BRANCHING BROWNIAN MOTION IN A TIME DEPENDENT BALL

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ABSTRACT

In this thesis, we study a d-dimensional branching Brownian motion (BBM) evolving inside a subdiffusively expanding ball. BBM is a randomly branching stochastic mathematical model where each particle moves under Brownian motion independently of each other. In this model, the particles of BBM are deactivated instantly upon touching the boundary of an expanding ball, but they will be reactivated at a later time provided their ancestral lines are fully inside the expanding ball at that later time. The main quantity of interest is the large-time mass of the BBM inside this time-dependent restricted domain. The study aims at obtaining sharp asymptotic results as time tends to infinity on the large-deviation probability that the mass inside the expanding ball is atypically small. A phase transition is identified concerning the optimal strategies to realize the aforementioned large-deviation event.

Keywords: Branching Brownian motion, expanding ball, large-deviations, deactivating boundary.

ÖZETÇE

Bu tezde, subdiffusive olarak genişleyen bir top içinde evrimleşen d-boyutlu dallanan Brown hareketi incelenmektedir. Dallanan Brown hareketi, parçacıkların birbirinden bağımsız şekilde davrandığı, her bir parçacığın Brown hareketi altında hareket ettiği ve rastlantısal olarak dallandığı stokastik bir matematiksel modeldir. Bu modelde, Dallanan Brown Hareketi'nin parçacıkları genişleyen bir topun sınırına temas ettiği anda devre dışı bırakılır, ancak ilerleyen bir zamanda atasal çizgileri tamamen genişleyen topun içinde bulunuyorsa tekrar aktif hale gelebilirler. İlgilenilen temel nicelik, bu kısıtlı alanda büyük zaman t içindeki aktif parçacık sayısı yani Dallanan Brown Hareketi'nin büyük zamanlardaki kütlesidir. Çalışma, zaman sonsuza gittiğinde, genişleyen topun içindeki kütlenin atipik olarak küçük olduğu büyük sapmalar olasılığının keskin asimptotik sonuçlarını elde etmektir. Daha önce bahsedilen büyük sapmalar olayını gerçekleştirmek için optimal stratejilere ilişkin bir faz geçişi belirlenmiştir.

Anahtar kelimeler: Dallanan Brown hareketi, genişleyen top, büyük sapmalar

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LIST OF NOTATION

$\mathbb{R}^d$	d-dimensional Euclidean space
β	branching rate
t	time
$X = (X_t)_{t\geq 0}$	standard Brownian motion in d dimensions
$Z = (Z_t)_{t\geq 0}$	branching Brownian motion in d dimensions
P	probability for branching process
E	expectation corresponding to P
$\mathbf{P}_x$	probability for standard Brownian motion in d-dimensions
$\mathbf{E}_x$	expectation corresponding to $\mathbf{P}_x$
λ_d	the principal Dirichlet eigenvalue for the open unit ball in $\mathbb{R}^d$
$B(x, r)$	open ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$

CHAPTER I

INTRODUCTION AND MAIN RESULTS

In this thesis, we study a d-dimensional branching Brownian motion (BBM) evolving inside a subdiffusively expanding ball. Branching Brownian motion is a randomly branching stochastic mathematical model where each particle moves under Brownian motion independently of each other. The process starts with a first particle at the origin which performs a standard Brownian motion in $\mathbb{R}^d$ along a random lifetime. The particle dies at the end of its lifetime and simultaneously gives birth to two offspring. New offspring are born at the position where their parent dies. The process evolves in this way, where each particle behaves stochastically identically as the parent.

We suppose that the particles of BBM are deactivated instantly upon touching the boundary of an expanding ball, but they will be reactivated at a later time provided their ancestral lines are fully inside the expanding ball at that later time. The main quantity of interest is the large-time mass of the BBM, that is, the number of *"active"* particles at a large time t inside this restricted domain. The study aims at obtaining asymptotic results as time tends to infinity on the large-deviation probability that the mass inside the expanding ball is atypically small.

1.1 The Model

In this chapter, we will develop our model and formulate our main problem in three stages, as follows:

1. Branching Brownian Motion: Branching Brownian motion is a randomly branching stochastic mathematical model where each particle moves under Brownian motion independently of each other. Let $Z = (Z_t)_{t\geq 0}$ represent a strictly dyadic d-dimensional branching Brownian motion with branching rate β, where $\beta > 0$ is a constant and t denotes time. When each particle produces exactly two offspring, the process is referred to as being strictly dyadic. The following is a description of the process. The process starts with a single particle at the origin, which performs a standard Brownian motion in $\mathbb{R}^d$ during a random lifetime. The particle dies at the end of its lifetime and simultaneously gives birth to two offspring. New offspring are born at the position where their parent dies. Each offspring repeats the same process independently of the others and its ancestors. The process continues to evolve in this way. The lifetime of each particle is exponentially distributed with constant parameter $\beta > 0$. The total mass of Z at time t is denoted by $|Z_t|$ for each $t \geq 0$, and we also use $N_t := |Z_t|$. Also, given a Borel set $A \subseteq \mathbb{R}^d$ and time $t \geq 0$, we write $Z_t(A)$ to denote the mass of Z that falls into A at time t.

2. Branching Brownian Motion with deactivation at a (moving) boundary: We now define a branching Brownian motion deactivated at a moving boundary. Let ∂A represent the boundary of the set A for any Borel set $A \subseteq \mathbb{R}^d$. Consider a family of Borel sets $B = (B_t)_{t\geq 0}$ indexed by time, and denote the BBM deactivated at ∂B is by $Z^B = (Z_t^{B_t})_{t\geq 0}$. The process at each time t, that is, $Z_t^{B_t}$, is obtained by starting with Z_t and deleting from it any particle whose ancestral line over $[0, t]$ has exited B_t. This means that $Z_t^{B_t}$ consists of particles at time t whose ancestral lines have stayed inside B_t (although they may have left B_s earlier).

The terminology *"deactivated"* can be explained as follows. Consider a particle w of Z_s. If w at time s is not a particle of $Z_s^{B_s}$, this means that its ancestral line has left B_s during the interval $[0, s]$ and is marked as "deactivated" at time s. This particle could reappear at a later time u, that is, it will be reactivated provided that its ancestral line becomes completely contained in B_u over the interval $[0, u]$; then it will be a part of $Z_u^{B_u}$.

We denote the law of a BBM starting with a single particle at $x \in \mathbb{R}^d$ is by P_x, and corresponding expectation as E_x. We also use similar notation for a BBM with deactivation at a boundary. For simplicity, we set $P = P_0$. We also denote the mass of a BBM at time t with deactivation at ∂B as,

$$n_t := |Z_t^{B_t}|. \tag{1}$$

3. Formulation of our problem: Let $B = (B_t)_{t\geq 0}$, where $B_t := B(0, r(t))$ with $r : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\lim_{t\to\infty} r(t) = \infty \ \text{ and } \ r(t) = o(\sqrt{t}).$$

We are interested in the large-time behavior of the probability $P(n_t < \gamma_t p_t e^{\beta t})$. In detail, consider a radius function $r : \mathbb{R}_+ \to \mathbb{R}_+$ which is subdiffusively increasing, in other words, $\lim_{t\to\infty} r(t) = \infty$ and $r(t) = o(\sqrt{t})$ as $t \to \infty$. A standard Brownian motion typically travels a distance of $\sim \sqrt{t}$ over the interval $[0, t]$, and this is called *diffusive* behavior. Since $r(t) = o(\sqrt{t})$ we say that the increase in r is subdiffusive. Furthermore, let p_t denote the probability of confinement for a standard Brownian motion to B_t over the interval $[0, t]$, which we can denote as

$$p_t = \mathbf{P}_0(\sigma_{B_t} \geq t).$$

Here, we use $\sigma_A = \inf\{s \geq 0 : X_s \notin A\}$ as the first exit time of a standard Brownian motion X out of a Borel set A. Let $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function such that $\lim_{t\to\infty} \gamma(t) = 0$. For simplicity, we set $\gamma_t = \gamma(t)$. The main quantity of interest is the large-time mass of the BBM with deactivation at the boundary of B_t. The study aims to obtain asymptotic results for the probability of large-deviations as time tends to infinity for

$$P\left(|Z_t^{B_t}| < \gamma_t p_t e^{\beta t}\right).$$

The event under consideration suggests that the mass inside the expanding ball is atypically small due to the following reason. It is standard by a first-moment argument to show that $E[n_t] = p_t e^{\beta t}$. We refer the reader to [Proposition 2.7, Chapter 2] for a proof. Since $\lim_{t\to\infty}\gamma_t = 0$, we suspect that for large times, $\gamma_t p_t e^{\beta t}$ is atypically small for the mass of a BBM that is deactivated at ∂B. Theorem 1.1 confirms that this is the case.

1.2 Previous Work

This section provides a brief literature review of some models that are similar to our model. We will review a selection of important research about BBM in restricted domains.

Historically, these kinds of models are traced back to Kesten's model in [2]. In 1978, Kesten studied a BBM with absorption at the origin. The process starts with a single particle at $x > 0$. In this process, each particle moves according to a Brownian motion with negative drift in one-dimension, and particles are killed upon touching the origin. In other words, the origin serves as a *"killing"* boundary. Kesten described the regions where the process is supercritical, critical, or subcritical, and provided the growth rate in the supercritical case, obtaining all results depending on the magnitude of the drift of the process. In addition, this article provided an asymptotic result on the number of particles in a given interval.

In [3], Berestycki et al. again considered the one-dimensional model of Kesten, and studied the survival probability of the BBM near the critical drift. The process starts with a single particle at $x > 0$, and the particles are killed upon hitting the origin. The critical value of $\sqrt{2}$ for the drift is important since it determines the transition point between survival and extinction. If the drift is larger than the critical value, the process tends to spread and survive, whereas if the drift is smaller than the critical value, the process become extinct with probability one.

Then, on the same model, in [4] Berestycki et al. demonstrated an analysis of the genealogy of the BBM by investigating the connections between the ancestors and descendants of the particles in the population. Various results are obtained, such as the numerical distribution, probability of absorption, and time distribution of the most recent common ancestor.

Later, in [5], Maillard studied a supercritical BBM. The process starts at the origin and with constant negative drift, and then, an absorbing one-sided barrier is added at the point $x > 0$. The process becomes extinct almost surely depending on a critical drift. Maillard obtained asymptotic results for large times about the number of absorbed individuals at the linear one-sided barrier.

Recently, in [6], Harris et al. studied a BBM with negative drift in one dimension, where particles are killed upon exiting a fixed-size interval. This means there is a two-sided barrier. Harris et al. have examined how the process changes as approaches a critical value where survival is impossible, and obtained exact asymptotics for the near-critical survival probability. In other words, they studied the interval width as it approaches the critical value where survival becomes impossible.

1.3 Main Results

Our main result is on the asymptotic behavior of the large-deviation probability that the mass inside a subdiffusively expanding ball is atypically small for large-time. The mass being atypically small means that $\{|Z_t^{B_t}| < \gamma_t p_t e^{\beta t}\}$ is a rare event. Expansion of a ball at a slower than the typical rate at which a single BM moves away from its starting point, is called subdiffusive expansion.

Theorem 1.1. (Lower large-deviations for mass of BBM in an expanding ball). *Let $r : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\lim_{t\to\infty} r(t) = \infty$ and $r(t) = o(\sqrt{t})$ as $t \to \infty$. Also, let $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $\gamma(t) = e^{-\kappa r(t)}$, where $\kappa > 0$ is a constant. For $t > 0$, set $B_t = B(0, r(t))$, $p_t = \boldsymbol{P}_0(\sigma_{B_t} \geq t)$, and $n_t = |Z_t^{B_t}|$. Then,*

$$\lim_{t\to\infty} \frac{1}{r(t)} \log P(n_t < \gamma_t p_t e^{\beta t}) = -(\kappa \wedge \sqrt{2\beta}). \tag{2}$$

Remark 1.2. *There is a continuous phase transition at a critical value $\kappa = \sqrt{2\beta}$ in the asymptotic behavior of $P(n_t < \gamma_t p_t e^{\beta t})$ in Theorem 1.1. The phase transition is* *explained in terms of the optimal strategies to realize the rare event* $\{n_t < \gamma_t p_t e^{\beta t}\}$*. This optimal strategy changes depending on the parameter* κ *in* $\gamma_t = e^{-\kappa r(t)}$*. Accordingly, the phase transition describes how the BBM realizes the large-deviation event* $\{n_t < \gamma_t p_t e^{\beta t}\}$*. There are two cases as follows:* $\kappa > \sqrt{2\beta}$ *and* $0 < \kappa \leq \sqrt{2\beta}$*.*

When $\kappa > \sqrt{2\beta}$*, the BBM suppresses the branching of the initial particle, and at the same time moves it outside* $B(0, r(t))$ *over the time interval* $[0, r(t)/\sqrt{2\beta}]$*. The event* $\{n_t = 0\}$ *is realized when the initial particle is carried outside* $B(0, r(t))$*, which means there is no need for further suppression of branching.*

When $0 < \kappa \leq \sqrt{2\beta}$*, the BBM precisely suppresses the branching over the time interval* $[0, (\kappa/\beta)r(t)]$*. In other words, it suppresses the branching of the initial particle for a sufficiently long time until the mass* $\gamma_t p_t e^{\beta t}$ *is no longer atypically small for the BBM to grow in the remaining interval. Then, the BBM behaves "normally" over the time interval* $[(\kappa/\beta)r(t), t]$*.*

The phase transition occurs at $\kappa = \sqrt{2\beta}$, as shown in Figure 1.

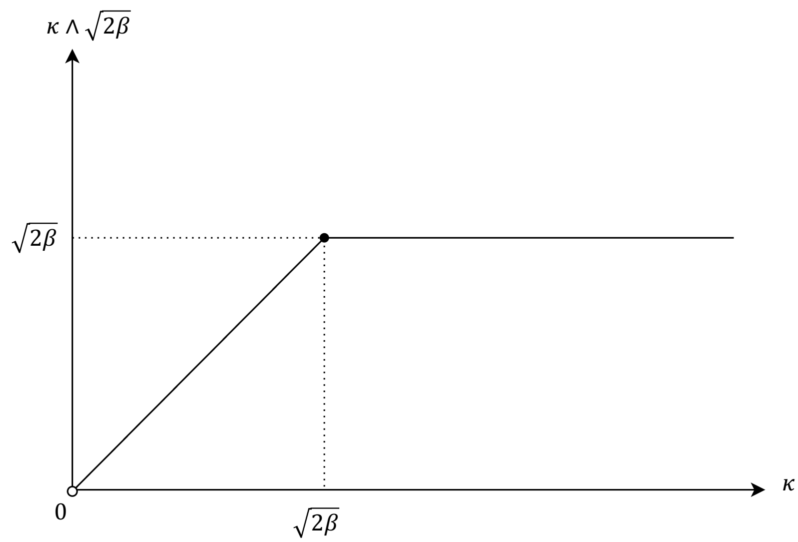

Figure 1: Phase transition occurs at $\kappa = \sqrt{2\beta}$.

Remark 1.3. *With* $\gamma_t = e^{-\kappa r(t)}$ *for* $\kappa > 0$, $P(n_t < \gamma_t p_t e^{\beta t})$ *indeed represents a large-deviation probability, and the reason is as follows. Note that* n_t *denotes the mass of BBM, so it cannot be smaller than zero. We can show that for all large* t, $P(n_t = 0)$ *decays as* $e^{-cr(t)}$ *for some constant* $c > 0$. *Since* $P(n_t = 0)$ *decays as* $e^{-cr(t)}$, *and similarly,* $P(n_t < \gamma_t p_t e^{\beta t})$ *also decay as* $e^{-cr(t)}$ *due to* (2), $P(n_t < \gamma_t p_t e^{\beta t})$ *is a large-deviation probability. In summary, with this choice of* $\gamma_t = e^{-\kappa r(t)}$, *both* $P(n_t < \gamma_t p_t e^{\beta t})$ *and* $P(n_t = 0)$ *decay as* $e^{-cr(t)}$ *for all large* t, *where the values of* c *are positive and may differ.*

We consider $\gamma_t = e^{-\kappa r(t)}$ *because of the following reason. We can show that* $E[n_t] = p_t e^{\beta t}$; *for a proof, see [Proposition 2.7, Chapter 2]. It can be shown that for all choices of* γ_t *with* $\gamma_t \to 0$, $P(n_t < \gamma_t p_t e^{\beta t}) \geq \delta \gamma_t$ *for some* $0 < \delta < 1$ *for all large* t *(see the proof of the lower bound of Theorem 1.1). Therefore, if* γ_t *decays slower than* $e^{-cr(t)}$ *so as to satisfy* $(\log \gamma_t)/r(t) \to 0$ *as* $t \to \infty$, *then*

$$\liminf_{t\to\infty} \frac{1}{r(t)} \log P(n_t < \gamma_t p_t e^{\beta t}) = 0.$$

This shows that in cases where γ_t *decays slower than* $e^{-cr(t)}$, *the event* $\{n_t < \gamma_t p_t e^{\beta t}\}$ *will not be a large-deviation event. We now explain with a concrete example. For instance, if* $\gamma_t = e^{-\kappa\sqrt{r(t)}}$, *we obtain* $P(n_t < \gamma_t p_t e^{\beta t}) > \delta e^{-\kappa\sqrt{r(t)}}$. *Since* $e^{-\kappa\sqrt{r(t)}} \to 0$ *slower than* $e^{-\kappa r(t)}$, *this means for the choice* $\gamma_t = e^{-\kappa\sqrt{r(t)}}$, $\{n_t < \gamma_t p_t e^{\beta t}\}$ *is not an LD-event. In order to be an LD-event it needs to decay to zero as* $\sim e^{-cr(t)}$ *for some* $c > 0$.

Theorem 1.4. (Law of large numbers for mass of BBM in an expanding ball). *Let* $r : \mathbb{R}_+ \to \mathbb{R}_+$ *be subdiffusively increasing such that* $r(t) \to \infty$ *as* $t \to \infty$ *and* $r(t) = o(\sqrt{t})$. *For* $t > 0$, *set* $B_t = B(0, r(t))$, $p_t = \mathbf{P}_0(\sigma_{B_t} \geq t)$, *and* $n_t = |Z_t^{B_t}|$. *Then,*

$$\lim_{t\to\infty} (r(t))^2 \left(\frac{\log n_t}{t} - \beta \right) = -\lambda_d \quad \text{in } P-\text{probability}. \tag{3}$$

Remark 1.5. *Since* $E(n_t) = p_t e^{\beta t}$, *from Proposition 2.3, we have*

$$\lim_{t\to\infty} (r(t))^2 \left(\frac{\log E(n_t)}{t} - \beta \right) = -\lambda_d.$$

Theorem 1.4 says that a similar limit result also holds for the process n_t itself. Therefore, Theorem 1.4 is called a law of large numbers. That is, in a loose sense,

$$\frac{\log n_t}{t} \approx \frac{\log E(n_t)}{t}, \quad t \to \infty.$$

1.4 Outline

The rest of the thesis is organized as follows. In Chapter 2, we first review some well-known results on Brownian motion and on branching Brownian motion, including results on the growth of mass and speed of BBM. Then we prove some preparatory results on the BBM with deactivation at the boundary of $B(0, r(t))$. In Chapter 3, we present the proof of the main theorem, that is, Theorem 1.1. In Chapter 4, we present a result on the large time probability of the event $\{n_t < \gamma_t p_t e^{\beta t}\}$ in the case of a fixed ball $B(0, R)$, where $R > 0$ is a constant. In Chapter 5, we present some simulations of BBM and some information about the simulation that has been realized, such as the interpretation of some output and the illustrations.

CHAPTER II

SOME BACKGROUND AND PRELIMINARY RESULTS

In this chapter, we first review some background on BM and BBM. Then, we obtain some preliminary results (see Proposition 2.7 and Proposition 2.8) on the mass of BBM with deactivation at the boundary of a subdiffusive expanding ball. We will later use these results in the proof of the main theorem.

2.1 Background on Brownian Motion

We start by reviewing the definitions of one-dimensional BM and d-dimensional BM.

Definition 2.1. *A real-valued stochastic process* $\{B(t) : t \geq 0\}$ *is called a* one-dimensional Brownian motion that starts at $x \in \mathbb{R}$ *if the following hold:* [7, Chapter 1]

1. $B(0) = x$,
2. *the process has independent increments, i.e., for all times* $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ *the increments* $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), \ldots, B(t_2) - B(t_1)$ *are independent random variables,*
3. *for all* $t \geq 0$ *and* $h > 0$*, the increments* $B(t+h) - B(t)$ *are normally distributed with variance* h *and expectation zero, that is,* $B(t+h) - B(t)$ *has transition density function given by*
$$P_h(x) = \frac{1}{\sqrt{2\pi h}} \exp\left(-\frac{x^2}{2h} \right),$$
4. *almost surely, the function* $t \to B(t)$ *is continuous.*

It is called a standard Brownian motion *if* $x = 0$.

It is not obvious that properties 2-4 are compatible with each other. It was shown by Wiener in [8] that standard Brownian motion indeed exists.

Definition 2.2. *An $\mathbb{R}^d$-valued stochastic process $\{B(t) : t \geq 0\}$ with $B(t) = (B_1(t), B_2(t), \ldots, B_d(t))$ in component form is called a d-*dimensional Brownian motion*, if $B_1, B_2, \ldots, B_d$ are independent one-dimensional Brownian motions.*

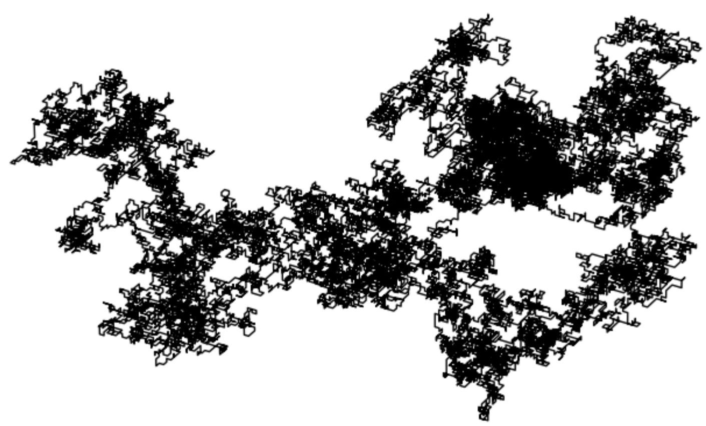

Figure 2: An illustration of a 2-dimensional Brownian motion.

In the rest of this thesis, let $X = (X(t))_{t\geq 0}$ be a generic d-dimensional BM. The following proposition is on the large-time asymptotic probability of atypically large Brownian displacements, expressed as the events $\{\sup_{0\leq s\leq t} |X(s)| > kt\}$. We refer the reader to [9, Lemma 5] for a proof.

Proposition 2.1. (Linear Brownian displacements). *For $k > 0$,*

$$\mathbf{P}_0\left(\sup_{0\leq s\leq t} |X(s)| > kt\right) = \exp\left[-\frac{k^2 t}{2}(1 + o(1))\right].$$

As before, we use $\sigma_A = \inf\{s \geq 0 : X_s \notin A\}$ as the first exit time of X out of A.

The following lemma is a standard result on the Brownian confinement in balls (see [10]).

Lemma 2.2. (Lower Bound on Probability of Confinement). *Let $B = B(0,r)$ be the open ball in $\mathbb{R}^d$ with radius r, centered at the origin. Let $X = (X_t)_{t\geq 0}$ a the standard Brownian motion, and $(\mathbf{P}_x, x \in \mathbb{R}^d)$ be the corresponding probabilities for processes that start at $x \in \mathbb{R}^d$. Then, for each $t > 0$,*

$$\mathbf{P}_0(\sigma_{B_t} \geq t) \geq c_d e^{-\frac{\lambda_d}{r^2}t},$$

where λ_d is the principal Dirichlet eigenvalue for the unit ball in $\mathbb{R}^d$ and c_d is a positive constant that depends on dimension.

The lemma above provides a lower bound on the probability of confining a Brownian particle within a ball centered at the starting position of the particle. Lemma 2.2 can be extended as follows.

Proposition 2.3. (Brownian confinement in small balls). *For $t > 0$, let $B_t = B(0, r(t))$, where $r : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $r(t) \to \infty$ as $t \to \infty$ and $r(t) = o(\sqrt{t})$. Then, as $t \to \infty$,*

$$\mathbf{P}^0(\sigma_{B_t} \geq t) = \exp\left[-\frac{\lambda_d t}{r^2(t)}(1 + o(1))\right].$$

We use $\mathbf{P}^y$ for the law of a Brownian motion that starts at a distance y from the origin. For $y > 0$, set $\tau_y = \sigma_{B(0,y)}$ for ease of notation. Then, we have the following result comparing the probabilities of confinement in balls for Brownian motions starting at the center of the ball and any other point inside the ball. We refer the reader to [12].

Proposition 2.4. (Brownian confinement in balls, comparison). *Let $b > 0$. Then, for every $0 < a < b$, there exists a positive constant $D = D(b/a, d)$ such that*

$$D\mathbf{P}^a(\tau_b \geq t) \geq \mathbf{P}^0(\tau_b \geq t) \tag{4}$$

for all large t.

Proof. Let $b > 0$ and ν represent the index of the Bessel process, and J_ν denote the Bessel function of the first kind. For $0 < a < b$, we denote by $\tau_b^{(\nu)}$ the first hitting time b of the Bessel process with index ν, then the Bessel process is identical in law to the radial component of a $(2\nu+2)$-dimensional Brownian motion. As a result, $2\nu + 2$ is referred to as the Bessel process dimension. It then follows from (2.7) and (2.8) in [13], that as $t \to \infty$,

$$\mathbf{P}^0(\tau_b \geq t) = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} e^{-\frac{j_{\nu,1}^2 t}{2b^2}}(1 + o(1)) \tag{5}$$

and

$$\mathbf{P}^a(\tau_b \geq t) = 2\left(\frac{b}{a}\right)^\nu \frac{J_\nu(a j_{\nu,1}/b)}{j_{\nu,1} J_{\nu+1}(j_{\nu,1})} e^{-\frac{j_{\nu,1}^2 t}{2b^2}}(1 + o(1)), \tag{6}$$

where $0 < a < b$, J_μ is the Bessel function of the first kind of order μ and $\{j_{\mu,k}\}_{k=1}^\infty$ is the increasing sequence of positive zeros of J_μ, and we suppress the dependence of τ on ν (hence on d) in notation. From (5) and (6) it follows that there exists $t_0 > 0$ and a positive constant $D = D(b/a, d)$ such that for all $t \geq t_0$,

$$D\mathbf{P}^a(\tau_b \geq t) \geq \mathbf{P}^0(\tau_b \geq t).$$

□

2.2 Background on Branching Brownian Motion

Branching Brownian motion is a randomly branching stochastic mathematical model where each particle moves under Brownian motion independently of each other. Let $Z = (Z_t)_{t \geq 0}$ represent a strictly dyadic d-dimensional branching Brownian motion with branching rate β, where $\beta > 0$ is a positive constant and t denotes time. Recall that the total mass of Z at time t is denoted by $|Z_t|$, set $N_t = |Z_t|$. One natural random variables associated to a BBM is the number of particles present, that is, the total mass of process. We present a standard result on the growth of mass of a BBM.

Proposition 2.5. (Distribution of mass in branching systems). *For a strictly dyadic continuous-time branching process $N = (N_t)_{t\geq 0}$ with constant branching rate $\beta > 0$, the probability distribution at time t is given by*

$$P(N_t = k) = e^{-\beta t}(1 - e^{-\beta t})^{k-1}, \qquad k \geq 1,$$

from which it follows that

$$P(N_t > k) = (1 - e^{-\beta t})^k, \tag{7}$$

and

$$E[N_t] = e^{\beta t}. \tag{8}$$

One of the most interesting properties related to branching Brownian motion is how far particles move away from the origin in time. Now, we introduce a notion of *speed* for branching Brownian motion, and then we review a well-known result on this speed.

For $t \geq 0$, define $M_t := \inf\{r > 0 : \mathrm{supp}(Z(t)) \subset B(0,r)\}$ to be the radius of the minimal ball that contains the support of BBM at time t. Observe that M_t quantifies the spatial spread of the BBM, therefore M_t/t is a measure of the speed for the BBM. We refer the reader to [14].

Theorem 2.6. (Speed of BBM; $d = 1$, [15]; $d \geq 2$, [16]) *Let Z be a strictly dyadic BBM in $\mathbb{R}^d$ with branching $\beta > 0$. Then, in any dimension,*

$$M_t/t \to \sqrt{2\beta} \quad \textit{in probability as } t \to \infty.$$

Theorem 2.6 implies in particular that typically for large t and any $0 < \varepsilon < 1$, at time t there will be particles outside $B(0, (\sqrt{2\beta} - \varepsilon)t)$ but particles outside $B(0, (\sqrt{2\beta} + \varepsilon)t)$.

2.3 Branching Brownian Motion with deactivation at a boundary

Recall from Section 1.1 the model of BBM with deactivation at a boundary. In this section, we present two preparatory propositions that will be useful in the proofs of the main theorem. Recall that $n_t = |Z_t^{B_t}|$ was introduced in (1).

Proposition 2.7. *For each $t \geq 0$, $E[n_t] = p_t e^{\beta t}$.*

Proof. Let the event A_u represent the probability that the ancestral line of uth particle stays within the ball up to time t. Using the definition of n_t and conditioning on N_t,

$$\begin{aligned} E[n_t] &= \sum_{u=1}^{\infty} E[n_t | N_t = u] P[N_t = u] \\ &= \sum_{u=1}^{\infty} E[\mathbb{1}_{A_1} + \mathbb{1}_{A_2} + \ldots + \mathbb{1}_{A_u}] P[N_t = u] \\ &= \sum_{u=1}^{\infty} \left[E[\mathbb{1}_{A_1}] + E[\mathbb{1}_{A_2}] + \ldots + E[\mathbb{1}_{A_u}] \right] P[N_t = u] \\ &= \sum_{u=1}^{\infty} u E[\mathbb{1}_{A_1}] P[N_t = u], \end{aligned} \tag{9}$$

where we have used linearity of expectation in the second equation, and sums of random variables in the third equation. Then, p_t is the probability of confinement of a single Brownian motion to $B(0, r(t))$ over the interval $[0, t]$, which yields $E[\mathbb{1}_{A_1}] = p_t$ in (9),

$$p_t \sum_{j=1}^{\infty} u P[N_t = u]$$

Recall that $E[N_t] = e^{\beta t}$ from (8), and then which yields $\sum_{u=1}^{\infty} u P[N_t = u] = E[N_t]$,

$$E[n_t] = p_t e^{\beta t}.$$

□

The following result is on the atypically small growth of mass of a BBM with deactivation at the boundary of subdiffusively expanding ball. Here, the BBM starts with a single particle at a distance $\delta r(t)$ from the origin, where $0 < \delta < 1$, define P_δ as the law of a branching Brownian motion that starts with a single particle at a distance δ from the origin.

We will use this theorem in the proof of Theorem 1.1 in the section on the upper bound and that the bootstrap argument and Proposition 2.8 in proof of the upper bound.

For $0 < \delta < 1$, $\phi \geq 0$ and $B_t = B(0, r(t))$, let

$$\widetilde{p}_t = P^{\delta r(t)}(\sigma_{B_t} \geq t - \phi r(t)). \tag{10}$$

Applying Brownian scaling to (10), we obtain $\widetilde{p}_t = P^{\delta}(\sigma_{B(0,1)} \geq (t - \phi r(t))/(r(t))^2)$.

Proposition 2.8. *Let $0 < \delta < 1$, $0 \leq \phi_1 < \phi_2$ and $r : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing such that $\lim_{t\to\infty} r(t) = \infty$ and $r(t) = o(\sqrt{t})$ as $t \to \infty$. Let $\gamma_t = e^{-\kappa r(t)}$, where $\kappa > 0$ is a constant and $\widetilde{p}_t = \widetilde{p}_t(\phi)$ be as in (10). For $t > 0$, set $B_t = B(0, r(t))$ and $n_t = |Z_t^{B_t}|$. Then, there exists a constant $c = c(\delta, \kappa, \beta) > 0$ such that for all large t,*

$$\sup_{\phi_1 \leq \phi \leq \phi_2} P^{\delta r(t)}\left(n_{t-\phi r(t))} < \gamma_t \widetilde{p}_t e^{\beta(t-\phi r(t))}\right) \leq e^{-kr(t)}. \tag{11}$$

Proof. Let $\phi \in [\phi_1, \phi_2]$ and set $g_t = 2\gamma_t$. For $t > 0$, start with the estimate

$$P^{\delta r(t)}(\,\cdot\,) \leq P^{\delta r(t)}\left(\,\cdot\,\Big|\, N_{t-\phi r(t)} > e^{\beta(t-\phi r(t))} g_t\right) + P^{\delta r(t)}\left(N_{t-\phi r(t)} \leq e^{\beta(t-\phi r(t)} g_t\right) \tag{12}$$

with $g_t \to 0$ as $t \to \infty$. Proposition 2.5 yields $P(N_{t-\phi r(t)} \leq w) = 1 - (1 - e^{-\beta(t-\phi r(t))})^w \leq w e^{-\beta(t-\phi r(t))}$ for any $w \geq 1$, and then by setting $w = \lfloor e^{\beta(t-\phi r(t))} g_t \rfloor$, we obtain

$$\begin{aligned} P\left(N_{t-\phi r(t)} \leq e^{\beta(t-\phi r(t))} g_t\right) &= P\left(N_{t-\phi r(t)} \leq \left\lfloor e^{\beta(t-\phi r(t))} g_t \right\rfloor\right) \\ &\leq \left\lfloor e^{\beta(t-\phi r(t))} g_t \right\rfloor e^{-\beta(t-\phi r(t))} \leq g_t. \end{aligned} \tag{13}$$

This bounds the second term on the right-hand side of equation (12) from above. Given that $\phi_2 > 0$ is fixed and assuming that $r(t) = o(\sqrt{t})$, it is clear that there exists a $t_0 > 0$ such that $\lfloor e^{\beta(t-\phi_2 r(t))} g_t \rfloor \geq 1$ and $t - \phi_2 r(t) > 0$ for all $t \geq t_0$. Fix this t_0, and for $t > 0$, define the law $\widetilde{P}_t$ via

$$\widetilde{P}_t(\,\cdot\,) = P^{\delta r(t)}\Big(\,\cdot\,\Big|\, N_{t-\phi r(t)} > e^{\beta(t-\phi r(t))} g_t\Big).$$

We now bound the first term on the right-hand side of (12) from above. Let $\widetilde{E}_t$ denote the expectation corresponding to $\widetilde{P}_t$ and $\mathcal{N}_t$ represent the set of particles belonging to Z that are alive at time t. Observe that $N_t = |\mathcal{N}_t|$. Let us denote the ancestral line of particle u until time t as $(Y_u(s))_{0\leq s\leq t}$ for $u \in \mathcal{N}_t$. We randomly choose $\lfloor e^{\beta(t-\phi r(t))} g_t \rfloor$ particles from $\mathcal{N}_t$, conditional on the event $\{N_{t-\phi r(t)} > e^{\beta(t-\phi r(t)))} g_t\}$. This choice is independent of their genealogy and position. Set $M_t := \lfloor e^{\beta(t-\phi r(t))} g_t \rfloor$. Let us denote by $\mathcal{M}_t$ this collection of particles, so that $M_t = |\mathcal{M}_t|$. We now define

$$\hat{n}_t = \sum_{u\in\mathcal{M}_t} \mathbb{1}_{A_u},$$

where $A_u = \{Y_u(s) \in B_t \quad \forall\ 0 \leq s \leq t - \phi r(t)\}$. Observe that $\hat{n}_t$ counts the particles out of $\mathcal{M}_t$ whose ancestral lines stay within B_t over $[0, t]$. Each particle in the collection $\mathcal{M}_t$ has a Brownian ancestral line $(Y_u(s))_{0\leq s\leq t}$ since $\mathcal{M}_t$ was selected independently of the motion process. The many-to-one lemma then implies that for $t > 0$,

$$\widetilde{E}_t[\hat{n}_t] = \widetilde{p}_t M_t = \widetilde{p}_t \big\lfloor e^{\beta(t-\phi r(t))} g_t \big\rfloor, \tag{14}$$

where $\widetilde{p}_t = P^{\delta r(t)}(\sigma_{B_t} \geq t - \phi r(t))$. Observe that $\hat{n}_t \leq n_t$. Then, we use Chebyshev's inequality together with (12), (13), and (14) to obtain

$$
\begin{aligned}
P^{\delta r(t)}(n_{t-\phi r(t)} < \gamma_t \widetilde{p}_t e^{\beta(t-\phi r(t))}) &\leq \widetilde{P}_t\left(\hat{n}_t < \gamma_t \widetilde{p}_t e^{\beta(t-\phi r(t))}\right) + g_t \\
&\leq \widetilde{P}_t\left(\widetilde{E}_t[\hat{n}_t] - \hat{n}_t > \widetilde{E}_t[\hat{n}_t] - \gamma_t \widetilde{p}_t e^{\beta(t-\phi r(t))}\right) + g_t \\
&\leq \widetilde{P}_t\left(|\hat{n}_t - \widetilde{E}_t[\hat{n}_t]| > \widetilde{p}_t \lfloor e^{\beta(t-\phi r(t))} g_t \rfloor - \gamma_t \widetilde{p}_t e^{\beta(t-\phi r(t))}\right) + g_t \\
&\leq \frac{\widetilde{\mathrm{Var}}_t(\hat{n}_t)}{[(g_t - \gamma_t)\,\widetilde{p}_t e^{\beta(t-\phi r(t))} - \widetilde{p}_t]^2} + g_t. \qquad (15)
\end{aligned}
$$

Let the variance associated to $\widetilde{P}_t$ be $\widetilde{\mathrm{Var}}_t$, and let $A = \{X(s) \in B_t \ \ \forall\, 0 \leq s \leq t\}$. Now, we estimate $\widetilde{\mathrm{Var}}_t(\hat{n}_t)$. We start with

$$
\begin{aligned}
\widetilde{\mathrm{Var}}_t(\hat{n}_t) &= \widetilde{\mathrm{Var}}_t\left(\sum_{u \in \mathcal{M}_t} \mathbb{1}_{A_u}\right) \\
&= M_t \mathrm{Var}(\mathbb{1}_A) + \sum_{1 \leq i \neq j \leq M_t} \widetilde{\mathrm{Cov}}_t(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}) \\
&= M_t(\widetilde{p}_t - \widetilde{p}_t^2) + M_t(M_t - 1)\frac{\sum_{1 \leq i \neq j \leq M_t} \widetilde{\mathrm{Cov}}_t(\mathbb{1}_{A_i}, \mathbb{1}_{A_j})}{M_t(M_t-1)} \\
&\leq g_t e^{\beta(t-\phi r(t))}(\widetilde{p}_t - \widetilde{p}_t^2) + g_t e^{\beta(t-\phi r(t))}(g_t e^{\beta(t-\phi r(t))} - 1)\left[(\mathcal{E} \otimes \widetilde{P}_t)(A_i \cap A_j) - \widetilde{p}_t^2\right].
\end{aligned}
\qquad (16)
$$

The probability of randomly selecting the pair (i, j) from the $M_t(M_t - 1)$ possible pairs in $\mathcal{M}_t$ is denoted by $\mathcal{P}$, and the corresponding expectation is denoted by $\mathcal{E}$. That is, the average value of $\widetilde{P}_t(A_i \cap A_j)$ from the $M_t(M_t - 1)$ possible pairs in the randomly selected set $\mathcal{M}_t$ is denoted by $(\mathcal{E} \otimes \widetilde{P}_t)(A_i \cap A_j)$. Let us also denote the distribution of the splitting time of the ith and jth particles most recent common ancestor under $\mathcal{E} \otimes \widetilde{P}_t$ by $Q^{(t)}$. Define

$$
\widetilde{p}^{(t)}(x, s, dy) := \mathbf{P}_x(X(s) \in \ dy \mid X(z) \in B_t \ \ \forall\, 0 \leq z \leq s), \qquad (17)
$$

$$
p^t_{s,x} := \mathbf{P}_x(\sigma_{B_t} \geq s). \qquad (18)
$$

When the Markov property is used at the aforementioned splitting time, we have

$$
(\mathcal{E} \otimes \widetilde{P}_t)(A_i \cap A_j) = \widetilde{p}_t \int_0^{t-\phi r(t)} \int_{B_t} \widetilde{p}_{t-\phi r(t)-s,x}\widetilde{p}^{(t)}(\delta r(t)\hat{e}, s, dx) Q^{(t)}(ds), \qquad (19)
$$

where $\widetilde{p}^{(t)}(x, s, dy)$ and $\widetilde{p}_{s,\delta r(t)}$ are defined in (17) and (18), and $\hat{e}$ is a unit vector in $\mathbb{R}^d$. When the Markov property of a standard BM is used at time s with $0 < s < t$, we obtain

$$\widetilde{p}_t = p^t_{s,\delta r(t)\hat{e}} \int_{B_t} p^t_{t-\phi r(t)-s,y} \widetilde{p}^{(t)}(\delta r(t)\hat{e}, s, dy). \tag{20}$$

For simplicity, we will use as $\widetilde{p}_{s,\delta r(t)} = \widetilde{p}_{s,\delta r(t)\hat{e}}$ from now on. Then, it follows from (19) and (20) that

$$(\mathcal{E} \otimes \widetilde{P}_t)(A_i \cap A_j) = \widetilde{p}_t^2 \int_0^{t-\phi r(t)} \frac{1}{p^t_{s,\delta r(t)}} Q^{(t)}(ds). \tag{21}$$

For $t > 0$, we define

$$J_t := \int_0^{t-\phi r(t)} \frac{1}{p^t_{s,\delta r(t)}} Q^{(t)}(ds).$$

Then, by (16) and (21), we obtain for all large t,

$$\widetilde{\mathrm{Var}}_t(\hat{n}_t) \leq g_t \widetilde{p}_t e^{\beta(t-\phi r(t))} + g_t^2 \widetilde{p}_t^2 e^{2\beta(t-\phi r(t))}(J_t - 1). \tag{22}$$

Observe that $J_t - 1 \geq 0$. Now, we bound $J_t - 1$ from above. Recall that $r(t)$ is a distance scale. We will utilize $kr(t)$ as a time scale for fixed $k > 0$. It is worth noting that for large t, a Brownian motion starting at the origin is unlikely to escape $B_t = B(0, \delta r(t))$ during the interval $[0, kr(t)]$. For t large enough so that $t - \phi_2 r(t) > kr(t)$, we split J_t as

$$J_t = \int_0^{kr(t)} \frac{1}{p^t_{s,\delta r(t)}} Q^{(t)}(ds) + \int_{kr(t)}^{t-\phi r(t)} \frac{1}{p^t_{s,\delta r(t)}} Q^{(t)}(ds),$$

and define

$$J_t^{(1)} := \int_0^{kr(t)} \frac{1}{p^t_{s,\delta r(t)}} Q^{(t)}(ds), \qquad J_t^{(2)} := \int_{kr(t)}^{t-\phi r(t)} \frac{1}{p^t_{s,\delta r(t)}} Q^{(t)}(ds).$$

Now, we will use that the integrand $1/p^t_{s,\delta r(t)}$ is small enough over the first time interval $[0, kr(t)]$. The distribution of $Q^{(t)}$ puts a small weight on the second time interval $[kr(t), t - kr(t)]$.

We first bound $J_t^{(1)} - 1$ from above. Observe that $p^t_{s,\delta r(t)}$ is nonincreasing in s, by which we estimate $J_t^{(1)}$ as

$$J_t^{(1)} = \int_0^{kr(t)} \frac{1}{p^t_{s,\delta r(t)}} Q^{(t)}(ds) \le \frac{1}{p^t_{kr(t),\delta r(t)}}. \tag{23}$$

From Proposition 2.1,

$$1 - p^t_{kr(t),\delta r(t)} = \exp\left[-\frac{(1-\delta)^2 r(t)}{2k}(1+o(1))\right]. \tag{24}$$

It then follows from (23) and (24) that

$$\begin{aligned} J_t^{(1)} - 1 &\le \frac{1 - p^t_{kr(t),\delta r(t)}}{p^t_{kr(t),\delta r(t)}} \\ &\le \exp\left[-\frac{(1-\delta)^2 r(t)}{2k}(1+o(1))\right]. \end{aligned} \tag{25}$$

We now bound the $J_t^{(2)}$ from above. We will utilize the following fact on the distribution $Q^{(t)}$ from [[17], Prop. 5] to bound $J_t^{(2)}$ from above: $Q^{(t)}$ is absolutely continuous with respect to the Lebesgue measure, denoted by ds, and its density function, denoted by $g^{(t)}$, satisfies

$$\exists\, C > 0,\ s_0 > 0 \ \text{ such that } \ \forall\, s \ge s_0,\ g^{(t)}(s) \le Cse^{-\beta s}. \tag{26}$$

Recall that λ_d denotes the principal Dirichlet eigenvalue of $(-1/2)\Delta$ on the unit ball in d dimensions. It is also clear that $0 < \delta < 1$, implying that the motion begins at a distance of δ from the origin. $D = D(\delta, d)$ is a positive constant that is dependent on dimension and δ. It follows from [[13],(2.8)] (since $2\lambda_d = j^2_{\nu,1}$ therein),

$$p^t_{s,\delta r(t)} = \mathbf{P}_{\delta r(t)}(\sigma_{B_t} \ge s) = \mathbf{P}_\delta\left(\sigma_{B(0,1)} \ge \frac{s}{(r(t))^2}\right) \ge De^{-\frac{\lambda_d s}{(r(t))^2}}.$$

Then,

$$\begin{aligned} J_t^{(2)} = \int_{kr(t)}^{t-\phi r(t)} \frac{1}{p^t_{s,\delta r(t)}} Q^{(t)}(ds) &\le \int_{kr(t)}^{t-\phi r(t)} \left(\frac{1}{D}\right) e^{-\frac{\lambda_d s}{(r(t))^2}} Cse^{-\beta s} ds \\ &\le \frac{C}{D}\int_{kr(t)}^{\infty} s \exp\left[-\left(\beta - \frac{\lambda_d}{(r(t))^2}\right)\right] ds. \end{aligned}$$

Let $\left(\beta - \frac{\lambda_d}{(r(t))^2}\right) := A$ and $\beta > \frac{\lambda_d}{(r(t))^2}$. Taking $s = u, dv = \exp(-As)ds$, $ds = du, v = \frac{\exp(-As)}{-A}$, we have

$$
\begin{aligned}
uv - \int vdu &= s\frac{\exp(-As)}{-A} - \int_{kr(t)}^{\infty} \frac{\exp(-As)}{-A} ds \\
&= s\frac{\exp(-As)}{-A} - \frac{\exp(-As)}{A^2}\bigg|_{kr(t)}^{\infty} \\
&= \frac{C}{DA} e^{-Akr(t)} \left(kr(t) + \frac{1}{A}\right) \qquad (27)
\end{aligned}
$$

where we have used integration by parts. It then follows from (25) and (27), that

$$
J_t - 1 = J_t^{(1)} - 1 + J_t^{(2)} \leq \exp\left[-\frac{(1-\delta)^2 r(t)}{2k}(1+o(1))\right] + \exp[-\beta kr(t)(1+o(1))]. \quad (28)
$$

To optimize the smallest exponent on the right-hand side of (28), choose k so that

$$
\frac{(1-\delta)^2 r(t)}{2k} = \beta kr(t).
$$

This yields $k = \frac{(1-\delta)}{\sqrt{2\beta}}$. With this choice of k, we obtain

$$
J_t - 1 \leq \exp\left[-\sqrt{\frac{\beta}{2}}(1-\delta)(1+o(1))\right].
$$

It then follows from (15) and (22) that

$$
\begin{aligned}
P^{\delta r(t)}(n_{t-\phi r(t)} < \gamma_t \widetilde{p}_t e^{\beta(t-\phi r(t))}) &\leq \frac{\widetilde{Var}_t(\widehat{n}_t)}{[(g_t - \gamma_t)\widetilde{p}_t e^{\beta(t-\phi r(t))} - \widetilde{p}_t]^2} + g_t \\
&\leq \frac{g_t \widetilde{p}_t e^{\beta(t-\phi r(t))} + g_t^2 \widetilde{p}_t^2 e^{2\beta(t-\phi r(t))}(J_t - 1)}{[(g_t - \gamma_t)\widetilde{p}_t e^{\beta(t-\phi r(t))} - \widetilde{p}_t]^2} + g_t \\
&\leq \frac{2g_t \widetilde{p}_t}{(g_t - \gamma_t)^2 \widetilde{p}_t^2} e^{-\beta(t-\phi r(t))} + \frac{2g_t^2 \widetilde{p}_t^2 (J_t - 1)}{(g_t - \gamma_t)^2 \widetilde{p}_t^2} + g_t \\
&\leq \frac{2g_t}{(g_t - \gamma_t)^2 \widetilde{p}_t} e^{-\beta(t-\phi r(t))} + \\
&\quad \frac{2g_t^2}{(g_t - \gamma_t)^2} e^{-\sqrt{\beta/2}(1-\delta) r(t)(1+o(1))} + g_t. \qquad (29)
\end{aligned}
$$

Given the assumption that $r(t) \to \infty$ and $r(t) = o(\sqrt{t})$ as $t \to \infty$, we can continue (29) with

$$
P^{\delta r(t)}(n_{t-\phi r(t)} < \gamma_t \widetilde{p}_t e^{\beta(t-\phi r(t))}) \leq \frac{4}{\gamma_t \widetilde{p}_t} e^{-\beta(t-\phi r(t))} + 8e^{-\sqrt{\beta/2}(1-\delta) r(t)(1+o(1))} + g_t. \quad (30)
$$

Recall that $\widetilde{p}_t \to 0$ as $t \to \infty$ and we choose $g_t = 2\gamma_t = 2e^{-\kappa r(t)}$ with $\kappa > 0$. Since $\phi \in [\phi_1, \phi_2]$, we can continue with

$$\begin{aligned} P^{\delta r(t)}\big(n_{t-\phi r(t)} < \gamma_t \widetilde{p}_t e^{\beta(t-\phi r(t))}\big) &\leq \frac{4}{\gamma_t \widetilde{p}_t} e^{-\beta(t-\phi r(t))} + 8e^{-\sqrt{\beta/2}(1-\delta)r(t)(1+o(1))} + g_t \\ &\leq 11e^{(-k(\delta,\beta)\wedge\kappa)r(t)}. \end{aligned}$$

This completes the proof of the Proposition 2.8.

$\square$

CHAPTER III

LARGE DEVIATIONS IN SUBDIFFUSIVELY EXPANDING BALL

3.1 *Proof of Theorem 1.1*

In this section, we present the proof of the main theorem. The proof of the lower bound was given in [18]. We review it for completeness.

3.1.1 Proof of the lower bound

Recall that $B = (B_t)_{t\geq 0}$ with $B_t = B(0, r(t))$, and $n_t := |Z_t^{B_t}|$ is the mass of a BBM at time t that has been deactivated at ∂B, and $p_t = \mathbf{P}_0(\sigma_{B_t} \geq t)$, and $\gamma_t = e^{-\kappa r(t)}$, where $\kappa > 0$.

The proof of the lower bound of Theorem 1.1 is based on finding an optimal strategy to realize the event $\{n_t < \gamma_t p_t e^{\beta t}\}$. There are two cases: $0 < \kappa \leq \sqrt{2\beta}$ and $\kappa > \sqrt{2\beta}$. The strategy of the lower bound presented for $\{n_t < \gamma_t p_t e^{\beta t}\}$ and $\{n_t = 0\}$ turn out to be the same when $\kappa > \sqrt{2\beta}$, but different when $0 < \kappa \leq \sqrt{2\beta}$.

Start with estimate

$$P(n_t < \gamma_t p_t e^{\beta t}) \geq P(n_t = 0).$$

To realize $\{n_t = 0\}$, it is enough to completely suppress branching and move the initial particle out of $B_t := B(0, r(t))$ up to time $kr(t)$. Let $k > 0$ be a constant and note that $r(t) < t$ for all large t. The probability of realizing the aforementioned strategy is

$$\exp\left[-\beta k r(t) - \frac{r(t)}{2k}(1 + o(1))\right]. \tag{31}$$

In (31), the first term corresponds to suppressing the branching, and the second term corresponds to a linear Brownian displacement. To minimize the absolute value of the exponent, we find the critical value of k by setting $\beta k r(t) = r(t)/(2k)$. This yields $k = 1/(\sqrt{2\beta})$ and when we substitute this value of k in (31), we arrive at

$$P(n_t < \gamma_t p_t e^{\beta t}) \geq P(n_t = 0) \geq \exp\Big[- \sqrt{2\beta} r(t)(1 + o(1))\Big]. \tag{32}$$

We will define $f : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $f(t) = o(t)$.The path of the initial particle and the particle's lifetime are denoted by τ_1 and $(Y_1(s))0 \leq s \leq \tau 1$, respectively. Then, $\tau_1 \geq f(t)$. Furthermore, define for $t > 0$ the events

$$E_t = \{n_t < \gamma_t p_t e^{\beta t}\}, \quad A_t = \{N_{f(t)} = 1\}, \quad D_t = \{Y_1(z) \in B_t \; \forall \; 0 \leq z \leq f(t)\}.$$

We continue with estimate

$$P(E_t) \geq P(E_t \cap A_t) = P(E_t \,|\, A_t) P(A_t). \tag{33}$$

Now, we will show that $P(E_t^c \,|\, A_t)$. Conditional on A_t, it is clear that $0 \leq z \leq f(t)$ due to $Y_1(z) \notin B_t$ for some $z \in [0, f(t)]$. Then,

$$E[n_t \,|\, A_t] = E[n_t \mathbb{1}_{D_t} \,|\, A_t] + E[n_t \mathbb{1}_{D_t^c} \,|\, A_t] = E[n_t \,|\, A_t, D_t] P(D_t \,|\, A_t), \tag{34}$$

where the second term in the middle expression vanishes because on the event D_t^c, and $n_t = 0$. We have,

$$\begin{aligned} E[n_t \,|\, A_t, D_t] &= \int_{B_t} E[n_t \,|\, A_t, D_t, Y_1(f(t)) = y] P(Y_1(f(t)) \in dy \,|\, A_t, D_t) \\ &= \int_{B_t} E[n_t \,|\, A_t, D_t, Y_1(f(t)) = y] \widetilde{p}^{(t)}(0, f(t), dy), \end{aligned} \tag{35}$$

and

$$P(D_t \,|\, A_t) = \widetilde{p}_{f(t),0}. \tag{36}$$

where $\widetilde{p}^{(t)}(x, s, dy)$ and $p_{s,\delta r(t)}$ are as defined in (17) and (18). On the other hands,

$$p_t = p_{s,0} \int_{B_t} p_{t-s,y} p^{(t)}(0, s, dy) \tag{37}$$

is obtained by applying the Markov property of a standard Brownian motion at time s when $0 < s < t$. Additionally, replacing s by $f(t)$ in (37), we obtained from (34), (35), and (36) that

$$E[n_t \,|\, A_t] = p_{f(t),0} \int_{B_t} E[n_t \,|\, A_t, D_t, Y_1(f(t)) = y] p^{(t)}(0, f(t), dy) \tag{38}$$

Now, we use the many-to-one lemma (we refer to reader to [19, Lemma 1.6] for instance) and apply the Markov property of BBM at time $f(t)$ to arrive at

$$E[n_t \,|\, A_t, D_t, Y_1(f(t)) = y] = p_{t-f(t),y} e^{\beta(t-f(t))}, \quad y \in B_t. \tag{39}$$

From (37), (38) and (39), we have

$$\begin{aligned} E[n_t \,|\, A_t] &= e^{\beta(t-f(t))} p_{f(t),0} \int_{B_t} p_{t-f(t),y} p^{(t)}(0, f(t), dy) \\ &= e^{\beta(t-f(t))} p_t. \end{aligned}$$

From the Markov inequality, we obtain

$$P(E_t^c \,|\, A_t) \leq \frac{E[n_t \,|\, A_t]}{\gamma_t p_t e^{\beta t}} = \gamma_t^{-1} e^{-\beta f(t)}. \tag{40}$$

Choosing $f(t) = -(1/\beta)\log((1-\delta)\gamma_t)$ with $0 < \delta < 1$. Given this f, (40) implies that $P(E_t \,|\, A_t) \geq \delta$. It then follows from (33), we obtain

$$P(E_t) \geq \delta e^{-\beta f(t)} = \delta(1-\delta)\gamma_t = e^{-\kappa r(t)(1+o(1))},$$

due to $P(A_t) = e^{-\beta f(t)}$. This completes the proof of the lower bound of Theorem 1.1.

3.1.2 Proof of the upper bound

For the proof of the upper bound, we use a bootstrap argument, similar to the one used in [14]. We divide the time interval $[0, (\kappa/\beta)r(t)]$ into n pieces, and condition the process on in which piece $(\kappa/\beta)r(t)$ falls. The branching is suppressed to produce only a polynomial number of particles, which causes an exponential probability cost. On the other hand, because there aren't exponentially many particles, we can keep them all in close proximity to the origin without any additional cost over $[0, (\kappa/\beta)r(t)]$.

To find the optimal value of ρ_t, we first divide the time interval $[0, t]$ into two parts. The first part is the interval $[0, (\kappa/\beta)r(t)]$ interval and the second part is $[(\kappa/\beta)r(t), t]$. We then divide the interval $[0, (\kappa/\beta)r(t)]$ into n pieces of equal length, where n is taken to be large.

Recall that $N_t := Z_t(\mathbb{R}^d)$ is the total mass at time t, $\gamma_t = e^{-\kappa r(t)}$ and $p_t = \mathbf{P}_0(\sigma_{B_t} \geq t)$. For $t > 0$, define the random variable

$$\rho_t = \sup\{\rho \in [0,1] : N_{\rho(\kappa/\beta)r(t)} \leq \lfloor r(t) \rfloor\}.$$

Notice that for $x \in [0,1]$, we have $\{\rho_t \geq x\} \subseteq \{N_{x(\kappa/\beta)r(t)} \leq \lfloor r(t) \rfloor + 1\}$. For $t > 0$, define the event

$$A_t = \{n_t < \gamma_t p_t e^{\beta t}\}.$$

We start by conditioning on ρ_t. For every $n \geq 2$,

$$\begin{aligned} P(A_t) &= \sum_{i=0}^{n-2} P\left(A_t \cap \left\{\frac{i}{n} \leq \rho_t < \frac{i+1}{n}\right\}\right) + P\left(A_t \cap \left\{\rho_t \geq 1 - \frac{1}{n}\right\}\right) \\ &\leq \sum_{i=0}^{n-2} \exp\left[-\frac{i}{n}\kappa r(t) + o(r(t))\right] P_t^{(i,n)}(A_t) + \\ &\qquad \exp\left[-\kappa r(t)\left(1 - \frac{1}{n}\right) + o(r(t))\right], \end{aligned} \tag{41}$$

where we have used Proposition 2.1, which yields $P(N_{(i/n)(\kappa/\beta)r(t)} \leq \lfloor r(t) \rfloor + 1) = \exp[-(i/n)\kappa r(t) + o(r(t))]$, to bound $P(\frac{i}{n} \leq \rho_t < \frac{i+1}{n})$ and $P(\rho_t \geq 1 - 1/n)$ from above. Now, we introduce the conditional probabilities

$$P_t^{(i,n)}(\,\cdot\,) = P\left(\,\cdot\,\middle|\, \frac{i}{n} \leq \rho_t < \frac{i+1}{n}\right), \quad i = 0, 1, 2, \ldots, n-2.$$

Define the interval

$$I^{(i,n)} := \left[\frac{i}{n}, \frac{i+1}{n}\right)$$

for each pair (i,n) with $n \geq 2$ and $i = 0, 1, \ldots, n-2$. According to the definition of ρ_t and conditional on the event $\rho_t \in I^{(i,n)}$, there is a moment in $[(i/n)(\kappa/\beta)r(t), (i + 1/n)(\kappa/\beta)r(t)]$, that is $\rho_t t$, where the system has precisely $\lfloor r(t) \rfloor + 1$ particles. Let E_t be the event that there is at least one particle outside $B(0, \delta r(t))$ at some instant s, where $0 \leq s \leq \rho_t(\kappa/\beta)r(t)$. We continue with the estimate

$$\begin{aligned} P_t^{(i,n)}(A_t) &= P_t^{(i,n)}(A_t \cap E_t^c) + P_t^{(i,n)}(A_t \cap E_t) \\ &\leq P_t^{(i,n)}(A_t \mid E_t^c) P_t^{(i,n)}(E_t^c) + P_t^{(i,n)}(E_t) \\ &\leq P_t^{(i,n)}(A_t \mid E_t^c) + P_t^{(i,n)}(E_t). \end{aligned} \tag{42}$$

We first consider the second term on the right-hand side of (42). The second term on the right-hand side of (42) can be estimated via Proposition 2.1. According to the definition of ρ_t the system has precisely $\lfloor r(t) \rfloor + 1$ particles present at time $\rho_t(\kappa/\beta)r(t)$. Moreover, under the law $P_t^{(i,n)}$, $\rho_t < (i+1)/n$. Therefore, by Proposition 2.1 and the union bound, it follows that

$$\begin{aligned} P_t^{(i,n)}(E_t) &\leq (\lfloor r(t) \rfloor + 1)\mathbf{P}_0\left(\sup_{0 \leq s \leq \frac{i+1}{n}\frac{\kappa}{\beta}r(t)} |X(s)| > \delta r(t) \right) \\ &= \exp\left[-\frac{1}{2}\left(\frac{\delta r(t)}{\frac{i+1}{n}\frac{\kappa}{\beta}r(t)} \right)^2 \frac{i+1}{n}\frac{\kappa}{\beta} r(t)(1 + o(1)) \right] \\ &= \exp\left[-\frac{1}{2}\left(\frac{\delta \beta n}{(i+1)\kappa} \right)^2 \frac{i+1}{n}\frac{\kappa}{\beta} r(t)(1 + o(1)) \right] \\ &= \exp\left[-\frac{\delta^2 n \beta r(t)}{2(i+1)\kappa}(1 + o(1)) \right]. \end{aligned} \tag{43}$$

We now bound the first term on the right-hand side of (42) from above. Conditional on $\{\frac{i}{n} \leq \rho_t \leq \frac{i+1}{n}\} \cap E_t^c$, during the time interval $[(i/n)(\kappa/\beta)r(t), ((i+1)/n)(\kappa/\beta)r(t)]$ no particle cannot leave the $B(0, \delta r(t))$. In this time interval, there are exactly $\lfloor r(t) \rfloor + 1$ particles. All these particles have ancestral lines confined to $B(0, \delta r(t))$ over the interval $[0, \rho_t(\kappa/\beta)r(t)]$.

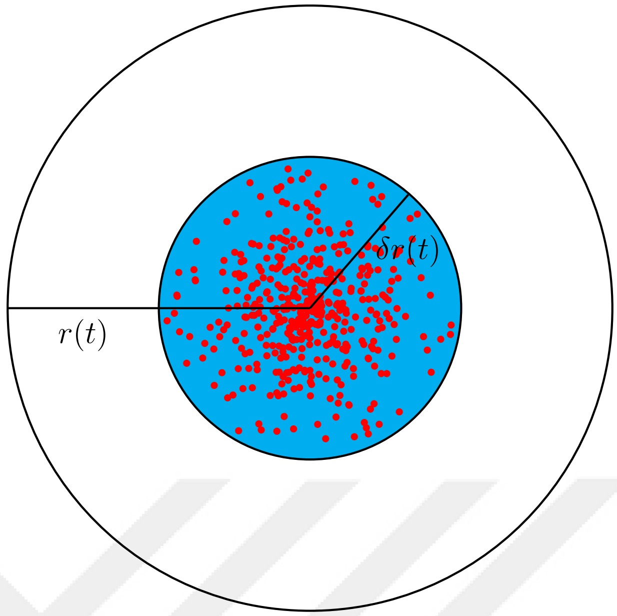

Figure 3: This is a snapshot of the branching Brownian motion at time $\rho_t(\kappa/\beta)r(t)$, conditional on the event of E_t^c. The radius of the large ball is $r(t)$, and the radius of the small ball is $\delta r(t)$ with $0 < \delta < 1$. The red dots represent the particles whose ancestral lines have been confined to the small ball over $[0, \rho_t(\kappa/\beta)r(t)]$.

Branching Brownian motion can be described from this snapshot of the process in Figure 4. Each of these particles initiates a sub-BBM from within $B(0, \delta r(t))$ at time $\rho_t(\kappa/\beta)r(t)$. Each of these sub-BBMs continue to evolve until time t.

For an upper bound on $P_t^{(i,n)}(A_t \mid E_t^c)$ in (42), we assume the "worst case scenario", that is, all of the particles in $B(0, \delta r(t))$ are on the boundary of $B(0, \delta r(t))$ at time $\rho_t(\kappa/\beta)r(t)$. The distance between $B(0, \delta r(t))$ and the starting point of the sub-BBM is too short for sub-BBM to send less than $e^{\beta t}$ particles to $B(0, \delta r(t))$ in the remaining time, which corresponding to $((i+1)/n(\kappa/\beta)r(t))$. Furthermore, the strong Markov property of the BBM at time $\rho_t(\kappa/\beta)r(t)$, and the independence of the sub-BBMs started at that time, gives the estimate

$$P_t^{(i,n)}(A_t \mid E_t^c) \leq \left(\sup_{\frac{i}{n} \leq x \leq \frac{i+1}{n}} \left[P^{\delta r(t)}(n_{t-x(\kappa/\beta)r(t)} < \gamma_t p_t e^{\beta t}) \right] \right)^{\lfloor r(t) \rfloor}. \tag{44}$$

Now, we find a convenient upper bound for $P^{\delta r(t)}(n_{t-x(\kappa/\beta)r(t)} < \gamma_t p_t e^{\beta t})$ which holds uniformly over the interval $x \in [i/n, (i+1)/n]$. Assume that $0 \leq \phi_1 < \phi_2$. Given $\kappa(1-(\phi_2/\kappa)\beta) > 0$, there exists $t_0 > 0$ such that for all $t \geq t_0$ and $\phi \in [\phi_1, \phi_2]$, we may bound $\gamma_t p_t e^{\beta t}$ from above as

$$
\begin{aligned}
\gamma_t p_t e^{\beta t} &= e^{-\kappa r(t)} \mathbf{P}_0(\sigma_{B_t} \geq t) e^{\beta t} \\
&= e^{-\kappa r(t)} \mathbf{P}_0(\sigma_{B_t} \geq t) e^{\beta t} e^{-\beta\phi r(t)} e^{\beta\phi r(t)} \\
&= e^{-\kappa r(t)(1-(\phi/\kappa)\beta)} \mathbf{P}_0(\sigma_{B_t} \geq t) e^{\beta(t-\phi r(t))} \\
&\leq D \overline{\gamma}_t \mathbf{P}_{\delta r(t)}(\sigma_{B_t} \geq t) e^{\beta(t-\phi r(t))} \\
&\leq D \overline{\gamma}_t \mathbf{P}_{\delta r(t)}(\sigma_{B_t} \geq t - \phi r(t)) e^{\beta(t-\phi r(t))} \\
&\leq D \overline{\gamma}_t \mathbf{P}_{\delta}\left(\sigma_{B(0,1)} \geq \frac{t-\phi r(t)}{(r(t))^2}\right) e^{\beta(t-\phi r(t))} \\
&\leq e^{-\kappa r(t)(1-(\phi_2/\kappa)\beta)/2} \widetilde{p}_t e^{\beta(t-\phi r(t))} \\
&\leq \widetilde{\gamma}_t \widetilde{p}_t e^{\beta(t-\phi r(t))},
\end{aligned}
\tag{45}
$$

where we have defined $\bar{\gamma}_t := e^{-\kappa r(t)(1-(\phi/\kappa)\beta)}$ and $D = D(\delta, d)$ is a positive constant. We used Proposition 2.4 in the first inequality, and used the monotonicity of $\mathbf{P}(\sigma_{B_t} \geq s)$ according to s in the second inequality, and used Brownian scaling in the third inequality. Then, we replaced $\widetilde{\gamma}_t := e^{-\kappa r(t)(1-(\phi_2/\kappa)\beta)/2}$ in the final inequality. We now set $\phi_1 = (i/n)(\kappa/\beta)$ and $\phi_2 = ((i+1)/n)(\kappa/\beta)$. We will replace γ_t with $\widetilde{\gamma}t$ in (11) from Proposition 2.8 and set $x = \phi\beta/\kappa$. In this situation, there exists a positive constant $c = c(\delta, \kappa, \beta) > 0$ and $t_1 > 0$. Then, according to (45) and (11) with ϕ_1 and ϕ_2 such that for all large t and $t \geq t_1$,

$$
P^{\delta r(t)}\left(n_{t-x(\kappa/\beta)r(t)} < \gamma_t p_t e^{\beta t}\right) \leq P^{\delta r(t)}\left(n_{t-x(\kappa/\beta)r(t)} < \widetilde{\gamma}_t \widetilde{p}_t e^{\beta(t-x(\kappa/\beta)r(t))}\right) \leq e^{-cr(t)}.
\tag{46}
$$

From (42),(43) and (46), for all large t we obtain

$$
P_t^{(i,n)}(A_t) \leq \exp\left[-\frac{\delta^2 n\beta r(t)}{2(i+1)\kappa}(1+o(1))\right] + \left(e^{-cr(t)}\right)^{\lfloor r(t) \rfloor}.
\tag{47}
$$

The first term on the right hand-side of (47) is the dominating term for large t. Substituting (47) into (41), and optimizing over $i \in \{0, 1, \ldots, n-2\}$ results in

$$\limsup_{t\to\infty} \frac{1}{r(t)} \log P(A_t) \le -\beta\left[\min_{i\in\{0,1,\ldots,n-2\}} \left\{\frac{i}{n}\frac{\kappa}{\beta} + \frac{\delta^2}{2\kappa(i+1)/n}\right\} \wedge \frac{\kappa}{\beta}\left(1-\frac{1}{n}\right)\right]. \tag{48}$$

Let $\delta \to 1$ on the right-hand side of (48), and then set $\rho = i/n$, we obtain

$$\limsup_{t\to\infty} \frac{1}{r(t)} \log P(A_t) \le -\beta\left[\min_{\rho\in\{0,1/n,\ldots,(n-2)/n\}} \left\{\frac{\rho\kappa}{\beta} + \frac{1}{2\kappa\rho + 2\kappa/n}\right\} \wedge \frac{\kappa}{\beta}\left(1-\frac{1}{n}\right)\right].$$

The value of ρ is divided into n equal pieces between 0 and 1. Let $n \to \infty$, the interval of ρ becomes continuous and turns into the interval $(0, 1]$. Infimum is used when the minimum of a continuity function is taken. Therefore, we use the continuity of the functional form which the minimum is taken to obtain

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{r(t)} \log P(A_t) &\le -\beta\left[\min_{\rho\in\{0,1/n,\ldots,(n-2)/n\}} \left\{\frac{\rho\kappa}{\beta} + \frac{1}{2\kappa\rho + 2\kappa/n}\right\} \wedge \frac{\kappa}{\beta}\left(1-\frac{1}{n}\right)\right] \\ &\le -\beta\left[\inf_{\rho\in(0,1]} \left\{\frac{\rho\kappa}{\beta} + \frac{1}{2\kappa\rho}\right\} \wedge \frac{\kappa}{\beta}\right].\end{aligned} \tag{49}$$

Define the function f by

$$f(\rho) = \frac{\rho\kappa}{\beta} + \frac{1}{2\kappa\rho}$$

for $\rho \in (0, 1]$.

To find the minimum of f over $(0, 1]$. This gives the minimal value, which depends on the take by the possible values of ρ over $(0, 1]$. There are two cases for the second term on the right-hand side in (49) to find the minimum of κ/β. For $\kappa > \sqrt{2\beta}$, f is minimized at $\bar{\rho} = \sqrt{\beta/(2\kappa^2)}$ in the interval $(0, 1]$, where $0 < \bar{\rho} < 1/2$. The minimum value of f is $\sqrt{2/\beta}$. In this case, the first term is the minimum value because it was smaller than the second term. For $\kappa \le \sqrt{2\beta}$ the minimum value of f is $\sqrt{2/\beta}$. In this case, the second term gives the minimum value in (49) since $\kappa \le \sqrt{2\beta}$. That is, due to $\kappa/\beta \le \sqrt{2/\beta}$, the second term on the right-hand side under the minimum in (49) gives the output of this minimum. Hence,

$$\limsup_{t\to\infty} \frac{1}{r(t)} \log P(A_t) \leq -(\kappa \wedge \sqrt{2\beta}).$$

This completes the proof of the Theorem 1.1.

$\square$

CHAPTER IV

GROWTH OF BRANCHING BROWNIAN MOTION IN A FIXED BALL

4.1 Growth of a Branching Brownian Motion in a Fixed Ball

In this section, we consider the growth of the BBM in a fixed ball. We present a result on the probability of the event $\{n_t < \gamma_t p_t e^{\beta t}\}$ for large t in the case of a fixed ball $B(0, R)$, where $R > 0$ is a constant. For any $\varepsilon > 0$, we will find that there exists $R = R(\varepsilon) > 0$ sufficiently large such that $P(n_t < \gamma_t p_t e^{\beta t}) \leq \varepsilon$, but $P(n_t < \gamma_t p_t e^{\beta t})$ does not tend to 0 as $t \to \infty$.

Proposition 4.1. *Fix $R > 0$, and let $B = B(0, R)$. Then, there exists $\delta = \delta(R)$ such that $P(n_t < \gamma_t p_t e^{\beta t}) \geq \delta$ for all large t.*

Remark 4.2. *The proposition above implies in particular that $P(n_t < \gamma_t p_t e^{\beta t}) \nrightarrow 0$ as $t \to \infty$ in the case of a fixed ball.*

Proof. Let the first particle not split in the interval $[0, 1]$ and at the same time exit the ball $B(0, R)$. Recall that for any $t > 0$, $e^{-\beta t}$ is the probability of not splitting over the interval $[0, t]$.

Start with

$$P(n_t < \gamma_t p_t e^{\beta t}) \geq P(n_t = 0).$$

We now find a lower bound for $P(n_t = 0)$. Let P_R be the probability that the initial particle exits $B(0, R)$ over $[0, 1]$. It is clear that $P_R > 0$ since $R > 0$ is fixed. Hence, it follows that

$$P(n_t = 0) \geq P_R e^{-\beta} > 0.$$

$\square$

The proof of the Proposition 4.3 is similar to the proof of the upper bound of Theorem 1.1. We refer the reader to [Section 3.1.2, Chapter 3]. Recall that p_t denote the probability of confinement for a standard Brownian motion.

Proposition 4.3. *For any given $\varepsilon > 0$, $\exists\, R > 0$ such that*

$$P(n_t < \gamma_t p_t e^{\beta t}) \leq \varepsilon,$$

where $p_t = \boldsymbol{P}_0(\sigma_B \geq t)$ with $B = B(0, R)$.

Proof. Start with estimate

$$P(\,\cdot\,) \leq P(\,\cdot\,|N_t > e^{\beta t} g_t) + P(N_t \leq e^{\beta t} g_t) \tag{50}$$

with $g_t \to 0$ as $t \to \infty$. From (7) in Proposition 2.5 that $P(N_t \leq k) = 1-(1-e^{-\beta t})^k \leq ke^{-\beta t}$ for $k \geq 1$, and then by setting $k = \lfloor e^{\beta t} g_t \rfloor$, we obtain

$$P(N_t \leq e^{\beta t} g_t) = P(N_t \leq \lfloor e^{\beta t} g_t \rfloor) \leq \lfloor e^{\beta t} g_t \rfloor e^{-\beta t} \leq g_t. \tag{51}$$

This provides to bound the second term on the right-hand side of equation (50) from above. Now, for $t > 0$, we have

$$\widetilde{P}_t(\,\cdot\,) = P(\,\cdot\,|N_t > e^{\beta t} g_t).$$

We bound on the right-hand side of (51) in the first term from above. Let $\widetilde{E}_t$ denote the expectation and $\mathcal{N}_t$ represent the set of particles belonging to Z that are alive at time t. Observe that $N_t = |\mathcal{N}_t|$. Let us denote the ancestral line of particle u until time t as $(Y_u(s))_{0 \leq s \leq t}$ for $u \in \mathcal{N}_t$. We randomly choose $\lfloor e^{\beta t} g_t \rfloor$ particles from $\mathcal{N}_t$, conditional on event $\{N_t > e^{\beta t} g_t\}$. This choice is independent of their genealogy and position, and we have $M_t := \lfloor e^{\beta t} g_t \rfloor$. Let $\mathcal{M}_t$ represent this collection of particles, and by setting $M_t := |\mathcal{M}_t|$, we obtain

$$\hat{n}_t = \sum_{u \in M_t} \mathbb{1}_{A_u},$$

where $A_u = \{Y_u(s) \in B_t \ \forall \ 0 \leq s \leq t\}$ and $\hat{n}_t$ counts the particles out of $\mathcal{M}_t$ whose ancestral lines stay within B_t over $[0,t]$. Each particle in the collection $\mathcal{M}_t$ has a Brownian ancestral line $(Y_u(s))_{0\leq s\leq t}$ since $\mathcal{M}_t$ was selected independently of the motion process. The many-to-one lemma the implies that for $t > 0$,

$$\widetilde{E}_t[\hat{n}_t] = \widetilde{p}_t M_t = \widetilde{p}_t \lfloor e^{\beta t} g_t \rfloor, \tag{52}$$

where $p_t = \mathbf{P}_0(\sigma_{B_t} \geq t)$. Observe that $\hat{n}_t \leq n_t$. Then, we use Chebyshev's inequality together with (50), (51) and (52), we obtain

$$\begin{aligned}
P(n_t < \gamma_t p_t e^{\beta t}) &\leq \widetilde{P}_t\left(\hat{n}_t < \gamma_t p_t e^{\beta t}\right) + g_t \\
&\leq \widetilde{P}_t\left(\widetilde{E}_t[\hat{n}_t] - \hat{n}_t > \widetilde{E}_t[\hat{n}_t] - \gamma_t p_t e^{\beta t}\right) + g_t \\
&\leq \widetilde{P}_t\left(|\hat{n}_t - \widetilde{E}_t[\hat{n}_t]| > p_t \lfloor e^{\beta t} g_t \rfloor - \gamma_t p_t e^{\beta t}\right) + g_t \\
&\leq \frac{\widetilde{\mathrm{Var}}_t(\hat{n}_t)}{[(g_t - \gamma_t)\widetilde{p}_t e^{\beta t} - p_t]^2} + g_t.
\end{aligned} \tag{53}$$

Let the variance be denoted by $\widetilde{\mathrm{Var}}_t$, and the variance related to $\widetilde{P}_t$. Now, we estimate $\widetilde{\mathrm{Var}}_t(\hat{n}_t)$. Then,

$$\begin{aligned}
\widetilde{\mathrm{Var}}_t(\hat{n}_t) &= \widetilde{\mathrm{Var}}_t\left(\sum_{u \in \mathcal{M}_t} \mathbb{1}_{A_u}\right) \\
&= M_t \mathrm{Var}(\mathbb{1}_A) + \sum_{1\leq i\neq j\leq M_t} \widetilde{\mathrm{Cov}}_t(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}) \\
&= M_t(p_t - p_t^2) + M_t(M_t - 1)\frac{\sum_{1\leq i\neq j\leq M_t} \widetilde{\mathrm{Cov}}_t(\mathbb{1}_{A_i}, \mathbb{1}_{A_j})}{M_t(M_t-1)} \\
&\leq g_t e^{\beta t}(p_t - p_t^2) + g_t e^{\beta t}(g_t e^{\beta t} - 1)\left[(\mathcal{E} \otimes \widetilde{P}_t)(A_i \cap A_j) - p_t^2\right].
\end{aligned} \tag{54}$$

The probability of randomly selected the pair (i,j) from the $M_t(M_t - 1)$ possible pairs in $\mathcal{M}_t$ is denoted by $\mathcal{P}$, and expectation of the probability of randomly selected the pair is denoted by $\mathcal{E}$. The average value of $\widetilde{P}_t(A_i \cap A_j)$ from the $M_t(M_t - 1)$ possible pairs in the randomly selected set $\mathcal{M}_t$ is denoted by $(\mathcal{E} \otimes \widetilde{P}_t)(A_i \cap A_j)$. The distribution of the splitting time of the ith and jth particles most recent common ancestor under $\mathcal{E} \otimes \widetilde{P}_t$ is denoted by $Q^{(t)}$.

Define

$$\widetilde{p}^{(t)}(x, s, dy) := \mathbf{P}_x(X(s) \in \ dy \mid X(z) \in B_t \ \ \forall\, 0 \leq z \leq s), \tag{55}$$

$$p^t_{s,x} := \mathbf{P}_x(\sigma_{B_t} \geq s). \tag{56}$$

When the Markov property is used at this splitting time, we have

$$(\mathcal{E} \otimes \widetilde{P}_t)(A_i \cap A_j) = p_t \int_0^t \int_{B_t} p_{t-s,x} \widetilde{p}^{(t)}(0, s, dx) Q^{(t)}(ds), \tag{57}$$

where $\widetilde{p}^{(t)}(x, s, dy)$ and $\widetilde{p}_{s,\delta r(t)}$ are defined in (55) and (56). When the Markov property is used a standard Brownian motion at time s with $0 < s < t$ in (55) and (56), we obtain

$$p_t = p_{s,0} \int_{B_t} p_{t-s,y} \widetilde{p}^{(t)}(0, s, dy). \tag{58}$$

For simplicity, we will use as $p_s = p_{s,0}$ from now on. Then, it follows from (57) and (58) that

$$(\mathcal{E} \otimes \widetilde{P}_t)(A_i \cap A_j) = \widetilde{p}_t^2 \int_0^t \frac{1}{p_s} Q^{(t)}(ds). \tag{59}$$

For $t > 0$, we define

$$J_t := \int_0^t \frac{1}{p_s} Q^{(t)}(ds).$$

Then, by (54) and (59), we obtain

$$\widetilde{\mathrm{Var}}_t(\hat{n}_t) \leq g_t p_t e^{\beta t} + g_t^2 p_t^2 e^{2\beta t}(J_t - 1). \tag{60}$$

Observe that $J_t - 1 \geq 0$. And now, we bound $J_t - 1$ from above. Note that R is a distance scale and we will use $kr(t)$ as a time scale when $k = 1$, fix it. For t large enough, we split J_t,

$$J_t = \int_0^R \frac{1}{p_s} Q^{(t)}(ds) + \int_R^t \frac{1}{p_s} Q^{(t)}(ds),$$

and define

$$J_t^{(1)} := \int_0^R \frac{1}{p_s} Q^{(t)}(ds), \qquad J_t^{(2)} := \int_R^t \frac{1}{p_s} Q^{(t)}(ds).$$

J_t is separated into $J_t^{(1)} + J_t^{(2)}$. Then, we will take use of fact them the integration of the $1/p_s$ is small enough over the first time interval $[0, R]$. We first bound $J_t^{(1)} - 1$ from above. Observe that p_s is nonincreasing in s, estimate $J_t^{(1)}$ as

$$J_t^{(1)} = \int_0^R \frac{1}{p_s} Q^{(t)}(ds) \le \frac{1}{p_R} \tag{61}$$

From Proposition 2.1,

$$1 - p_R = \mathbf{P}_0 \left(\sup_{0 \le s \le R} |X(s)| > R \right) \quad = \quad \exp\left[-\frac{R}{2}(1 + o(1)) \right] \tag{62}$$

as $R \to \infty$. Then, follows from (61) and (62) that

$$\begin{aligned} J_t^{(1)} - 1 &\le \frac{p_R}{1 - p_R} \\ p_R &\le e^{-R/3} \end{aligned} \tag{63}$$

We will utilize the following fact on the distribution $Q^{(t)}$ from [[17], Prop. 5] to bound $J_t^{(2)}$ from above: $Q^{(t)}$ is absolutely continuous in terms of the Lebesgue measure, denoted by ds, and its density function, denoted by $g^{(t)}$, satisfies

$$\exists\ C > 0, s_0 > 0 \text{ such that } \forall\ s \ge s_0,\ g^{(t)}(s) \le Cse^{-\beta s}.$$

We bound the second term from above. From Lemma 2.2,

$$p_s = \mathbf{P}_0 \left(\sigma_{B(0,R)} \ge s \right) \ge c_d e^{-\frac{\lambda_d s}{R^2}}$$

$$\frac{1}{p_s} \le \left(\frac{1}{c_d} \right) e^{\frac{\lambda_d s}{R^2}}$$

Then, we choose $R \ge s_0$

$$\begin{aligned} J_t^{(2)} = \int_R^t \frac{1}{p_s} g^{(t)}(s) ds &\le \int_R^t \left(\frac{1}{c_d} \right) e^{\frac{\lambda_d s}{R^2}} Cse^{-\beta s} ds \\ &\le C \left(\frac{1}{c_d} \right) \int_R^\infty s \exp\left[-\left(\beta - \frac{\lambda_d}{R^2} \right) s \right] ds. \end{aligned}$$

Let $\left(\beta - \frac{\lambda_d}{R^2}\right) := A$ and $\beta > \frac{\lambda_d}{R^2}$. Taking $s = u,\ dv = \exp(-As)ds,\ ds = du,\ v = \frac{\exp(-As)}{-A}$,

$$\begin{aligned} uv - \int v du &= s\frac{\exp(-As)}{-A} - \int_R^\infty \frac{\exp(-As)}{-A} ds \\ &= s\frac{\exp(-As)}{-A} - \frac{\exp(-As)}{A^2}\Big|_R^\infty \\ &= \left(\frac{Re^{(-AR)}}{A} + \frac{e^{(-AR)}}{A^2}\right) \\ &= \left(\frac{C}{c_d}\right)\left(\frac{Re^{(-AR)}}{A} + \frac{e^{(-AR)}}{A^2}\right), \end{aligned} \tag{64}$$

where we have used integration by parts. From (63) and (64), we obtain

$$J_t - 1 = J_t^{(1)} - 1 + J_t^{(2)} \leq 2e^{-R/3} + \frac{C}{c_d A} e^{-AR}\left(R + \frac{1}{A}\right) \tag{65}$$

It then follows from (81) and (85) in [18] that

$$\begin{aligned} P(n_t < \gamma_t p_t e^{\beta t}) &\leq \frac{\widetilde{Var}_t(\widehat{n}_t)}{[(g_t - \gamma_t)p_t e^{\beta t} - p_t]^2} + g_t \\ &\leq \frac{g_t p_t e^{\beta t} + g_t^2 p_t^2 e^{2\beta t}(J_t - 1)}{[(g_t - \gamma_t)p_t e^{\beta t} - p_t]^2} + g_t \\ &\leq \frac{e^{\beta t} + g_t(J_t - 1)}{(g_t - \gamma_t)^2}. \end{aligned} \tag{66}$$

Then, let us substitute $(J_t - 1)$,

$$P(n_t < \gamma_t p_t e^{\beta t}) \leq \frac{e^{\beta t} + 2g_t\left(2e^{-R/3} + \frac{C}{c_d A}e^{-AR}\left(R + \frac{1}{A}\right)\right)}{(g_t - \gamma_t)^2} \tag{67}$$

Given the assumption that $\gamma_t = e^{-\kappa r(t)}$ and $\kappa > 0$ and we choose $g_t = 2\gamma_t$. Then, we can continue (67) with

$$P(n_t < \gamma_t p_t e^{\beta t}) \leq \frac{e^{\beta t} + 4\gamma_t\left(2e^{-R/3} + \frac{C}{c_d A}e^{-AR}\left(R + \frac{1}{A}\right)\right)}{\gamma_t^2} \tag{68}$$

Then, using that $g_t = 2\gamma_t = 2e^{-\kappa r(t)}$, we obtain

$$P(n_t < \gamma_t p_t e^{\beta t}) \leq \begin{cases} e^{-\kappa r(t)}, & 0 < \kappa \leq -R/3 \\ e^{-R/3}, & \kappa > -R/3 \end{cases}$$

This completes the proof of the Proposition 4.3.

$\square$

Conclusion: The probability of the event $\{n_t < \gamma_t p_t e^{\beta t}\}$ for large t in the case of a fixed ball $B(0, R)$ has no tail.

CHAPTER V

SIMULATIONS

In this section, we present some simulations on our model of BBM in an expanding ball. We present some information about the code, such as the interpretation of some output and the illustrations. Furthermore, we present the algorithm that enables us to see the branching Brownian motion appear clearer with illustrations.

The model of branching Brownian motion was described in Section 1.1.

Here, we begin by introducing the basic terms about the simulation. The step size is a deterministic, and equal the length. The direction of each step is randomly chosen from the $2d$ nearest neighbors. Also, a larger step size determines that the particle moves much faster and covers longer distances. The scale λ is a parameter that determines the frequency of occurrence of new branching point during the simulation. This scale corresponds to the branching rate, that is, use of λ in the exponential distribution this was previously called β in Chapter 1. When λ increases, fewer branching events occur and longer walking routes result.

We now present the results of a simulation according to branching Brownian motion (BBM) algorithm.

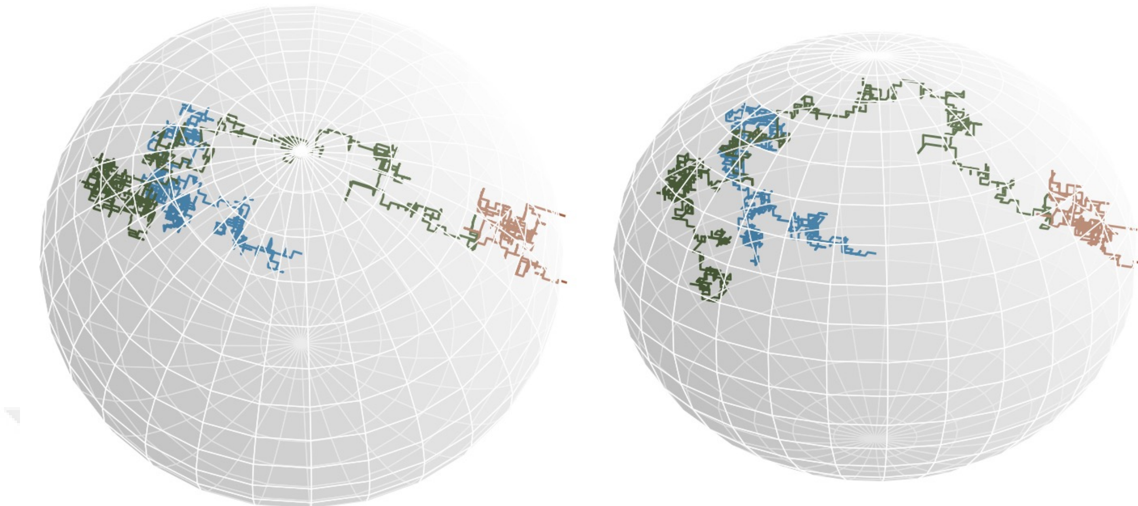

Figure 4: An illustration from different perspectives of a 3-dimensional BBM deactivated at the boundary of a ball. Here, the BBM is started at the origin with blue color code.

The parameters of the simulation in Figure 4 are as follows:

The radius of the ball is 100.

The step size to be taken for this BBM is 3.

The branching decision is taken based on the λ value of 1100 for this BBM.

We now describe the simulation in Figure 4.

1. Movement begins at the starting point (0,0,0), and after progressing 152 steps, initial point arrives the point (51, -9, 6). This point is added to the queue and waits to be processed later. Let us give a more detailed explanation of the term "queue". Each branching point is added to the queue and waits to be processed afterwards. Each time step updating is done the control of active and decative for each ancestral line in the system. The decision to add a point to the queue depends on this control.

2. Starting from (51,-9,6), and after progressing 1312 steps, branching point arrived the point (87, -24, -48). As this branching point deactivated at the boundary of a ball, indicating that no further branches can occur, it is removed from the queue and not processed further.

3. Starting from (51,-9,6), after progressing 371 steps, branching point arrives at (-66, -15, -18). This point is added to the queue and waits to be processed later.

4. Starting from (-66, -15, -18), after progressing 284 steps, branching point arrives at (-96, -12, -33. As this branching point deactivated at the boundary of a ball, indicating that no further branches can occur, it is removed from the queue and not processed further.

5. Starting from (-66, -15, -18), after progressing 882 steps, branching point arrives at (-90, -33, -30). As this branching point deactivated at the boundary of a ball, indicating that no further branches can occur, it is removed from the queue and not processed further.

Consequently, branching has occurred at points (51,-9,6) and (-66,-15,-18), and a total of 5 offspring are born in this simulation. In this way, each point in the branching process is added to the queue to be processed, and then is taken to be processed sequentially or is removed from the queue due to deactivated at the boundary of a ball. This approach neatly facilitates following branching events of simulation on our model of BBM in an expanding ball.

In this explanation are presented the result of a simulation on our model of BBM in an expanding ball. In Figure 5, we kept to fix the the step size while decreasing the value of λ.

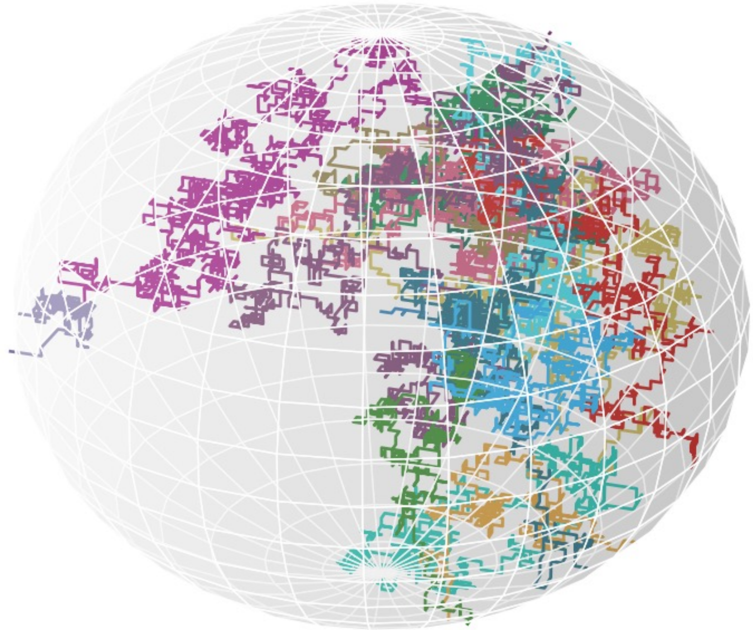

Figure 5: Another illustration from different perspectives of a 3-dimensional branching Brownian motion, using different parameters confined to a ball.

The parameters of the simulation in Figure 5 are as follows:

The radius of the ball is 100.

The step size to be taken for this BBM is 3.

The branching decision is taken based on the λ value of 750 for this BBM.

We now describe the simulation in Figure 5.

A total of 31 offspring are born starting from the initial point at (0,0,0) with BBM. We took the λ value as 750 in Figure 6, this means that branching is expected to happen each 750 steps during this process on average. More branching happened due to the smaller value of λ in Figure .

Now, let us explain why the first particle is always blue in the illustration. The rule is that the color changes when branching happens. The initial color was set to blue in our code, but it changes randomly after the first branching. Therefore, the color code has no significance. The color code denotes that these are from the same parent.

5.1 Algorithms

In this section, we present the algorithm that enables us to see the branching Brownian motion appear clearer with illustrations.

Algorithm 1: Branching Brownian Motion (BBM)

Input : Radius r, step size s, scale λ

Output: Walking history

```
Initialize empty history list H;
Initialize steps to branch n with exponential distribution parameter λ;
Initialize empty walking history list W;
Initialize empty queue Q;
Set bound contact to False;
Initialize empty branching record list B;
while Q is not empty do
    foreach checkpoint c in Q do
        Append c to H;
        Append c to B;
        while not bound contact do
            for i in range(n) do
                Take a step according to a random direction;
                if step is within the ball then
                    Append step to H;
                end
                else
                    Set bound contact to True;
                    Append step to H;
                    Break;
                end
            end
        end
        if not bound contact then
            Push c twice to Q;
        end
        Clear H;
    end
    Update n based on exponential distribution with parameter λ;
end
```

An explanation of the algorithm is step by step as follows:

Input and Output:

The algorithm requires input parameters for the walk within the ball: the radius of the ball r, the step size s, the scale λ. The output is the walking history.

Initialization:

An empty history list is initiliazed H. Steps to branch is initilized n. This means that the number of branches can change depending on the previous steps. The number of branches n has an exponential distribution, which is generated according to the parameter λ. An empty history list is initialized W. The empty queue is initiliazed Q. This is used to track where the movement branches come from. It is indicated that there is no contact with the boundary at the initial point. Empty branching record list is initialized B.

Main Loop:

The history list H and the branching record B, for each checkpoint c in queue, is added. Inner Loop is activated as long as there is no contact with the boundary, and n steps are taken. Every steps are gone random direction. If the step is within the ball, this step is added the history list H. If the step is outside the ball, this means that there is contact with the boundary. In this situation, bound contact flag is set to "True", this step is added to the history list, and it is removed from the inner loop. If there is no contact with the boundary, the checkpoint is added to the queue twice. The history list is cleansed. Steps to branch n are updated using an exponential distribution according to the λ parameter.

End of the Algorithm:

Walking history list is returned as the output when the main loop is completed.

CHAPTER VI

CONCLUSION

In this thesis, we study a d-dimensional BBM evolving inside a subdiffusively expanding ball. In particular, we investigate the large-time probability of the large-deviation event $\{n_t < \gamma_t p_t e^{\beta t}\}$ given in Theorem 1.1. The main output of this thesis is Theorem 1.1, in which we observe a phase transition in $P(n_t < \gamma_t p_t e^{\beta t})$, which can be explained in terms of the optimal strategies to realize the rare event $\{n_t < \gamma_t p_t e^{\beta t}\}$. This optimal strategy changes depending on the parameter κ in $\gamma_t = e^{-\kappa r(t)}$. Accordingly, the phase transition describes how the BBM realizes the large-deviation event $\{n_t < \gamma_t p_t e^{\beta t}\}$. There are two cases as follows: $\kappa > \sqrt{2\beta}$ and $0 < \kappa \leq \sqrt{2\beta}$.

When $\kappa > \sqrt{2\beta}$, the BBM suppresses the branching of initial particle, and at the same time moves it outside $B(0, r(t))$ over the time interval $[0, r(t)/\sqrt{2\beta}]$. The event $\{n_t = 0\}$ is realized when the initial particle is carried outside $B(0, r(t))$, which means there is no need for further suppression of branching.

When $0 < \kappa \leq \sqrt{2\beta}$, the BBM precisely suppresses the branching over the time interval $[0, (\kappa/\beta)r(t)]$. In other words, it suppresses the branching of the initial particle for a sufficiently long time until the mass $\gamma_t p_t e^{\beta t}$ is no longer atypically small for the branching Brownian motion to grow in the remaining interval. Then, the BBM behaves *"normally"* over the time interval $[(\kappa/\beta)r(t), t]$.

The main original contribution of this thesis is the proof of the upper bound of Theorem 1.1.

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VITA

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