{t-1}^2 = C(1)^2 T^{-2}\sum_{t=1}^{T}(w_{t-1}^*)^2 + o_p(1)$ and (c) $T^{-1}\sum_{t=1}^{T} u_{t,k}^2 = T^{-1}\sum_{t=1}^{T}(u_t^*)^2 + o_p(1)$.

By using Lemma A5 in Cavaliere and Xu (2014) and Lemma 3 in Cavaliere and Taylor (2007), we can also conclude that as $T \to \infty$ (a)$||(T^{-1}(\sum_{t=1}^{T} Z_{t,k}Z_{t,k}')^{-1}|| = O_p(1)$; (b)$||\sum_{t=1}^{T} Z_{t,k}X_{t-1})|| = O_p(Tk^{1/2})$ and (c) $||(T^{-1}\sum_{t=1}^{T} Z_{t,k}u_{t,k}|| = O_p(1)$.

Also, by Lemma 2 in Cavaliere and Taylor (2007) and by proof of Lemma 4, we can conclude that $T^{-1}\sum_{t=1}^{T}(u_t^*)^2 \xrightarrow{p} \bar{\omega}^2$. Then we can proceed as in Chang and Park (2002) to prove that

$$ADF_t = \overline{\omega}^{-1}(T^{-2}\sum_{t=1}^{T}(w_{t-1}^*)^2)^{-1/2}(T^{-1}\sum_{t=1}^{T}w_{t-1}^*v_t^*) + o_p(1)$$

Then Theorem 1, the continous mapping theorem [CMT] imply

$$T^{-1}\sum_{t=1}^{T}w_{t-1}^*v_t^* = \frac{1}{2T}w_T^{*2} - \frac{1}{2T}v_t^{*2} \xrightarrow{w} \frac{\bar{\omega}^2}{2}(B_\eta^{c,\overline{c}}(1)^2 - 1)$$

$$T^{-2}\sum_{t=1}^{T}w_{t-1}^{*2} \xrightarrow{w} \overline{\omega}^2\int_0^1 B_\eta^{c,\overline{c}}(s)^2 ds$$

which completes the proof for ADF_t. Following similiar manner,

$$\begin{aligned} ADF_\alpha &= \frac{T(\hat{\alpha}-1)}{\hat{\alpha}(1)} \\ &= (T^{-2}\sum_{t=1}^{T}w_{t-1}^{*2})^{-1}(T^{-1}\sum_{t=1}^{T}w_{t-1}^*v_t^*) + o_p(1) \end{aligned}$$

Under Lemma A5 of Cavaliere and Xu (2014) and Theorem 1 of Cavaliere and Taylor (2007), we can conclude that $\hat{\alpha}(1) \xrightarrow{p} 1/C(1)$. This completes the proof of Part (i), consistency of s_{AR}^2, and ADF_α. □

Proof of Lemma 3. $\hat{\eta}(s)$ and $\eta(s)$ are both continuous, bounded between $[0, 1]$ and monotone for all $s \in [0, 1]$, we only need to show point wise convergence as in Cavaliere and Taylor (2007). Consider

$$\hat{\eta}(s) = \frac{\sum_{t=1}^{\lfloor sT \rfloor}\epsilon_t^2 + (sT - \lfloor sT \rfloor)\epsilon_t^2}{\sum_{t=1}^{T}\epsilon_t^2}$$

since $\frac{1}{T}\sum_{t=1}^{\lfloor sT \rfloor}(\Delta\hat{x}_t)^2 = \sum_{t=1}^{\lfloor sT \rfloor}\epsilon_t^2 + o_p(1)$. Then we can write,

$$\sup_{s\in[0,1]}\left|\hat{\eta}(s) - \frac{\sum_{t=1}^{\lfloor sT \rfloor}\epsilon_t^2}{\sum_{t=1}^{T}\epsilon_t^2}\right| \leq \left(\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2\right)^{-1}\max_{1\leq t\leq T}\frac{\epsilon_t^2}{T}$$

But $\max_{1\leq t\leq T}\frac{\epsilon_t^2}{T} = (\max_{1\leq t\leq T}\frac{\epsilon_t^2}{T^{-1/2}})^2$ is $o_p(1)$ from Theorem 1. For the first factor, notice that

$$\sum_{t=1}^{\lfloor sT \rfloor}\epsilon_t^2 = \sum_{t=1}^{\lfloor sT \rfloor}(\psi(L)u_t + \underline{\xi}_t - \overline{\xi}_t)^2$$

Define $r_t = \underline{\xi}_t - \overline{\xi}_t$ and $u_t^* = \psi(L)u_t$ then,

$$\frac{1}{T}\sum_{t=1}^{\lfloor sT \rfloor}\epsilon_t^2 = \frac{1}{T}\sum_{t=1}^{\lfloor sT \rfloor}(u_t^* + r_t)^2 = \frac{1}{T}\sum_{t=1}^{\lfloor sT \rfloor}(u_t^*)^2 + \frac{1}{T}\sum_{t=1}^{\lfloor sT \rfloor}(r_t)^2 + \frac{2}{T}\sum_{t=1}^{\lfloor sT \rfloor}u_t^* r_t$$

From Cavaliere and Taylor (2007) $\frac{1}{T}\sum_{t=1}^{\lfloor sT \rfloor}(u_t^*)^2 \xrightarrow{w} \kappa\eta(s)$ where $\kappa = \omega^2\psi(1)$. However, $\frac{1}{T}(\sum_{t=1}^{\lfloor sT \rfloor} r_t^2 + u_t^* r_t) = o_p(1)$ since:

$$\begin{aligned}\left|\frac{1}{T}(\sum_{t=1}^{\lfloor sT \rfloor} r_t^2 + u_t^* r_t)\right| &\leq \frac{1}{T}\sum_{t=1}^{\lfloor sT \rfloor}|r_t^2 + u_t^* r_t| \\ &\leq \frac{1}{T}\left[max|u_t^*| + max|r_t|\right]\sum_{t=1}^{T}2|r_t| = o_p(1)\end{aligned}$$

from assumption 2.4 we have $max|u_t|$, $max|r_t|$ are $o_p(T^{-1/2})$ and $\sum_{t=1}^{T}2|r_t| = o_p(T^{1/2})$.

Since $\frac{1}{T}\sum_{t=1}^{T}(u_t^*)^2 \xrightarrow{w} \kappa\eta(1) = \kappa$, the $\hat{\eta}(s) \xrightarrow{w} \frac{\kappa\eta(s)}{\kappa} = \eta(s)$. □

Proof of Lemma 4. The proof of this lemma is based on the consistency proof of $s^2_{AR}(k)$. Remember from Theorem 2 of the consistency of $s^2_{AR}(k)$ was proven by using the consistency of $\hat{\sigma}$ and $\hat{\alpha}(1)$ Therefore, we have a conclusion that $\hat{\underline{c}} \xrightarrow{p} \underline{c}$ and $\hat{\overline{c}} \xrightarrow{p} \overline{c}$. □

Proof of Theorem 3. For proof, we need to first show the difference process $\Delta B_{\hat{\eta},T}(t/T)$ is identical to $dB_{\eta}(s)$ in the limit. In order to show this, we will use a siliar structure as in Cavaliere and Taylor (2007). First, define usual partial sum process $S_T(s) = \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor Ts \rfloor} e_t$. Then $B_{\hat{\eta},T}(s) = S_T(\hat{\eta}(\lfloor sT \rfloor/T))$. Cavaliere and Taylor (2007) already showed that $S_T(\hat{\eta}(\lfloor sT \rfloor/T)) \xrightarrow{w} B(\eta(s)) = B_{\eta}(s)$. Now we can write:

$$\Delta B_{\hat{\eta},T}(t/T) = S_T(\hat{\eta}(t/T)) - S_T(\hat{\eta}(t/T - 1/T))$$

Define $h = 1/T$ then

$$\begin{aligned}
\lim_{T\to\infty} \Delta B_{\hat{\eta},T}(t/T) &= S_T(\hat{\eta}(s)) - S_T(\hat{\eta}(s-h)) \\
\lim_{T\to\infty, h\to 0} \Delta B_{\hat{\eta},T}(t/T) &= S_T(\hat{\eta}(s)) - S_T(\hat{\eta}(s-h)) = dS_T(\hat{\eta}(s)) \\
&\xrightarrow{w} dB(\eta(s)) = dB_{\eta}(s)
\end{aligned}$$

Last line follows from joint convergence of $(S_T(\cdot), \hat{\eta}(\cdot))$ to $(B(\cdot), \eta(t))$ which is shown in Cavaliere and Taylor (2007). Rest is application of CMT and conjuncture which is described in Theorem 1. Simply replacing the increments of $S_T^*(s)$ in equation 20 with $\Delta B_{\hat{\eta},T}(t)$, we obtain

$$B_{\eta}^{\underline{c},\bar{c}}(s) = B_{\eta}(s) + L(s) - Q(s)$$

where $L(s)$ and $Q(s)$ are continuous regulators. $\square$

B Proofs for Chapter 3

Throughout the proofs, we will utilize the circulant matrices. These matrices can be defined as in del Barrio Castro et al. (2012):

$$C_0 = v_0' v_0 = \mathbf{Circ}[1, 1, 1, ..., 1]$$
$$C_{S/2} = v_{S/2}' v_{S/2} = \mathbf{Circ}[1, -1, 1, ..., -1, 1]$$
$$C_i = v_i' v_i = \mathbf{Circ}[cos(0), cos(\omega_i, cos(2\omega_i)), ..., cos\left((S-1)\omega_i\right)]$$
$$C_i^* = v_i^{*'} v_i^* = \mathbf{Circ}[sin(0), sin(\omega_i, sin(2\omega_i)), ..., sin\left((S-1)\omega_i\right)]$$

with

$$v_0 = \begin{bmatrix} 1, 1, ..., 1 \end{bmatrix}$$
$$v_{S/2} = \begin{bmatrix} -1, 1, -1, ..., 1 \end{bmatrix}$$
$$v_i = \begin{bmatrix} cos(w_j[1-S]) & cos(w_j[2-S]) & ... & cos(0) \\ sin(w_j[1-S]) & sin(w_j[2-S]) & ... & sin(0) \end{bmatrix}$$
$$v_i^* = \begin{bmatrix} -sin(w_j[1-S]) & -sin(w_j[2-S]) & ... & -sin(0) \\ cos(w_j[1-S]) & cos(w_j[2-S]) & ... & cos(0) \end{bmatrix}$$

where vs are the associated row vectors and matrices that create the circulant matrices. These matrices above have the following properties:

$$C_0 \Psi(1) = \psi(1) C_0$$
$$C_{S/2} \Psi(1) = \psi(-1) C_{S/2}$$
$$C_j \Psi(1) = b_j C_j + a_j C_j^* \text{ for all } j = 1, ..., S^*$$
$$C_j^* \Psi(1) = b_j C_j^* - a_j C_j \text{ for all } j = 1, ..., S^*$$

On the other hand, the interaction of the circulant matrices with each other engender interesting results:

$$
\begin{aligned}
C_0 C_0 &= S C_0 \\
C_{S/2} C_{S/2} &= S C_{S/2} \\
C_j C_j &= (S/2) C_j \textit{ for all } j = 1, ..., S^* \\
C_j^* C_j^* &= (S/2) C_j \textit{ for all } j = 1, ..., S^* \\
C_j C_j^* &= C_j^* C_j = (S/2) C_j^* \textit{ for all } j = 1, ..., S^*
\end{aligned}
$$

with other cross products are all zero matrices.

Proof of Lemma 5. The proof of this lemma can be found in Lemma 1 of del Barrio Castro et al. (2012). □

Proof of Lemma 6. The proof follows from the same argument in proof of Lemma 6.(d) of Nielsen (2010). Since the operator Δ_+^{-d} satisfies $\Delta_+^{-d}\Delta_+^{-d_1}Y_t = \Delta_+^{-d_1}\Delta_+^{-d}Y_t = \Delta_+^{-d-d_1}Y_t$, we can write

$$\tilde{X}_r = \Delta_+^{-d}\Delta_+^{-1}E_r = \Delta_+^{-1}\Delta_+^{-d}E_r = \sum_{i=1}^{r} \Delta_+^{-d}E_r$$

Now define $V_r = \Delta_+^{-d}E_r$ which satisfies the conditions of Theorem 1 in Robinson (2003), we obtain :

$$N^{1/2-d} \sum_{i=1}^{\lfloor rN \rfloor} V_i \Rightarrow \sigma\Psi(1)W_{d+1}(r)$$

□

Proof of Theorem 4. Consider numerator of $\tau_j(d)$ for $j = 0, S/2$, this object is given in del Barrio Castro et al. (2012) as:

$$T^{-2}\sum_{t=1}^{N}\sum_{s=1-S}^{0}x_{j,St+s}^{2} = T^{-2}\sum_{t=1}^{N}SX_t'C_jX_t + o_p(1) \quad for \quad j=0,S/2$$
$$\Rightarrow \frac{1}{S}\int_0^1 W(r)'\Psi(1)'C_j\Psi(1)W(r)dr$$
$$= \begin{cases} \frac{\sigma^2\psi(1)^2}{S}\int\limits_0^1 W(r)'C_0W(r)dr \ if\ j=0 \\ \frac{\sigma^2\psi(-1)^2}{S}\int\limits_0^1 W(r)'C_{S/2}W(r)dr \ if\ j=S/2 \end{cases} \tag{21}$$

The first equality stems from the properties of circulant matrices C_0 and $C_{S/2}$. The weak convergence is a result of Lemma 5 and CMT. But also note that the term $\frac{1}{S}$ appears since the partial sum convergence is over N not $T = NS$, thus we have an extra term $\frac{1}{S^2}$. The final convergences differ for $j = 0, S/2$ because of the circulant matrices again. Similarly, for $j = 1, ..., S^*$ del Barrio Castro et al. (2012) show that:

$$T^{-2}\sum_{t=1}^{N}\sum_{s=1-S}^{0}x_{j,St+s}^{2} = T^{-2}\sum_{t=1}^{N}\frac{S}{2}X_t'C_jX_t + o_p(1) \quad \textbf{for} \quad j=1,...,S^*$$
$$\Rightarrow \frac{1}{2S}\int_0^1 W(r)'\Psi(1)'C_j\Psi(1)W(r)dr$$
$$= \frac{\sigma^2(a_j^2+b_j^2)}{2S}\int\limits_0^1 W(r)'C_jW(r)dr \ if\ j=1,...,S^* \tag{22}$$

and

$$T^{-2}\sum_{t=1}^{N}\sum_{s=1-S}^{0}x_{j,St+s}^{*2} = T^{-2}\sum_{t=1}^{N}\frac{S}{2}X_t'C_j^*X_t + o_p(1) \quad \textbf{for} \quad j=1,...,S^*$$
$$\Rightarrow \frac{1}{2S}\int_0^1 W(r)'\Psi(1)'C_j^*\Psi(1)W(r)dr$$
$$= \frac{\sigma^2(a_j^2+b_j^2)}{2S}\int\limits_0^1 W(r)'C_j^*W(r)dr \ if\ j=1,...,S^* \tag{23}$$

Remark that in del Barrio Castro et al. (2016), it is shown that the objects in (22) and (23) are the same. Now, we can focus on the denominators and apply the same procedure:

$$
\begin{aligned}
T^{-2-2d}\sum_{t=1}^{N}\sum_{s=1-S}^{0}\tilde{x}_{j,St+s}^{2} &= T^{-2-2d}\sum_{t=1}^{N} S\tilde{X}_t' C_j \tilde{X}_t + o_p(1) \quad for \quad j=0,S/2 \\
&\Rightarrow \frac{1}{S}\int_0^1 W_{d+1}(r)'\Psi(1)'C_j\Psi(1)W_{d+1}(r)dr \\
&= \begin{cases} \frac{\sigma^2\psi(1)^2}{S}\int\limits_0^1 W_{d+1}(r)'C_0W_{d+1}(r)dr \ \ if\ j=0 \\ \frac{\sigma^2\psi(-1)^2}{S}\int\limits_0^1 W_{d+1}(r)'C_{S/2}W_{d+1}(r)dr \ \ if\ j=S/2 \end{cases}
\end{aligned} \tag{24}
$$

The week convergence stem from Lemma 2 and the rest can derived from the above results. Finally, we can also demonstrate,

$$
\begin{aligned}
T^{-2-2d}\sum_{t=1}^{N}\sum_{s=1-S}^{0}\tilde{x}_{j,St+s}^{2} &= T^{-2-2d}\sum_{t=1}^{N} \frac{S}{2}\tilde{X}_t' C_j \tilde{X}_t + o_p(1) \quad \textbf{for} \quad j=1,...,S^* \\
&\Rightarrow \frac{1}{2S}\int_0^1 W_{d+1}(r)'\Psi(1)'C_j\Psi(1)W_{d+1}(r)dr \\
&= \frac{\sigma^2(a_j^2+b_j^2)}{2S}\int\limits_0^1 W_{d+1}(r)'C_jW_{d+1}(r)dr \ \ if\ j=1,...,S^*
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
T^{-2-2d}\sum_{t=1}^{N}\sum_{s=1-S}^{0}\tilde{x}_{j,St+s}^{*2} &= T^{-2-2d}\sum_{t=1}^{N} \frac{S}{2}\tilde{X}_t' C_j^* \tilde{X}_t + o_p(1) \quad \textbf{for} \quad j=1,...,S^* \\
&\Rightarrow \frac{1}{2S}\int_0^1 W_{d+1}(r)'\Psi(1)'C_j\Psi(1)W_{d+1}(r)dr \\
&= \frac{\sigma^2(a_j^2+b_j^2)}{2S}\int\limits_0^1 W_{d+1}(r)'C_j^*W_{d+1}(r)dr \ \ if\ j=1,...,S^*
\end{aligned} \tag{26}
$$

where the asymptotic objects in (25) and (26) are same from the similar arguments made for numerator counterparts. After obtaining these fundamental objects, for all tests $\tau_0(d)$, $\tau_{S/2}(d)$, $\tau_j(d)$, $\tau_j^*(d)$, $\tau_j(d)$, $\tau_{1,...,S/2}(d)$ and $\tau_{0,...,S/2}(d)$

the results follow by invoking CMT. Also note that the scales for the numerator and denominator of these are the same. Thus, the division cancels the impact of the serial correlation and other complicated dynamics appearing in the asymptotic objects.

Further, following Hylleberg et al. (1990), we can write $W(s)'C_jW(s) = W(s)'v_j'$ $v_jW(s)$ for $j = 0, S/2$ and $W(s)'C_iW(s) = W(s)'v_i'v_iW(s)$. However, $W(s)'v_j'$ is just linear combination of independent Brownian motions. For $i = 1, ..., S^*$, $W(s)'v_i$ is two dimensional Brownian motion. Instead, we can define two new linear combination of independent Brownian motions as $W(s)'c_i$ and $W(s)'c_i^*$ where c_i and c_i^* are the first rows of v_j and v_j^* respectively, for all $j = 1, ..., S^*$. Moreover, we can make the same arguments for the Fractional Brownian motions appearing in the denominators which is clear from equation (3.19) for all cases.

□

C Proofs for Chapter 4

Preliminaries

In this appendix, by following Cavaliere et al. (2017), we will define the some crucial definitions and make use of them.

First of all, let us define c_0 , $c_{S/2}$, c_k, $\tilde{c}_k$, $k = 1, ..., S^*$, which denote the (mutually) orthogonal $S \times 1$ selection vectors as:

$$\begin{aligned} c_0 &= [1, 1, 1, ..., 1]' \\ c_{S/2} &= [1, -1, 1, -1, .., 1]' \\ c_k &= [cos(\omega_k[1-S]), cos(\omega_k[2-S]), ..., cos(0)] \\ \tilde{c}_k &= [sin(\omega_k[1-S]), sin(\omega_k[2-S]), ..., sin(0)] \end{aligned}$$

We also introduce the $S \times S$ circulant matrices as follows:

$$
\begin{aligned}
C_0 &= circ[1, 1, 1, ..., 1]' \\
C_{S/2} &= circ[1, -1, 1, -1, .., 1]' \\
C_i &= circ[cos(0), cos(\omega_i), ..., cos((S-1)\omega_i)] \\
\tilde{C}_i &= circ[sin(0), sin(\omega_i), ..., sin((S-1)\omega_i)], \omega_i = 2\pi i/S, i = 1, ..., S^*
\end{aligned}
$$

As discussed in Smith et al. (2009) and Cavaliere et al. (2017), these matrices are mutually orthogonal and have the following properties: $C_0 C_0 = SC_0$, $C_{S/2}C_{S/2} = SC_{S/2}$, $C_j\tilde{C}_j = (S/2)\tilde{C}_j$, $\tilde{C}_j\tilde{C}_j = (S/2)C_j$, $j = 1, ..., S^*$. Also, the relationship between vectors and circulant matrices are: $C_0 = c_0 c_0'$, $C_{S/2} = c_{S/2}c_{S/2}'$, $C_j = c_j c_j'$, $\tilde{C}_j = c_j \tilde{c}_j'$, where

$$
c_j' = \begin{pmatrix} cos(\omega_j[1-S]), & cos(\omega_j[2-S]), & ... & , cos(0) \\ sin(\omega_j[1-S]), & sin(\omega_j[2-S]), & ... & , sin(0) \end{pmatrix}
$$

$$
\tilde{c}_j' = \begin{pmatrix} -sin(\omega_j[1-S]), & -sin(\omega_j[2-S]), & ... & , -sin(0) \\ cos(\omega_j[1-S]), & cos(\omega_j[2-S]), & ... & , cos(0) \end{pmatrix}
$$

Proof of Lemma 7. The proof of this lemma can be found in Cavaliere et al. (2017).

Now, we will present the proof of Lemma 4.2:

Proof of Lemma 8. We know from Boswijk et al. (2016) for $\varepsilon_t = \Omega_t E_t$, the following convergence result holds:

$$
Y_{\lfloor .N \rfloor} = N^{-1/2} \sum_{t=1}^{\lfloor .N \rfloor} \varepsilon_t \Rightarrow M(.)
$$

Now, first we need to prove the convergence result of the following fractional partial sum:

$$
N^{-1/2-d} \sum_{t=1}^{\lfloor .N \rfloor} \Delta_+^{-d} \varepsilon_t
$$

We know from Nielsen (2010) Lemma 6.d that Δ_+^{-d} satisfies $\Delta_+^{-d}\Delta_+^{-d_2}Y_t = \Delta_+^{-d_2}\Delta_+^{-d}Y_t = \Delta_+^{-d-d_2}Y_t$, we can write

$$\sum_{i=1}^{r} \Delta_+^{-d}\varepsilon_i = \Delta_+^{-d}\sum_{i=1}^{r}\varepsilon_i$$

Therefore,

$$\begin{aligned}
N^{-1/2-d}\sum_{t=1}^{\lfloor rN\rfloor}\Delta_+^{-d}\varepsilon_t &= N^{-1/2-d}\sum_{t=1}^{\lfloor rN\rfloor}\pi_{\lfloor rN\rfloor-t}(d)\sum_{i=1}^{t}\varepsilon_i \\
&= N^{-1/2-d}\sum_{t=1}^{\lfloor rN\rfloor}\pi_{\lfloor rN\rfloor-t}(d+1)\varepsilon_t \\
&= N^{-1/2-d}\sum_{t=1}^{\lfloor rN\rfloor}\frac{(\lfloor rN\lfloor -t)^d}{\Gamma(d+1)}\varepsilon_t \\
&= \sum_{t=1}^{\lfloor rN\rfloor}\frac{(r-t/N)^d}{\Gamma(d+1)}N^{-1/2}\varepsilon_t \\
&= \sum_{t=1}^{\lfloor rN\rfloor}\int_{(r-1)/N}^{r/N}\frac{(r-t/N)^d}{\Gamma(d+1)}dy_N(s) \\
&\Rightarrow \frac{1}{\Gamma(d+1)}\int_0^r (r-t)^d dM(t) = M_{d+1}(r)
\end{aligned}$$

Consider now, $\tilde{X}_{\lfloor rN\rfloor} = \sum_{t=1}^{\lfloor rN\rfloor}\Delta_+^{-d}U_t$. By Beveridge Nelson decomposition, we have $U_t = \Psi(1)\varepsilon_t + \varepsilon_{t-1}{}^* - \varepsilon_t{}^*$, where $\varepsilon_t{}^* = \Psi^*(L)\varepsilon_t$. Now we have

$$N^{-1/2-d}\sum_{t=1}^{\lfloor rN\rfloor}\Delta_+^{-d}U_t = \Psi(1)N^{-1/2-d}\sum_{t=1}^{\lfloor rN\rfloor}\Delta_+^{-d}\varepsilon_t + N^{-1/2-d}\Delta_+^{-d}(\varepsilon_0{}^* - \varepsilon_{\lfloor rN\rfloor}{}^*)$$

Under Assumptions $\mathcal{A}.1$ and $\mathcal{A}.2$, the second term on the above equation is $O_p(N^{(-1-2d)})$ uniformly in $r\in[0,1]$. Therefore, we have the desired result that

$$N^{-d-1/2}\tilde{X}_{\lfloor .N\rfloor} \Rightarrow \Psi(1)M_{d+1}(.)$$

□

Proof of Theorem 5. Now, we will obtain the limiting distribution of FSVR tests under non-stationary volatility. Firstly, notice that using Boswijk et al. (2016), Lemma 1 and CMT, the following results hold:

$$N^{-1}\sum_{t=2}^{N} X_{t-1}X'_{t-1} \Rightarrow \Psi(1)\int_0^1 M(r)M(r)'dr\Psi(1)' =: Q_1$$
$$N^{-1-2d}\sum_{t=2}^{N} \tilde{X}_{t-1}\tilde{X}'_{t-1} \Rightarrow \Psi(1)\int_0^1 M_{d+1}(r)M_{d+1}(r)'dr\Psi(1)' =: Q_2$$

Now, firstly consider the numerator of $\tau_j(d)$, for $i = 0, S/2$. By standard manipulations, Cavaliere et al. (2017) show that

$$T^{-2}\sum_{t=2}^{N}\sum_{s=1-S}^{0} x^2_{j,St+s} \Rightarrow \frac{1}{S}c'_j\Psi(1)\int_0^1 M(r)M(r)'dr\Psi(1)'c_j = \frac{1}{S}c'_jQ_1c_j, j = 0, S/2$$

Similiar manipulations for the harmonic pairs can be done with the denominator as it is shown in Proposition 1 of Cavaliere et al. (2017):

$$\begin{aligned} T^{-2}\sum_{t=2}^{N}\sum_{s=1-S}^{0} (x_{j,St+s})^2 &\Rightarrow \frac{1}{2S}(c'_j\Psi(1)\int_0^1 M(r)M(r)'dr\Psi(1)'c_j \\ &+ \tilde{c}'_j\Psi(1)\int_0^1 M(r)M(r)'dr\Psi(1)'\tilde{c}_j) \\ &= \frac{1}{2S}[c'_jQ_1c_j + \tilde{c}'_jQ_1\tilde{c}_j], j = 1, ..., S^* \end{aligned}$$

and

$$T^{-2}\sum_{t=2}^{N}\sum_{s=1-S}^{0} (x^*_{j,St+s})^2 \Rightarrow \frac{1}{2S}[c'_jQ_1c_j + \tilde{c}'_jQ_1\tilde{c}_j], j = 1, ..., S^*$$

Now, we focus on the denominators. For zero and Nyquist frequency, we have:

$$
\begin{aligned}
T^{-2-2d}\sum_{t=2}^{N}\sum_{s=1-S}^{0}\tilde{x}^2_{j,St+s} &= T^{-2-2d}\sum_{t=2}^{N}\tilde{X}'_{j,t}\tilde{X}_{j,t} \\
&= T^{-2-2d}\sum_{n=2}^{N}S(\tilde{X}'_{t-1}C_j\tilde{X}_{t-1}) + o_p(1) \\
&= T^{-2-2d}\sum_{t=2}^{N}S(\tilde{X}'_{t-1}c_jc'_j\tilde{X}_{t-1}) + o_p(1) \\
&= T^{-2-2d}\sum_{t=2}^{N}tr\{S(c'_j\tilde{X}_{t-1}\tilde{X}'_{t-1}c_j\} + o_p(1) \\
&= Sc'_jT^{-2-2d}\sum_{t=2}^{N}\tilde{X}_{t-1}\tilde{X}'_{t-1}c_j + o_p(1) \\
&\Rightarrow \frac{1}{S}c'_j\Psi(1)\int_0^1 M_{d+1}(r)M_{d+1}(r)'dr\Psi(1)'c_j \\
&= \frac{1}{S}c'_jQ_2c_j, j = 0, S/2
\end{aligned}
$$

For the harmonic frequency pairs, we have

$$
\begin{aligned}
T^{-2-2d}\sum_{t=2}^{N}\sum_{s=1-S}^{0}(\tilde{x}_{j,St+s})^2 &= T^{-2-2d}\sum_{t=2}^{N}\tilde{X}_{j,t}\tilde{X}_{j,t} \\
&= T^{-2-2d}\sum_{t=2}^{N}\frac{S}{2}(\tilde{X}'_{t-1}C_j\tilde{X}_{t-1}) + o_p(1) \\
&= T^{-2-2d}\sum_{t=2}^{N}\frac{S}{2}(\tilde{X}'_{t-1}c_jc'_j\tilde{X}_{t-1}) + o_p(1) \\
&= T^{-2-2d}\sum_{t=2}^{N}\frac{S}{2}tr\{(c'_j\tilde{X}_{t-1}\tilde{X}'_{t-1}c_j)\} + o_p(1) \\
&= T^{-2-2d}\sum_{t=2}^{N}\frac{S}{2}tr\{\begin{pmatrix} c'_j \\ \tilde{c}'_j \end{pmatrix}\tilde{X}_{t-1}\tilde{X}'_{t-1}\begin{pmatrix} c_j & \tilde{c}_j \end{pmatrix}\} + o_p(1)
\end{aligned}
$$

Now let us apply the multiplication of the vectors:

$$
\begin{aligned}
&= T^{-2-2d}\sum_{t=2}^{N}\frac{S}{2}tr\begin{pmatrix} c_j'\tilde{X}_{t-1}\tilde{X}_{t-1}'c_j & c_j'\tilde{X}_{t-1}\tilde{X}_{t-1}'\tilde{c}_j \\ \tilde{c}_j'\tilde{X}_{t-1}\tilde{X}_{t-1}'c_j & \tilde{c}_j'\tilde{X}_{t-1}\tilde{X}_{t-1}'\tilde{c}_j \end{pmatrix} + o_p(1) \\
&= \frac{S}{2}T^{-2-2d}\sum_{t=2}^{N}(c_j'\tilde{X}_{t-1}\tilde{X}_{t-1}'c_j + \tilde{c}_j'\tilde{X}_{t-1}\tilde{X}_{t-1}'\tilde{c}_j) + o_p(1) \\
&\Rightarrow \frac{1}{2S}(c_j'\Psi(1)\int_0^1 M_{d+1}(r)M_{d+1}(r)'dr\Psi(1)'c_j \\
&+ \tilde{c}_j'\Psi(1)\int_0^1 M_{d+1}(r)M_{d+1}(r)'dr\Psi(1)'\tilde{c}_j) \\
&= \frac{1}{2S}[c_j'Q_2c_j + \tilde{c}_j'Q_2\tilde{c}_j], j = 1, ..., S^*
\end{aligned}
$$

and

$$
\begin{aligned}
T^{-2-2d}\sum_{t=2}^{N}\sum_{s=1-S}^{0}(\tilde{x}_{j,St+s}^*)^2 &= T^{-2-2d}\sum_{t=2}^{N}\tilde{X}_{j,t}^*\tilde{X}_{j,t}^* \\
&= T^{-2-2d}\sum_{t=2}^{N}(S/2)(\tilde{X}_{t-1}'\tilde{C}_j\tilde{C}_j\tilde{X}_{t-1}) + o_p(1) \\
&= T^{-2-2d}\sum_{t=2}^{N}(S/2)(\tilde{X}_{t-1}'\tilde{C}_j\tilde{X}_{t-1}) + o_p(1) \\
&\Rightarrow \frac{1}{2S}[c_j'\tilde{Q}c_j + \tilde{c}_j'\tilde{Q}\tilde{c}_j], j = 1, ..., S^*
\end{aligned}
$$

Now, with the help of the S(normalised) variance transformed Brownian and type II fractional variance transformed Brownian motion processes,firstly, let us consider limiting distribution of $\tau_0(d)$ statistic under periodic nonstationary volatility:

$$
\begin{aligned}
\tau_0(d) &\Rightarrow \frac{c_0'Q_1c_0c_0'\Sigma(1)c_0}{c_0'Q_2c_0c_0'\Sigma(1)c_0} = \frac{c_0'\Psi(1)Q_1^*\Psi(1)'c_0c_0'\Sigma(1)c_0}{c_0'\Psi(1)Q_2^*\Psi(1)'c_0c_0'\Sigma(1)c_0} \\
&= \frac{\psi(1)c_0'Q_1^*c_0\psi(1)c_0'\Sigma(1)c_0}{\psi(1)c_0'Q_2^*c_0\psi(1)c_0'\Sigma(1)c_0} = \frac{c_0'(\int_0^1 M(r)M(r)'dr)c_0c_0'\Sigma(1)c_0}{c_0'(\int_0^1 M_{d+1}(r)M_{d+1}(r)'dr)c_0c_0'\Sigma(1)c_0} \\
&= \frac{\int_0^1 B_{\eta,0}^2(r)dr}{\int_0^1 B_{d+1,\eta,0}^2(r)dr}
\end{aligned}
$$

The preceeding result comes from the result of del Barrio Castro et al. (2012)

such that $c_0'\Psi(1) = \psi(1)c_0$. The proof for the limiting distribution of $\tau_{S/2}(d)$ at Nyquist frequency is similiar to the zero frequency except the corresponding result of Castro et al. (2012) such that $c_{S/2}'\Psi(1) = \psi(-1)c_{S/2}$ and that $c_{S/2}'\Sigma(1)c_{S/2} = c_0'\Sigma(1)c_0$.

For the frequency pair limiting distribution, let us introduce the results of Castro et al. (2012). We have $c_k'\Psi(1) = a_k\tilde{c}_k' + b_kc_k'$ and $\tilde{c}_k'\Psi(1) = b_k\tilde{c}_k' - a_kc_k'$ where $a_k = Im(\psi(exp(i\omega_k))$ and $b_k = Re(\psi(exp(i\omega_k))$, $k = 1, ..., S^*$, $Re(.)$ and $Im(.)$ denoting the real and imaginary parts of their arguments, respectively. Therefore, for $k = 1, .., S^*$, we have

$$
\begin{aligned}
c_k'Q_1c_k &= (a_k\tilde{c}_k' + b_kc_k')(\int_0^1 M(r)M(r)'dr)(a_k\tilde{c}_k' + b_kc_k') \\
\tilde{c}_k'Q_1\tilde{c}_k &= (b_k\tilde{c}_k' - a_kc_k')(\int_0^1 M(r)M(r)'dr)(b_k\tilde{c}_k' - a_kc_k') \\
c_k'Q_2c_k &= (a_k\tilde{c}_k' + b_kc_k')(\int_0^1 M_{d+1}(r)M_{d+1}(r)'dr)(a_k\tilde{c}_k' + b_kc_k') \\
\tilde{c}_k'Q_2\tilde{c}_k &= (b_k\tilde{c}_k' - a_kc_k')(\int_0^1 M_{d+1}(r)M_{d+1}(r)'dr)(b_k\tilde{c}_k' - a_kc_k')
\end{aligned}
$$

The final step for obtaining the limiting distribution for frequency pair is as follows:

$$
\begin{aligned}
c_k'Q_1c_k + \tilde{c}_k'Q_1\tilde{c}_k = \frac{S}{2}(a_k^2 + b_k^2)(\{c_k'\Sigma(1)c_k\} \int_0^1 B_{\eta,k}(r)^2dr \\
+ \{\tilde{c}_k'\Sigma(1)\tilde{c}_k\} \int_0^1 B_{\eta,k}(r)^{2*}dr)
\end{aligned}
$$

$$
\begin{aligned}
c_k'Q_2c_k + \tilde{c}_k'Q_2\tilde{c}_k = \frac{S}{2}(a_k^2 + b_k^2)(\{c_k'\Sigma(1)c_k\} \int_0^1 B_{d+1,\eta,k}(r)^2dr \\
+ \{\tilde{c}_k'\Sigma(1)\tilde{c}_k\} \int_0^1 B_{d+1,\eta,k}(r)^{2*}dr)
\end{aligned}
$$

The stated results for $\tau_k(d)$, $k = 1, ..., S^*$ statistics then follow after some routine algebra. $\square$

Proof of Theorem 6. Let us define $S_N^b(r) = N^{-1/2}X^*_{\lfloor rN \rfloor}$, where U_n^* is defined as $U_n^* = [u^*_{Sn-(S-1)}, u^*_{Sn-(S-2)}, ..., u^*_{Sn}]'$. We can express U_n^* by definition as $U_n^* = \hat{U}_n.w_n = (\Psi(L)\hat{E}_n)w_n$, where $\Psi(L)$ is the lag polynomial that creates serial correlation. By Cavaliere and Taylor (2009), we know that wild bootstrap kills the serial correlation in the innovations. Therefore, we have $U_n^* = \hat{E}_n w_n = E_n^*$.

Under Assumption $\mathcal{A}$, by using by Lemma 4 of Boswijk et al. (2016), we have

$$S_N^b(r) = N^{-1/2}\sum_{n=2}^{\lfloor rN \rfloor} U_n^* \Rightarrow M(r) \tag{27}$$

where $U_n^* = [u^*_{Sn-(S-1)}, u^*_{Sn-(S-2)}, ..., u^*_{Sn}]'$, because conditional on $\{\hat{U}_n\}_{n=1}^N$, $S_N^b(r)$ is a Gaussian process with independent increments and covariance kernel $E*(S_N^b(.)S_N^b(.)) = \sum_{n=2}^{\lfloor rN \rfloor}\hat{E}_n\hat{E}_n'$, where $\sum_{n=2}^{\lfloor rN \rfloor}\hat{E}_n\hat{E}_n' \to \Sigma(r)$ in probability uniformly for all $r \in [0,1]$.

Now consider $\tilde{S}_N^b(r) = N^{-1/2-d}\tilde{X}^*_{\lfloor rN \rfloor}$, where $\tilde{U}_n^* = [\tilde{u}^*_{Sn-(S-1)}, \tilde{u}^*_{Sn-(S-2)}, ..., \tilde{u}^*_{Sn}]'$. Now, we can write $\tilde{U}_n^*$ as $\tilde{U}_n^* = \Delta_+^{-d}U_n^* = \Delta_+^{-d}(\hat{U}_n.w_n) = \Delta_+^{-d}(\hat{E}_n.w_n) = \Delta_+^{-d}E_n^*$

Now, Under Assumption $\mathcal{A}$, by using the main result in Lemma 1, and (27), we can conclude that

$$\tilde{S}_N^b(r) = N^{-1/2-d}\sum_{n=2}^{\lfloor rN \rfloor}\tilde{U}_n^* \Rightarrow M_{d+1}(r) \tag{28}$$

Now, by using the main convergence results (27) and (28) related to the wild bootstrap, we can continue as follows:

$$N^{-1}\sum_{t=2}^{N} X_{t-1}^{*}X_{t-1}^{*} \Rightarrow \int_0^1 M(r)M(r)'dr =: Q_1^*$$
$$N^{-1-2d}\sum_{t=2}^{N} \tilde{X}_{t-1}^{*}\tilde{X}_{t-1}^{*} \Rightarrow \int_0^1 M_{d+1}(r)M_{d+1}(r)'dr =: Q_2^*$$

Therefore, we obtain

$$T^{-2}\sum_{t=2}^{N}\sum_{s=1-S}^{0} x_{j,St+s^*}^{2} \Rightarrow \frac{1}{S}c_j'Q_1^*c_j, j=0,S/2$$

$$T^{-2}\sum_{t=2}^{N}\sum_{s=1-S}^{0} (x_{j,St+s}^{**})^2 \Rightarrow \frac{1}{2S}[c_j'Q_1^*c_j+\tilde{c}_j'Q_1^*\tilde{c}_j], j=1,...,S^*$$
$$T^{-2}\sum_{t=2}^{N}\sum_{s=1-S}^{0} (x_{j,St+s}^{*})^2 \Rightarrow \frac{1}{2S}[c_j'Q_1^*c_j+\tilde{c}_j'Q_1^*\tilde{c}_j], j=1,...,S^*$$
$$T^{-2-2d}\sum_{t=2}^{N}\sum_{s=1-S}^{0} \tilde{x}_{j,St+s^*}^{2} \Rightarrow \frac{1}{S}c_j'Q_2^*c_j, j=0,S/2$$

$$T^{-2-2d}\sum_{t=2}^{N}\sum_{s=1-S}^{0} (\tilde{x}_{j,St+s}^{**})^2 \Rightarrow \frac{1}{2S}[c_j'Q_2^*c_j+\tilde{c}_j'Q_2^*\tilde{c}_j], j=1,...,S^*$$
$$T^{-2-2d}\sum_{t=2}^{N}\sum_{s=1-S}^{0} (\tilde{x}_{j,St+s}^{*})^2 \Rightarrow \frac{1}{2S}[c_j'Q_2^*c_j+\tilde{c}_j'Q_2^*\tilde{c}_j], j=1,...,S^*$$

After obtaining the convergence results above, the asymptotic null distributions of the wild bootstrap FSVR statistics given in the first part of this theorem can be obtained straightforwardly.

Now, we switch to the proof of the second part of the theorem. Firstly, consider the nonparametric wild bootstrap $\tau_0^*(d)$ statistic. From the result $\tau_0^*(d) \Rightarrow \xi_{\eta,0}(d)$ obtained in the first part of the theorem, we can conclude that, uniformly in probability, $G_{0,T}^*(.) \to G_0(.)$, where $G_0(.)$ is the cdf of $\xi_{\eta,0}(d)$. This result coupled with the proof of Corollary 1 in Hansen (2000) implies that P_T^* converges weakly to $U[0,1]$ under the conditions of Theorem 2. The corresponding results for the other FSVR statistics can be established similiarly. $\square$